Recurrent relations for branching coefficients of affine Lie algebra modules

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Simple and affine Lie algebras

Consider semisimple Lie algebra $\mathfrak{g}$ with commutator $[\cdot,\cdot]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$

Definition

Affine Lie algebra $\hat{\mathfrak{g}}$ is central extension of loop algebra, corresponding to semisimple Lie algebra $\mathfrak{g}$ with commutation relations

$$[Xt^n+\alpha c, Yt^m+\beta c]=t^{n+m}[X,Y]+(X,Y)n\delta_{n+m,0}c$$

$$Vir\subset U(\hat{\mathfrak{g}})\quad \text{(Sugawara construction)}$$

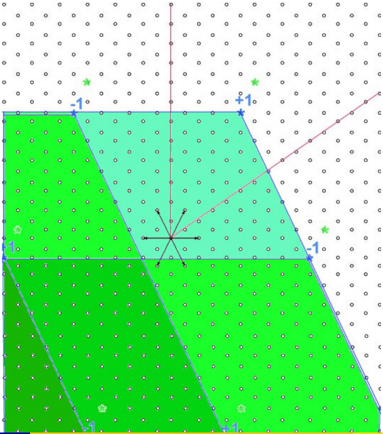
Current algebra of WZNW models in CFT is

$$\hat{\mathfrak{g}}\ltimes Vir$$

Primary fields are indexed by highest weights of irreducible $\hat{\mathfrak{g}}$-modules

Weyl character formula

$$ch\left(L^{\mu}\right) = \frac{\sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}; \qquad \frac{1}{R} = \frac{1}{\prod_{\alpha\in\Delta^+}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}}$$



Recurrent relation for weight multiplicities

We can use Weyl character formula to obtain recurrent relation on weight multiplicities which can be used for calculations:

$$m_{\xi} = -\sum_{w \in W \setminus e} \epsilon(w) m_{\xi-(w(\rho)-\rho)} + \sum_{w \in W} \epsilon(w) \delta_{(w(\mu+\rho)-\rho),\xi}. \quad (1)$$

The set of weights to sum over is obtained through expansion of denominator:

$$\sum_{w \in W} \epsilon(w) e^{w\rho-\rho} = \prod_{\alpha \in \Delta^{+}} \left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}$$

Branching

Consider reductive subalgebra $\hat{\mathfrak{g}} \supset \hat{\mathfrak{a}}$.
Each $\hat{\mathfrak{g}}$-module is also an $\hat{\mathfrak{a}}$-module, although $L^{\mu}_{\hat{\mathfrak{g}}}$ is not irreducible as $\hat{\mathfrak{a}}$-module. It can be decomposed into the sum of irreducible $\hat{\mathfrak{a}}$-modules:

$$L^{\mu}_{\hat{\mathfrak{g}}\downarrow\hat{\mathfrak{a}}} = \bigoplus_{\nu \in P^{+}_{\hat{\mathfrak{a}}}} b^{\mu}_{\nu} L^{\nu}_{\hat{\mathfrak{a}}}.$$

For characters we have

$$\pi_{\hat{\mathfrak{a}}} ch\left(L^{\mu}\right) = \sum_{\nu \in P^{+}_{\hat{\mathfrak{a}}}} b^{\mu}_{\nu} ch\left(L^{\nu}_{\hat{\mathfrak{a}}}\right).$$

We want to write recurrent relations.
Denote by $k^{(\mu)}_{\xi}$ signed branching coefficients.

$$\begin{array}{lll} k^{(\mu)}_{\xi} = b^{(\mu)}_{\xi} & \text{if} & \xi \in \bar{C}_{\hat{\mathfrak{a}}} \\ k^{(\mu)}_{\xi} = \epsilon(w) b^{(\mu)}_{w(\xi+\rho_{\hat{\mathfrak{a}}})-\rho_{\hat{\mathfrak{a}}}} & \text{where} & w \in W_{\hat{\mathfrak{a}}} : w(\xi + \rho_{\hat{\mathfrak{a}}}) - \rho_{\hat{\mathfrak{a}}} \in \bar{C}_{\hat{\mathfrak{a}}} \end{array}$$

The Weyl-Kac character formula leads to the relation

$$\pi_{\hat{\mathfrak{a}}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)=\sum_{\nu\in P^+_{\hat{\mathfrak{a}}}} b^{(\mu)}_{\nu}\frac{\sum_{\omega\in W_{\hat{\mathfrak{a}}}}\epsilon(\omega)e^{\omega(\nu+\rho_{\hat{\mathfrak{a}}})-\rho_{\hat{\mathfrak{a}}}}}{\prod_{\beta\in\Delta^+_{\hat{\mathfrak{a}}}}(1-e^{-\beta})^{\mathrm{mult}_{\hat{\mathfrak{a}}}(\beta)}}.$$

Now we want to multiply by R and rewrite this as recurrent relation.
Consider the roots orthogonal to $\Delta_{\hat{\mathfrak{a}}}$.
Let $\Delta^+_{\mathfrak{b}}=\left\{\alpha\in\Delta^+_{\mathfrak{g}}:\forall\beta\in\Delta_{\hat{\mathfrak{a}}};\alpha\perp\beta\right\}$ be the subset of positive roots of $\mathfrak{g}$ orthogonal to the root system of $\hat{\mathfrak{a}}$.
Denote by $W_{\mathfrak{b}}$ the subgroup of the Weyl group W generated by the reflections ω_β corresponding to the roots $\beta\in\Delta^+_{\mathfrak{b}}$.
The subsystem $\Delta_{\mathfrak{b}}$ determines the subalgebra $\mathfrak{b}=\mathfrak{a}_\perp\subset\mathfrak{g}$.
$\hat{\mathfrak{a}},\mathfrak{b}$ is the "orthogonal pair" of subalgebras in $\mathfrak{g}$.
Cartan subalgebra is decomposed $\mathfrak{h}=\mathfrak{h}_{\hat{\mathfrak{a}}}+\mathfrak{h}_{\mathfrak{d}}+\mathfrak{h}_{\mathfrak{b}}$.
Introduce

$$\mathcal{D}_{\hat{\mathfrak{a}}}:=\rho_{\hat{\mathfrak{a}}}-\pi_{\hat{\mathfrak{a}}}\rho.$$

$$\mathcal{D}_{\mathfrak{b}}:=\rho_{\mathfrak{b}}-\pi_{\mathfrak{b}}\rho.$$

Recurrent relation for branching coefficients

$$k_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)}\left(\sum_{u\in W/W_{\mathfrak{b}}} \epsilon(u)\,\dim\left(L_{\mathfrak{b}}^{\pi_{(\mathfrak{b})}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{b}}}\right)\right.$$

$$\left.\delta_{\xi-\gamma_0,\pi_{(\mathfrak{a}\oplus\mathfrak{h}_{\mathfrak{d}})}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{b}}} + \sum_{\gamma\in\Gamma_{\hat{\mathfrak{a}}\subset\hat{\mathfrak{g}}}} s(\gamma+\gamma_0)\,k_{\xi+\gamma}^{(\mu)}\right).$$

The recursion is goverened by the set $\Gamma_{\hat{\mathfrak{a}}\subset\hat{\mathfrak{g}}}$ of the weights $\{\xi\}$ appearing in the expansion

$$\prod_{\alpha\in\Delta^+\setminus\Delta_{\mathfrak{b}}^+}\left(1-e^{-\pi_{\hat{\mathfrak{a}}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\hat{\mathfrak{a}}}(\pi_{\hat{\mathfrak{a}}}\alpha)} = -\sum_{\gamma\in P_{\hat{\mathfrak{a}}}} s(\gamma)e^{-\gamma}$$

The weights are to be shifted by γ_0 – the lowest vector in $\{\xi\}$ – and the zero element is to be eliminated:

$$\Gamma_{\hat{\mathfrak{a}}\subset\hat{\mathfrak{g}}} = \{\xi-\gamma_0\}\setminus\{0\}\,.$$

Simple example: $A_1 \subset B_2$

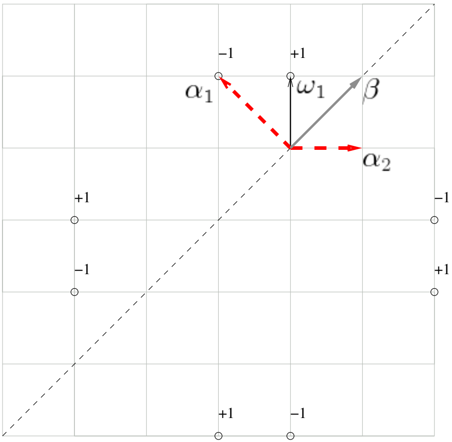


Figure: Roots of B_2, A_1 and Ψ^{ω_1}

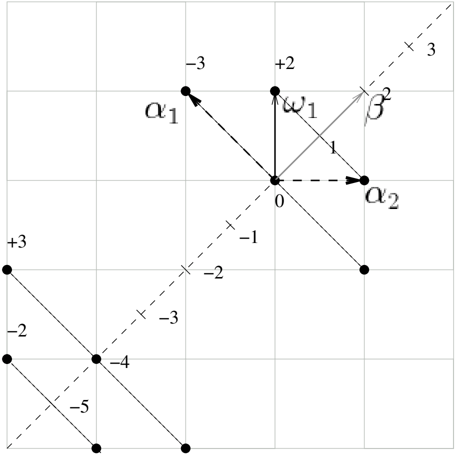


Figure: Orthogonal subalgebra $\mathfrak{b}$ and $\mathfrak{b}$-module dimensions

Surprises

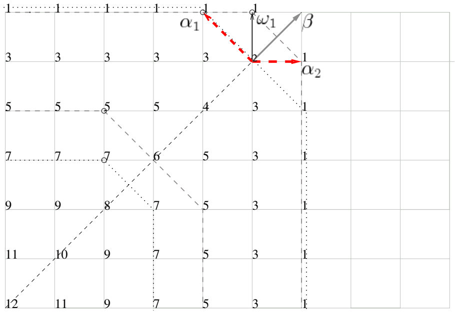


Figure: Embedding $A_1 \oplus u(1) \subset B_2$ and generalized Verma modules. Dashed – positive $\epsilon(u)$, dotted – negative.

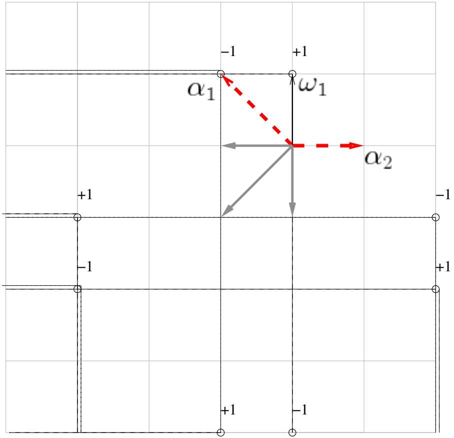
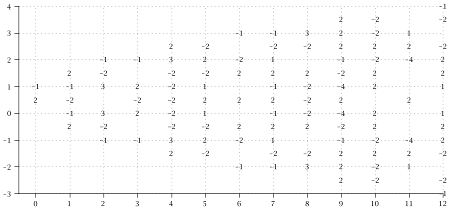


Figure: Embedding $A_1 \oplus u(1) \subset B_2$ and (deformed) A_2-modules



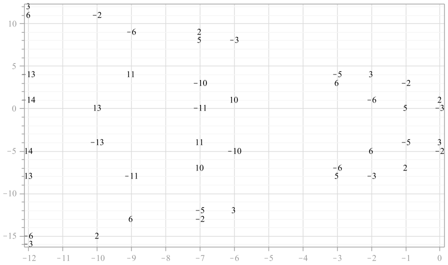
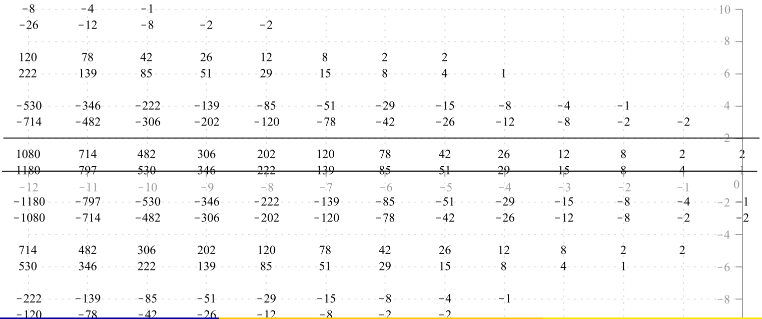


Figure: $\Gamma_{\hat{A}_1 \subset \hat{B}_2}$

Figure: $\pi_{\hat{A}_1}\left(\Psi^{\omega_1}_{\hat{B}_2}\right)$



Affine case

- Infinite number of generalized Verma modules – in each grade containing weights of singular element Ψ
- Not so simple relation with multiplicities of another algebra. $\Gamma_{\hat{\mathfrak{a}}\subset\hat{\mathfrak{g}}} \not\leftrightarrow \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}$ due to the difference in imaginary roots. E.g. $\widehat{A_1 \oplus u(1)} \subset \hat{B}_2$:

$$\frac{\cancel{\prod_{n>0}(1-e^{-n\delta})^2}\cancel{\prod_{n\geq 0}(1-e^{-\alpha_1-n\delta})}\dots}{\cancel{\prod_{n>0}(1-e^{-n\delta})^2}\cancel{\prod_{n\geq 0}(1-e^{-\alpha_1-n\delta})}\dots}$$

$$\prod_{n\geq 0}(1-e^{-\alpha_2-n\delta})\prod_{n\geq 0}(1-e^{-\alpha_1-\alpha_2-n\delta})\prod_{n\geq 0}(1-e^{-\alpha_1-2\alpha_2-n\delta})\dots$$

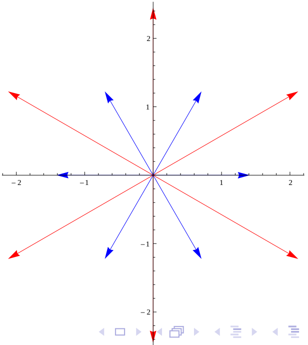
$$\not\leftrightarrow \prod_{n>0}(1-e^{-n\delta})^2\prod_{n\geq 0}(1-e^{-\alpha_1-n\delta})\prod_{n\geq 0}(1-e^{-\alpha_2-n\delta})$$

$$\prod_{n\geq 0}(1-e^{-\alpha_1-\alpha_2-n\delta})\dots$$

Conclusion

- Recurrent relation for branching coefficients works as in finite-dimensional case
- It can be used for computations
- We can relate recurrent relations for branching coefficients of one pair algebra-subalgebra with recurrent relation of another pair, but we can not interpret singular weights as highest weights of Verma modules.

E.g.: $\widehat{A_1 \oplus u(1)} \subset \hat{B}_2 \longleftrightarrow \widehat{u(1) \oplus u(1)} \subset \hat{A}_2$,
or
$\hat{A}_2 \subset \hat{G}_2 \longleftrightarrow \widehat{u(1) \oplus u(1)} \subset \hat{A}_2$



Thank you for your attention!