

Dubna, SQS'2011

July, 2011

Stueckelberg approach to conformal gravity

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We are going to discuss
conformal Weyl gravity theories

Our purpose:

**Second-derivative
gauge invariant formulation
for conformal gravity theories**

Ordinary-derivative formulation of conformal gauge fields

Requirements for ordinary-derivative formulation:

- i) **should be Lagrangian**
- ii) **second-derivative** (for bosonic)
- iii) **should retain on-shell D.o.F**
of generic higher-derivative theory
- iv) realization of global conformal symmetries **should be local**

Conformal Weyl gravity in 4d

$$\mathcal{L} = C_{\text{lin}}^{abce} C_{\text{lin}}^{abce}$$

$$C_{\text{lin}}^{abce} \quad \text{Weyl tensor}$$

$$g_{ab} = \eta_{ab} + \phi_{ab}$$

$$C_{abce} = R_{abce}$$

$$-\frac{1}{2}(g_{ac}R_{be} - g_{bc}R_{ae} + g_{be}R_{ac} - g_{ae}R_{bc})$$

$$+\frac{1}{6}R(g_{ac}g_{be} - g_{ae}g_{bc})$$

$$\mathcal{L} = \text{R}^{\text{ab}}\text{R}^{\text{ab}} - \frac{1}{3}\text{R}\text{R}$$

$$g_{ab} = \eta_{ab} + \phi_{ab}$$

$$\begin{aligned}
R^{ab} = & -\square\phi^{ab} + \partial^a\partial^c\phi^{bc} \\
& + \partial^b\partial^c\phi^{ac} - \partial^a\partial^b\phi^{cc}
\end{aligned}$$

$$R = \partial^a\partial^b\phi^{ab} - \square\phi^{aa}$$

$$\mathcal{L} = \phi^{ab} \square^2 \textcolor{blue}{P}^{ab\,ce} \phi^{ce}$$

$$\textcolor{blue}{P}^{ab\,ce} \equiv \frac{1}{2}(\pi^{ac}\pi^{be} + \pi^{ae}\pi^{bc}) - \frac{1}{3}\pi^{ab}\pi^{ce}$$

$$\pi^{ab} \equiv \eta^{ab} - \frac{\partial^a \partial^b}{\square}$$

Δ - conformal dimension

$$\Delta(\phi^{ab}) = 0$$

**Conformal dimensions are not
convenient to label fields**

Conformal charge k'

$$k' = \Delta - \Delta_0$$

Δ_0 canonical conformal dimension

$$\Delta_0 = \frac{d-2}{2}$$

$$d = 4$$

$$\text{conformal charge}(\phi^{ab}) = -1$$

$$\phi^{ab} \equiv \phi_{-1}^{ab}$$

$$d = 6$$

$$\text{conformal charge}(\phi^{ab}) = -2$$

$$\phi^{ab} \equiv \phi_{-2}^{ab}$$

General rule

for arbitrary conformal fields

Sum of conformal charges
in kinetic term $= 0$

Sum of conformal charges
in mass-like term $= 2$

invariance w.r.t dilatation symmetry

Step 1

use light-cone gauge

classify on-shell D.o.F by $so(2)$

On-shell D.o.F

Two traceless rank-2 tensor fields of
 $so(2)$

$$\phi_{-1}^{IJ} \quad \phi_1^{IJ}$$

one vector field

$$\phi_0^I$$

Step 2

In place of $\mathfrak{so}(2)$ fields
introduce Lorentz $\mathfrak{so}(3,1)$ fields

Two rank-2 tensor fields of $\mathfrak{so}(3,1)$

$$\phi_{-1}^{ab} \quad \phi_1^{ab}$$

one vector field

$$\phi_0^a$$

Step 3

Write most general second-order Lagrangian

$$\mathcal{L} = \phi_{-1}^{ab} \square \phi_1^{ab} + \phi_0^a \square \phi_0^a + \dots$$

and gauge transformations

$$\delta \phi_{-1}^{ab} = \partial^a \xi_{-2}^b + \partial^b \xi_{-2}^a + \eta^{ab} q_1 \xi_{-1}$$

$$\delta \phi_1^{ab} = \partial^a \xi_0^b + \partial^b \xi_0^a$$

$$\delta \phi_0^a = \partial^a \xi_{-1} + q_2 \xi_0^a$$

Requirement of gauge invariance

does not fix

Lagrangian uniquely

Step 4

Take into account restrictions
imposed by conformal algebra

Conformal algebra $so(d, 2)$

P^a translations

J^{ab} Lorentz rotations

D Dilatation

K^a conformal boosts

$$\mathbf{P}^a = \partial^a$$

$$\mathbf{J}^{ab} = x^a \partial^b - x^b \partial^a + S^{ab}$$

$$\mathbf{D} = x^a \partial^a + \Delta$$

**All that remains is to respect
conformal boost symmetries**

K^a

$$K^a = -\frac{1}{2}x^2\partial^a + x^a D + S^{ab}x^b + \mathbf{R}^a$$

higher-derivative: $\mathbf{R}^a = 0$

ordinary - derivative: $\mathbf{R}^a \neq 0$

Statement

**Requirements of gauge invariance
and conformal invariance
fix Lagrangian uniquely**

$$\mathcal{L} = \phi_{-1}^{ab} \mathbf{E}_{\text{EH}} \phi_1^{ab} + \phi_0^a \mathbf{E}_{\text{Max}} \phi_0^a$$

$$+ \phi_0^a (\partial^b \phi_1^{ba} - \partial^a \phi_1^{bb})$$

$$- \phi_1^{ab} \phi_1^{ab} + \phi_1^{aa} \phi_1^{bb}$$

$$\mathbf{E}_{\text{EH}} \phi^{ab} = \square \phi^{ab} - \partial^a \partial^c \phi^{bc} - \partial^b \partial^c \phi^{ac}$$

$$+ \partial^a \partial^b \phi^{cc} + \eta^{ab} (\partial^c \partial^e \phi^{ce} - \square \phi^{cc})$$

$$\mathbf{E}_{\text{Max}} \phi^a = \square \phi^a - \partial^a \partial^b \phi^b$$

Gauge transformations

$$\delta\phi_{-1}^{ab} = \partial^a \xi_{-2}^b + \partial^b \xi_{-2}^a + \eta^{ab} \xi_{-1}$$

$$\delta\phi_1^{ab} = \partial^a \xi_0^b + \partial^b \xi_0^a$$

$$\delta\phi_0^a = \partial^a \xi_{-1} + \xi_0^a$$

ϕ_0^a Stueckelberg field

Interrelation of ordinary-derivative and higher-derivative approaches to conformal fields

Two steps

- 1) Gauge away Stueckelberg fields
- 2) Exclude auxiliary fields via equations of motion

1) Gauge away Stueckelberg field ϕ_0^a

$$\mathcal{L} = \phi_{-1}^{ab} \mathbf{E}_{\mathbf{EH}} \phi_1^{ab} - \phi_1^{ab} \phi_1^{ab} + \phi_1^{aa} \phi_1^{bb}$$

2) EOM for auxiliary field ϕ_1^{ab} :

$$\phi_1^{ab} = \mathbf{E}_{\mathbf{EH}} \phi_{-1}^{ab}$$

$$\mathcal{L} = \phi_{-1}^{ab} \square^2 \mathbf{P}^{ab\,ce} \phi_{-1}^{ce}$$

Alternative ordinary-deriv approach (without vector Stueckelberg field)

$$\mathcal{L} = \phi_{-1}^{\text{ab}} \mathbf{E}_{\text{EH}} \phi_1^{\text{ab}} - \phi_1^{\text{ab}} \phi_1^{\text{ab}} + \phi_1^{\text{aa}} \phi_1^{\text{bb}}$$

which approach is preferable ?

quantization ?

Interacting Weyl gravity in 4d

Free

$$\phi_{-1}^{ab} \quad \phi_1^{ab}$$

$$\phi_0^a$$

Interacting

$$\phi_{-1}^{ab} \Longrightarrow e_{\mu}^a$$

$$g_{\mu\nu} = e_{\mu}^a e_{\nu a}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{-1 \mu\nu}$$

$$\phi_0^a \Longrightarrow b^a$$

Interacting

$$e_{\mu}^a$$

$$\phi_1^{ab}$$

$$b^a$$

Deformation procedure

$$\partial \Rightarrow D$$

$$\mathcal{L}_{int} = \mathcal{L}_0 + interaction$$

$$\delta_{int}\phi = \delta_0\phi + \dots$$

\mathcal{L}_{int} no more than two deriv

δ_{int} no more than one deriv

$$\mathcal{L} = \phi_1^{ab} \widehat{\mathbf{G}}^{ab} - \mathbf{F}^{ab} \mathbf{F}^{ab} - \phi_1^{ab} \phi_1^{ab} + \phi_1^{aa} \phi_1^{bb}$$

$$\widehat{G}^{ab} = R^{ab} - \frac{1}{2} \eta^{ab} R$$

$$+ \mathbf{D}^{\mathbf{a}} \mathbf{b}^{\mathbf{b}} + \mathbf{D}^{\mathbf{b}} \mathbf{b}^{\mathbf{a}} + \mathbf{b}^{\mathbf{a}} \mathbf{b}^{\mathbf{b}}$$

$$F^{ab} \;=\; D^a b^b - D^b b^a$$

1) Gauge away **Stueckelberg field**

$$b^a$$

$$\mathcal{L} = \phi_1^{\text{ab}} R^{ab} - \phi_1^{\text{ab}} \phi_1^{\text{ab}} + \phi_1^{\text{aa}} \phi_1^{\text{bb}}$$

2) EOM for auxiliary field ϕ_1^{ab} :

$$\phi_1^{\text{ab}} = R^{\text{ab}}$$

Higher derivative Lagrangian

$$\mathcal{L} = R^{ab} R^{ab} - \frac{1}{3} R R$$

$$\mathcal{L} = C^{abce} C^{abce}$$

Conformal Weyl gravity in 6d

$$\mathcal{L}_{free} = C_{\text{lin}}^{abce} \square C_{\text{lin}}^{abce}$$

$$g_{ab} = \eta_{ab} + \phi_{ab}$$

Step 1

use light-cone gauge

classify **on-shell D.o.F by** $so(4)$

Three traceless rank-2 tensor fields

$$\phi_{-2}^{IJ}$$

$$\phi_0^{IJ}$$

$$\phi_2^{IJ}$$

two vector fields

$$\phi_{-1}^I$$

$$\phi_1^I$$

one scalar field

$$\phi_0$$

Step 2

In place of $\mathfrak{so}(4)$ fields

introduce Lorentz $\mathfrak{so}(5,1)$ fields

Three rank-2 tensor fields

$$\phi_{-2}^{ab}$$

$$\phi_0^{ab}$$

$$\phi_2^{ab}$$

two vector fields

$$\phi_{-1}^a$$

$$\phi_1^a$$

one scalar field

$$\phi_0$$

Step 3

Write general second-order Lagrangian

$$\mathcal{L} = \phi_{-2}^{ab} \square \phi_2^{ab} + \phi_0^{ab} \square \phi_0^{ab}$$

$$+ \phi_{-1}^a \square \phi_1^a + \phi_0 \square \phi_0 + \dots$$

+ one derivative contrib.

+ mass – like contrib.

general form gauge transformations

$$\delta\phi_{-2}^{ab} = \partial^a \xi_{-3}^b + \partial^b \xi_{-3}^a + \eta^{ab} q_1 \xi_{-2}$$

$$\delta\phi_0^{ab} = \partial^a \xi_{-1}^b + \partial^b \xi_{-1}^a + \eta^{ab} q_2 \xi_0$$

$$\delta\phi_2^{ab} = \partial^a \xi_1^b + \partial^b \xi_1^a$$

$$\delta\phi_{-1}^a = \partial^a \xi_{-2} + q_3 \xi_{-1}^a$$

$$\delta\phi_1^a = \partial^a \xi_0 + q_4 \xi_1^a$$

$$\delta\phi_0 = q_5 \xi_0$$

Statement

**Requirements of gauge invariance
and conformal invariance
fix Lagrangian uniquely**

$$\mathcal{L} = \phi_2^{ab} \mathbf{E}_{\mathbf{EH}} \phi_{-2}^{ab} + \phi_0^{ab} \mathbf{E}_{\mathbf{EH}} \phi_0^{ab}$$

$$+ \phi_1^a \mathbf{E}_{\mathbf{Max}} \phi_{-1}^a + \phi_0 \square \phi_0$$

$$+ \phi_1^a \partial^{\mathbf{b}} \phi_0^{ab} + \phi_{-1}^a \partial^{\mathbf{b}} \phi_2^{ba}$$

$$+ \phi_0^{ab} \phi_2^{ab}$$

Interacting Weyl gravity in 6d

Free

$$\phi_{-2}^{ab}$$

$$\phi_0^{ab}$$

$$\phi_2^{ab}$$

$$\phi_{-1}^a$$

$$\phi_1^a$$

$$\phi_0$$

Interacting

$$\phi_{-2}^{ab} \Longrightarrow e_{\mu}^a$$

$$g_{\mu\nu} = e_{\mu}^a e_{\nu a}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{-2}{}_{\mu\nu}$$

$$\phi_{-1}^a \Longrightarrow b^a$$

Interacting

$$e^a_\mu$$

ϕ_0^{ab}

ϕ_2^{ab}

$$b^a$$

ϕ_1^a

$$\phi_0$$

Deformation procedure

$$\partial \Rightarrow D$$

$$\mathcal{L}_{int} = \mathcal{L}_0 + interaction$$

$$\delta_{int}\phi = \delta_0\phi + \dots$$

\mathcal{L}_{int} no more than two deriv

δ_{int} no more than one deriv

$$\mathcal{L} = \phi_2^{ab} \widehat{G}^{ab} - \phi_0^{ab} E_{EH} \phi_0^{ab} + R^{abce} \phi_0^{bc} \phi_0^{ae}$$

$$+ F^{ab}(b) F^{ab}(\phi_1) + \phi_0 \square \phi_0$$

$$+ \phi_2^{ab} \phi_0^{bc}$$

$$+ \phi_0^{\mathbf{a}\mathbf{b}} \phi_0^{\mathbf{b}\mathbf{c}} \phi_0^{\mathbf{c}\mathbf{a}} + \dots$$

$$\widehat{G}^{ab} = R^{ab} - \frac{1}{2} \eta^{ab} R + \mathbf{D}^{\mathbf{a}} \mathbf{b}^{\mathbf{b}} + \mathbf{D}^{\mathbf{b}} \mathbf{b}^{\mathbf{a}} + \dots$$

1) Gauge away **Stueckelberg fields**

$$b^a \quad \phi_1^a \quad \phi_0$$

$$\mathcal{L} = \phi_2^{\text{ab}} R^{ab} - \phi_0^{ab} E_{EH} \phi_0^{ab} + R^{abce} \phi_0^{bc} \phi_0^{ae} \\ + \phi_2^{\text{ab}} \phi_0^{bc} + \phi_0^{\text{ab}} \phi_0^{\text{bc}} \phi_0^{\text{ca}} + \dots$$

2) EOM for auxiliary field ϕ_2^{ab} :

$$\phi_0^{\text{ab}} = R^{\text{ab}}$$

Higher derivative Lagrangian

$$\mathcal{L} = R^{ab} \square R^{ab} + R \square R + C R^2 + R^3$$

C Weyl tensor

Conformal invariants in 6d

Bonora,et.al. 1986

$$I_1 = R^{ab} \square R^{ab} + R \square R + C R^2 + R^3$$

$$I_2 = C^3$$

$$I_3 = C^3$$

C Weyl tensor

$$I_1 = \mathcal{L}$$

$$I_2 = \mathbf{C}^3$$

$$I_2 = \mathbf{C}^3$$

$$\mathbf{C}^{\text{abce}} = \mathbf{C}^{\text{abce}} + \eta^{\text{bc}} \phi_0^{\text{ae}} + \dots$$

+ Stueckelberg fields

$$\mathbf{C}^{\text{abce}} \text{ inv. under Stueck. symm.}$$

Using auxiliary fields \mathbf{I}_1 can be cast into ordinary derivative form

\mathbf{I}_2 and \mathbf{I}_3 involve higher derivatives (auxiliary fields are not helpful)

Our definition of $6d$ conformal gravity

$$\mathcal{L} = \mathbf{I}_1$$

Problems for future

Quantization of ordinary-derivative
Weyl gravity

Quantum behavior of I_1 ???

Is I_1 renormalized

without involving I_2 and I_3

in renormalization procedure ???

Possible applications:

Einstein gravity from Weyl gravity

$$\mathcal{L}_{\text{Weyl}} = \frac{1}{g^2} (\phi_1^{\mu\nu} R^{\mu\nu} - \phi_1^{\mu\nu} \phi_1^{\mu\nu} + \phi_1^{\mu\mu} \phi_1^{\nu\nu})$$

EOM

$$E_{EH} \phi_1^{\mu\nu} + R^{\cdots} \phi^{\cdots} + \phi_{\dot{1}} \phi_{\dot{1}} = 0$$

$$R^{\mu\nu} + \phi_1^{\mu\nu} + \dots = 0$$

Solution of EOM

$$\bar{\phi}_1^{\mu\nu} = -2\rho\bar{g}^{\mu\nu}$$

$\bar{g}_{\mu\nu}$ -metric of (A)dS

$$R_{\mu\nu\lambda\sigma} = \rho(\bar{g}_{\mu\lambda}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\lambda})$$

$$\rho = -1 \quad \text{for AdS}$$

$$\rho = 1 \quad \text{for dS}$$

Relax restriction on metric tensor, consider

$$\phi_1^{\mu\nu} = -2\rho g^{\mu\nu}$$

and plug such $\phi_1^{\mu\nu}$ in ordinary derivative Weyl Lagrangian

$$\mathcal{L}_{\text{Weyl}} = -\frac{2\rho}{g^2}(R - 6\rho)$$

$$\mathcal{L}_{\text{EH}} = \frac{1}{2\kappa^2}(R - 6\rho)$$

$$\kappa^2 = -\frac{1}{4\rho}g^2$$

We are left with $\rho = -1$ **AdS**

Problems for future

AdS/CFT: Bulk (bosonic) fields satisfy second-deriv. equations of motion.

We expect that ordinary form of conformal fields should have interesting implications to AdS/CFT

ordinary-derivative formulation of $N = 4$ conformal supergravity.

Note that $N = 4$ conformal SUGRA couples with operators of \mathcal{N} SYM