Fuzzy Topology of Phase Space and Gauge Fields

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Motivations:

Study of geometric structures can be important for the construction of quantum space-time

Axioms of Set theory and Topology are the basis of any particular geometric structure

Examples:

Discrete space-time (Snyder, 1947)

Noncommutative geometry (Connes, 1991)

Sets, Topology and Geometry

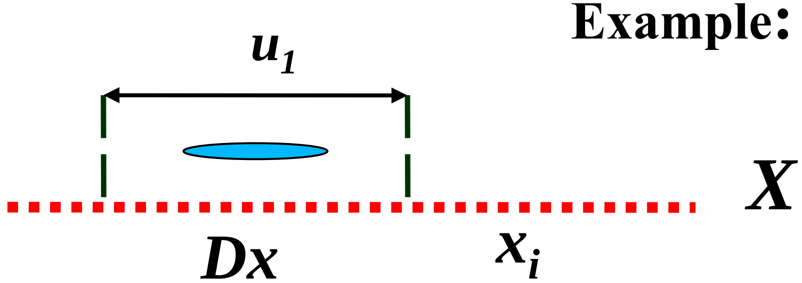
Example: 1-dimensional Euclidian geometry is constructed on ordered set of elements $X = \{ x_l \}$; x_l **- points**

$$\forall x_i , x_j \qquad x_i \le x_j .or. x_j \le x_i$$

Partial ordered set - *P*

Beside $x_i \leq x_j$ it can be also $x_i \sim x_j$

Example: $P = X \cup P^u; \quad P^u = \{u_l\}$



$u_1 \sim x_k, \qquad \forall x_k \in Dx$

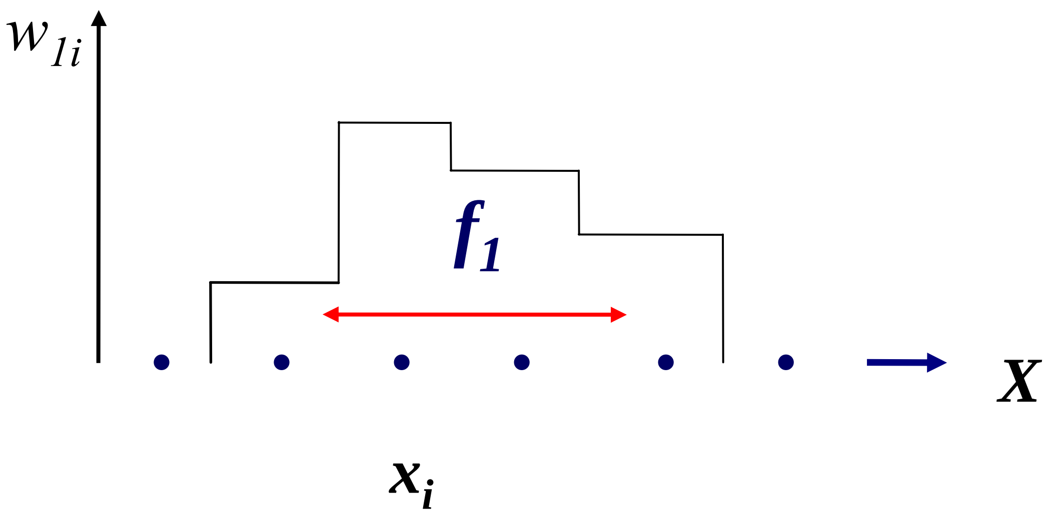
u_1, x_k are incomparable P elements

Fuzzy ordered set (foset) - $F = \{ f_i \}$

f_j - fuzzy points, *(Zeeman, 1968)*

$$f_i \sim f_k, \quad w_{i,k} > 0, \ \Sigma w_{jk} = 1$$

Example: $F_T = X \cup F$; $F = \{ f_1 \}$



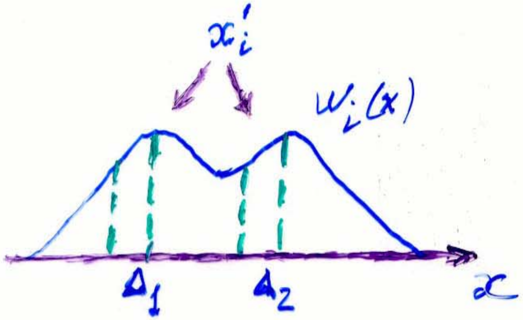
Continuous fuzzy ordered sets

Foset $F^x = O^x \cup P'$

O^x - is continious $R^1 = \{x_a\}$

$P' = \{x_i'\}, \; i = 1, N$ discrete set

$\forall x_i' \sim w_i(x) \geq 0 \; ; \; \int_{O^x} w_i(x)\,dx = 1$



w_i support O_s^x

$O_s^x \in O^x$

$\forall x_a \in O_s^x, \quad x_i' \parallel x_a$

$\forall \Delta_1, \Delta_2 \in O_s^x \; ; \quad x_i' \in \Delta_1$ and $x_i' \in \Delta_2$

Topological structure of F^x

F^x is nonprobabilistic structure!

Classical Mechanics

particle is ordered point $x(t)$ in O^X

its state: $|m\} = (x(t), \dot{x}(t))$

Fuzzy Mechanics (FM)

Particle is fuzzy point $m(t)$ in O^F

Fuzzy state $|m\} = (w(x,t), \dots ?)$

$? = Q_1(x), \dots Q_N(x);\ Q_2'(x, x'), \dots;\ Q_2''(x, x', x'')$

Evolution:

$\hat{N}(t)|m_0\} = |m(t)\}$

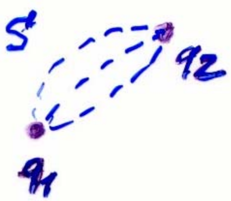
minimal FM

Fundamental laws of dynamics

a) Classical Mechanics: Minimal action

$$S = \int_{q_1}^{q_2} L(q,\dot{q})\,dt \quad - \text{min}$$

b) Quantum Mechanics - Path Integral



$$A(q_1,q_2) = e^{i\int_{q_1}^{q_2} S}[dq]$$

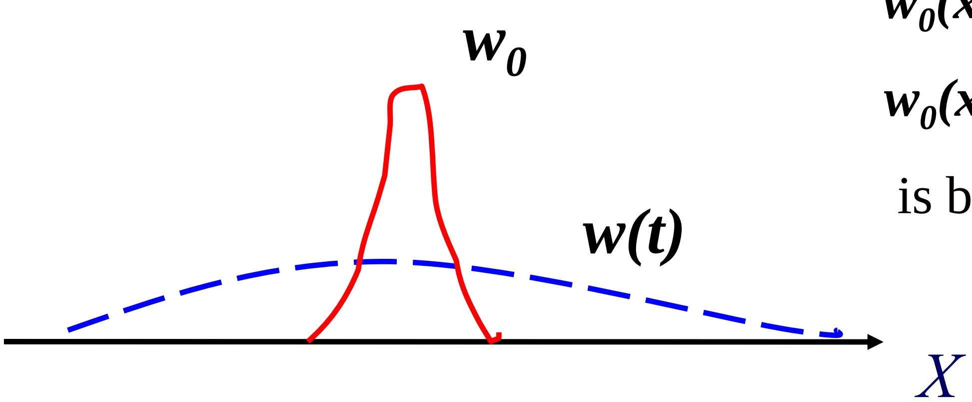
$$\Psi(t) = \hat{U}_t \Psi_0$$

Too complicated to be truly fundamental ?!

Fuzzy mechanics – space symmetry restoration

as the general low of free motion

Global (and local) symmetries are very important in microphysics



$w_0(x) = w_0(-x)$

$w_0(x)$: space shift symmetry is broken

Fuzzy *m* motion – topological diffusion

Low of free *m* evolution: space symmetry of *m* state is restored:

$w_0(x) \rightarrow const$ **at very large t**

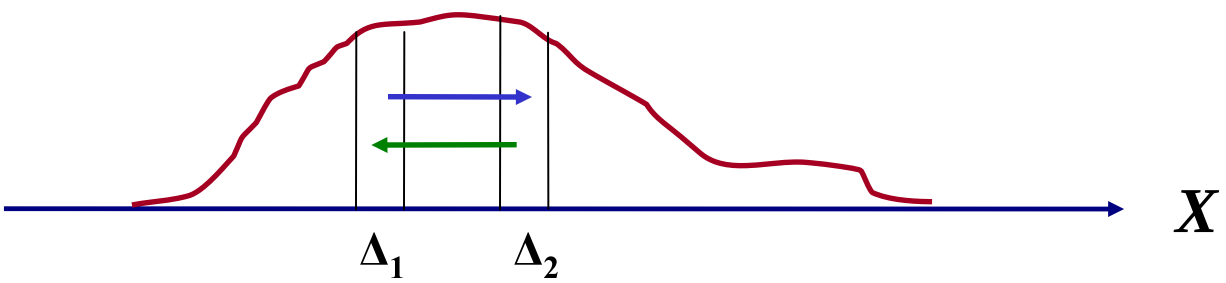
Free *m* evolution $U(t)$: $w_0(x) \rightarrow const$; as fast as possible

Problem: if $|g\}= w(x)$, then $\langle V_x \rangle = 0$, for any *m* state $|g\}$

Fuzzy (virtual) motion

Fuzzy state g **:** $m \,\|\, \Delta_i$**.** *and* **.** $m \,\|\, \Delta_{i+1}$**….**

Even if $dw(x,t) = 0$ **, there is virtual** m **motion**



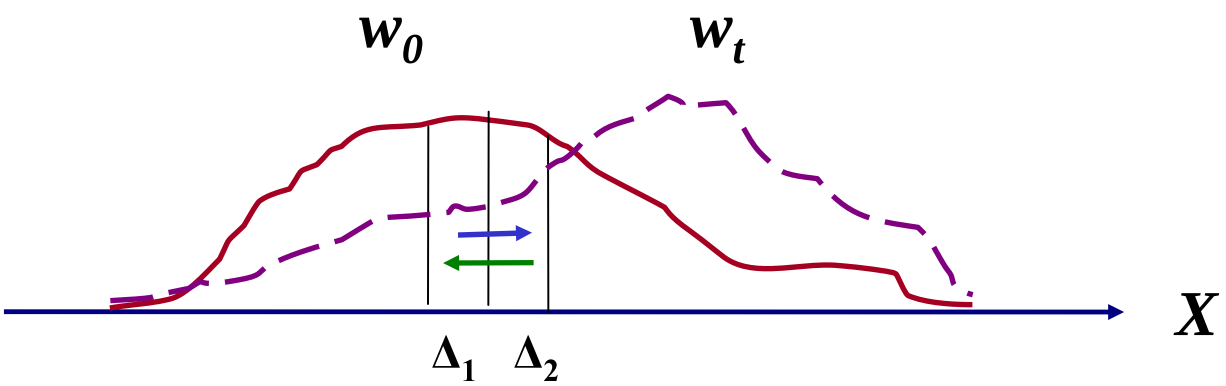
m **flow between** Δ_i**.** **and** Δ_2 **:** $f(x_1, x_2) \sim dw(x_1) - dw(x_2)$

In the simple case, if $w(x_1) = w(x_2)$ **then** $f(x_1, x_2) = 0$,

New parameter $\alpha(x)$ **, it can result in effective** w **flow:**

$$f(x_1, x_2) = \alpha(x_1) - \alpha(x_2) \neq 0$$

As the result *m* can move with $<V_x> \neq 0$, if $\alpha(x) \neq const$



m flow between Δ_i. and Δ_2 : $f(x_1, x_2) \sim dw(x_1) - dw(x_2)$

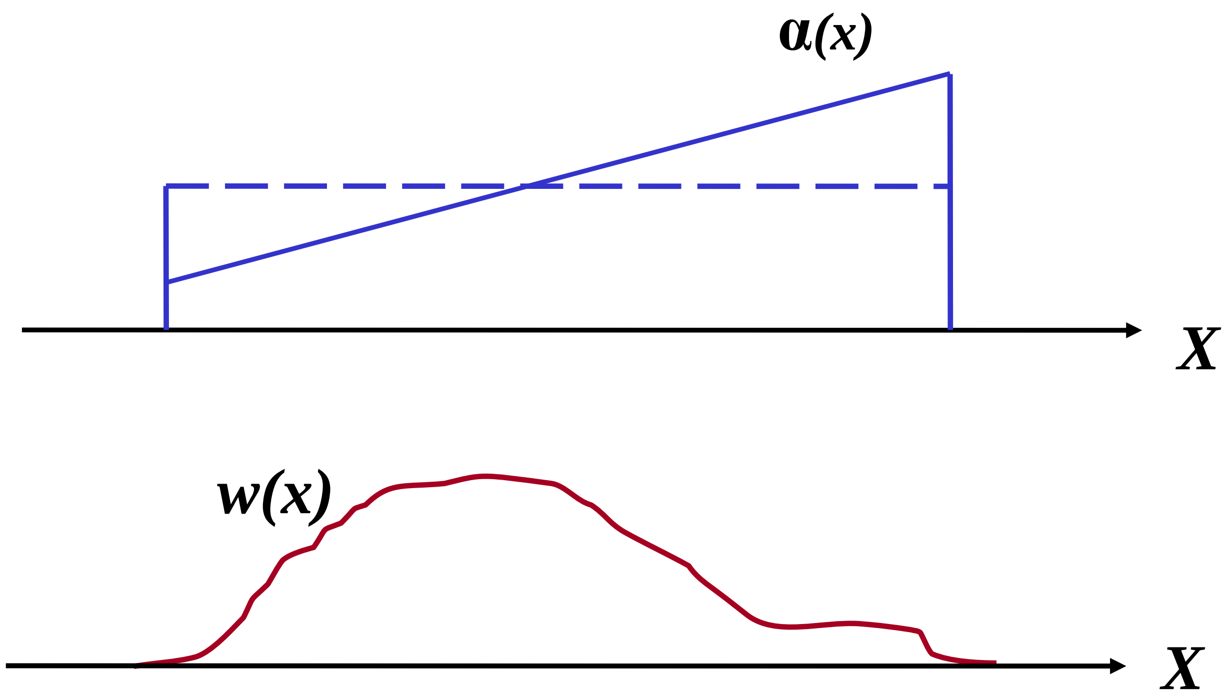
If $w(x_1)=w(x_2)$ then $f(x_1, x_2) = 0$,

New parameter $\alpha(x)$, it results in effective *w* flow:

$$f(x+dx, x) = \alpha(x+dx) - \alpha(x) \neq 0$$

m Velocity as the result of space symmetry violation for |*g*}

α*(x)* is analog of quantum phase



$$m \text{ velocity: } \; <V_x> \sim \; < \frac{d\alpha}{dx} >$$

α(x) is analog of quantum phase

m **particle's state** $|g\}$ $= \{ w(x), \alpha(x) \}$

$\alpha(x)$ **is analog of quantum phase**

The problem: to find dynamical $|g\}$ representation $g(x,t)$:

$$\frac{dg}{dt} = \hat{F}g$$

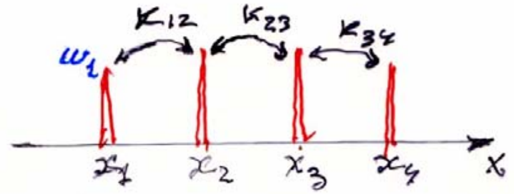
We have two degrees of freedom $g = \{ w(x), \alpha(x) \}$**, it can be transfered to:**

$g(x) = a(x) + ib(x),$

such that $a,b = 0,$ **if** $w(x) = 0$

this is symmetric g **representation**

a is $|g\}$ free parameter



$n = 4$

$$w_0(x) = \sum w_i \, \delta(x - x_i)$$

$n=2 \to a \sim k_{12}$ - correlation w_1 and w_2

if $k_{13} = k_{12} + k_{23}$

Then k_{ij} - phase $\alpha(x_i, x_j)$

$n \to \infty \quad |g\} = \{ w(x), \alpha(x) \}, \, x_0$

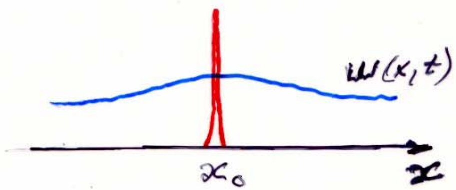
$$|g\} \to g(x,t) = \sqrt{w(x)}\, e^{i\alpha(x)} = g_1(x) + i g_2(x)$$

$$g(x) \to 0 \iff w(x) \to 0$$

$|g\} = \{ w(x), \alpha(x, x') \}$ - Bilocal state!

$g(x) = \{w(x), a(x)\} \to g_1(x) + i\, g_2(x)$

$g(x,t_0) = c\,\delta(x - x_0)$



$g(x) \sim \sum c_i\, \delta(x - x_i)$!

$g(t) = \hat{U}_t\, g(t_0)$; $\hat{U}_t$ - unitary

$U_{t_1+t_2} = U_{t_1} \cdot U_{t_2} \to U_t = e^{-i\hat{H} t}$, $\forall t$

$\hat{H}$ - time independent

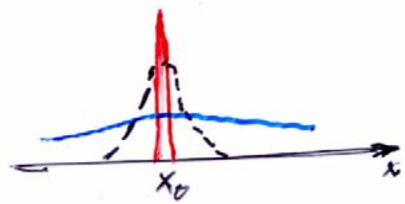
$\hat{V}_a$ - space shift ; $\hat{V}_a = e^{a\frac{\partial}{\partial x}}$

for free motion $[\hat{V}_a, \hat{U}_t] = 0$

fourier-transform : $\varphi(p,t) = \int g(x,t)\, e^{ipx} dx$

$[\hat{V}_a, \hat{H}] = 0 \to \hat{H}_p = F(p)$

for $g(x, t=0) = \delta(x - x_0) \to \varphi(p,t) = e^{-iF(p)t + ipx_0}$



$$g(x,t) \xrightarrow[t \to t_0+]{} \delta(x - x_0)$$

$$\int g(x, t_j)\, dx = 1$$

$g(x, t_j)$ – δ-sequence $\{t_j\} \to t_0$

$$z = \frac{x}{f(t)} \quad ; \quad f(t) \xrightarrow[t \to t_0]{} 0 \quad ; \quad g(x,t) = \frac{e^{i\gamma(z)}}{f(t)}$$

if $\int e^{i\gamma(z)} dz = 1 + O(t)$

$$\eta(p,t) = \int g(x,t) e^{ipx} dx = e^{-i\Gamma(pf(t))}$$

$$\varphi(p,t) = \eta(p,t) \to e^{-iF(p)t} = e^{-i\Gamma(pf(t))}$$

$$\hat{H}_p = F(p) = \frac{p^s}{2m_0} \quad ; \quad m_0 > 0$$

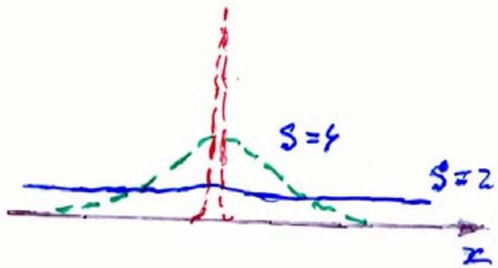
$$s = 2, 4, \dots, 2n, \dots$$

only $s = 2$ gives $w(x,t) \underset{x \to \infty}{\neq} 0$!

$$-i \frac{\partial g}{\partial t} = \hat{H}_0 g = \frac{\hat{p}^2}{2m_0} g$$ – Schrödinger Equation

$$\hat{H}_0 = \frac{p^s}{2m_0} \qquad s = 2, 4, 6, \ldots 2n$$

$$\varphi(p,t) = e^{-i\frac{p^s t}{2m}} \to g(x,t)$$



$$w(x,t) \underset{x\to\infty}{\neq} 0$$

$$s = 4$$

$$w \sim \frac{1}{(tx^2)^{\frac{1}{6}}}$$

$$s = 2 \to g_0 = \delta(x - x_0) \to g(x,t) = \frac{e^{-\frac{ix^2 m}{t}}}{c\sqrt{t}}$$

$$\psi(x) = \sum a_i g_i(x,t) \; ; \quad -i\frac{\partial \psi}{\partial t} = \frac{1}{2m}\frac{\partial^2}{\partial x^2}\psi$$

RF : $m \to \infty$ Galilean Transform.

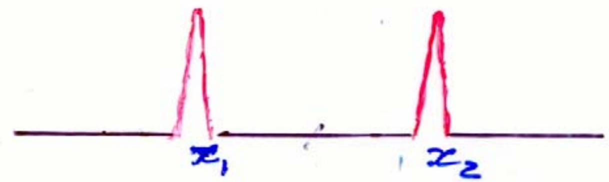
$c \neq \infty$, From ψ linearity

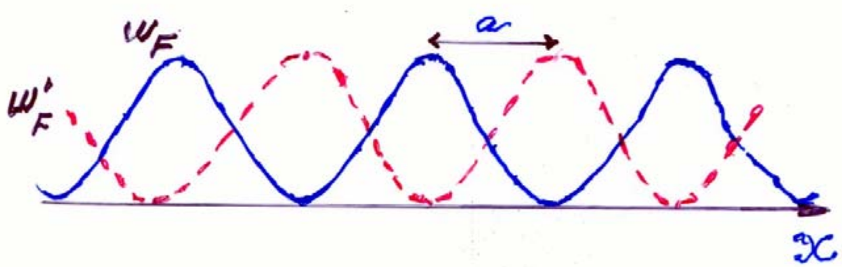
$$-i\frac{\partial \psi}{\partial t} = (\vec{\alpha}\hat{p} + \beta m)\psi \; :$$

ψ – 4-spinor

$$|g_0\} \sim w_0(x) = \frac{1}{2}\,\delta(x - x_1) + \frac{1}{2}\,\delta(x - x_2)$$

Does $|g_0\}$ include other parameters q_i?





for $|g_F\}$ $\langle x \rangle$ is undefined

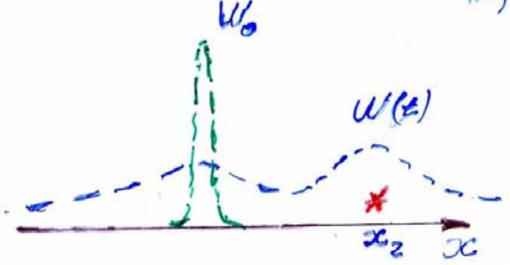
hence if $w_F(x,t)$ is solution

then $w'_F(x,t) = w_F(x+a,t)$ is solution

$\forall a$; $-\infty \leq a \leq \infty$

so, $|g_0\}$ has a-parameter
in addition to $w_0(x)$

Interactions on Fuzzy Manifold



m, m_2 interactions

$m_2 \to \infty$

$\sigma_2(x) \to 0$

$\hat{H}_{int} = Q_1 \cdot Q_2 \cdot \hat{B}_{int}$

$\hat{H}_{int}$ perturbs $g(x,t)$ restoration of R^3 symmetry by m free evolution

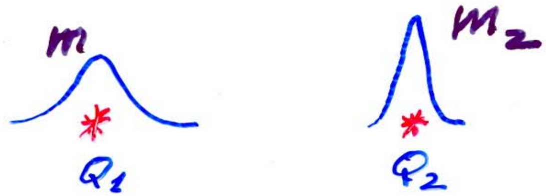
$g(x) \sim (w(x), \alpha(x)\}$ components

$$-i\frac{\partial g}{\partial t} = -i\frac{\partial}{\partial t}\left(\sqrt{w}\, e^{i\alpha(x)}\right) = \left(\hat{H}_0 + \hat{H}_{int}\right) g$$

$$\begin{cases} \dfrac{\partial \sqrt{w}}{\partial t} = \dfrac{\sqrt{w}}{2m}\dfrac{\partial^2 \alpha}{\partial x^2} + \dfrac{\partial \sqrt{w}}{\partial x} \cdot \dfrac{\partial \alpha}{\partial x} \\ \dfrac{\partial \alpha}{\partial t} = \dfrac{1}{2m}\dfrac{\partial^2 w^{\frac{1}{2}}}{\partial x^2} - \dfrac{\sqrt{w}}{2m}\left(\dfrac{\partial \alpha}{\partial x}\right)^2 + \hat{H}_{int} \end{cases}$$

$\hat{H}_{int} = Q_1 \cdot Q_2 \cdot F(r_{12}, t)$

$$\frac{\partial \alpha}{\partial t} = \left(\frac{\partial \alpha}{\partial t}\right)_{free} + Q_1 Q_2 F(r_{12}, t)$$

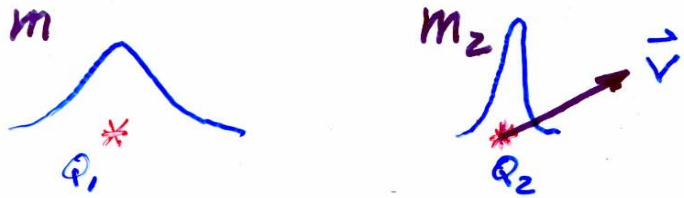


$$H_{int} = Q_1 Q_2 F(x_{12})$$

$$\frac{\partial \mathcal{L}(x)}{\partial t} = \left[\frac{\partial \mathcal{L}(x)}{\partial t}\right]_{free} + Q_1 \cdot Q_2 F(x_{12})$$

deformation of $\mathcal{H}$

Relativistic case



$$Q_2 \to J_\mu \sim \left\{ \frac{Q_2}{\sqrt{1-v^2}},\ \frac{Q_2 \vec{v}}{\sqrt{1-v^2}} \right\}$$

$$H_{int} \sim Q_2 F(x_{12}) \longrightarrow A_\mu \simeq \{A_0(x), \vec{A}(x)\}$$

$$A_0(x),\ \vec{A}(x) \sim \frac{Q_2}{\sqrt{1-V^2}},\ \frac{Q_2\vec{V}}{\sqrt{1-V^2}}$$

Then $\exists\ RF'(\vec{V}')\quad \vec{A}(x) \to \vec{A}'(x) = 0$

for m: $\vec{p} \to \vec{p}' = \dfrac{\vec{p} - m\vec{V}'}{\sqrt{1-V'^2}}$

hence: $\vec{p} \to \vec{p} + Q_1\vec{A}(x)$

$$\frac{\partial\alpha}{\partial t} \to \frac{\partial\alpha}{\partial t} + Q_1 A_0(x)\ ;\ \frac{\partial\alpha}{\partial\vec{x}} \to \frac{\partial\alpha}{\partial\vec{x}} + Q_1\vec{A}(x)$$

$$-i\frac{\partial\psi}{\partial t} = [\vec{\alpha}(\vec{p} + Q_1\vec{A}(x)) + \beta m + Q_1 A_0(x))\psi$$

Local $U(1)$ gauge invariance - QED

SU(2) Gauge Invariance

$$\Psi = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \sqrt{w(x)}\, e^{i\alpha(x) + i\vec{\theta}(x)\vec{\sigma}}\begin{pmatrix}1\\0\end{pmatrix}$$

$\alpha(x)$ $\vec{\theta}(x)$

R^3 $SU(2)$

3 phase parameters

$$-i\frac{\partial\psi}{\partial t} = -i\frac{\partial}{\partial t}\left(\sqrt{w}\,e^{i\alpha+\vec{\theta}\vec{\sigma}}\begin{pmatrix}1\\0\end{pmatrix}\right) = \left(\frac{1}{2m}\frac{\partial^2}{\partial x^2} + \hat{H}_{int}\right)\psi$$

$$\begin{cases}\dfrac{\partial\sqrt{w}}{\partial t} = \left(\dfrac{\partial w}{\partial t}\right)_{free}\\[2ex] \dfrac{\partial}{\partial t}(\alpha + \vec{\theta}\vec{\sigma}) = \dfrac{\partial}{\partial t}(\alpha + \vec{\theta}\vec{\sigma})_{free} + \hat{H}_{int}\end{cases}$$

$$\hat{H}_{int} = Q_1 \cdot Q_2\, \vec{A}(x,t)\,\vec{\sigma}$$

$\vec{S}$ $\vec{A}$ $\vec{S}_2$

Q_1 Q_2

$\vec{A} \parallel \vec{S}_2$

if $Q_2, |\vec{S}_2| \to \infty$

$$\langle \vec{S}_{\perp}(t) \rangle \to 0$$

Conclusions

1. Fuzzy topology is the most simple and natural formalism for introduction of quantization into physical theory
2. Shroedinger equation is obtained from simple assumptions
3. Gauge invariance of fields corresponds to dynamics on fuzzy manifold

Fuzzy ordered set - Foset F^x

$F^x \sim P^x$, but $\forall x_i, x_j \to w_{ij} \geq 0$

Example: $F^x = O^x \cup P'$

$x'_i \| x_j \to w_{ij}$

$\sum_j w_{ij} = 1$

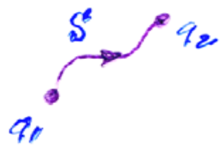
if O^x - is continuum ?

Fuzzy Geometry is consistent theory (Zeeman 1968
Zedson 1974)

Fuzzy points x'_i - are particles with uncertain coordinate x in O^x

Law of motion in Fuzzy Mechanics

a) Classical Mechanics: Minimal action



$$S = \int_{q_1}^{q_2} L(q, \dot{q})\,dt \; - \; min$$

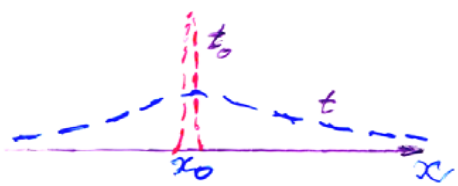
b) Quantum Mechanics – Path Integral

$$A(q_1, q_2) = e^{i\int_{q_1}^{q_2} S[dq]}$$

$$\Psi(t) = \hat{U}_t \Psi_0$$

c) Fuzzy Mechanics:

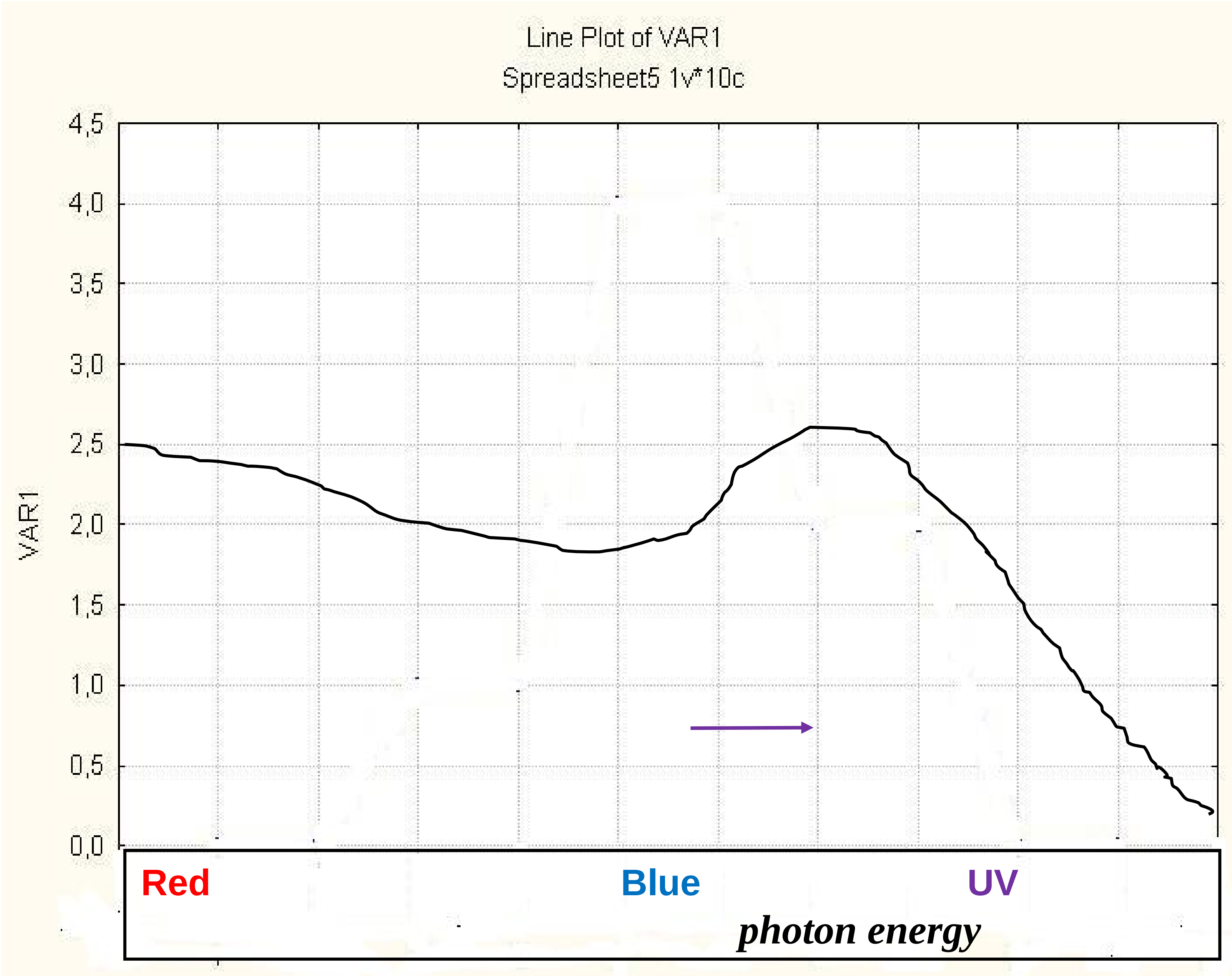
Fuzzy free state $g(x,t)$ tends to maximal space symmetry !?



$$g(x,t) = \hat{N}_t\, g(x, t_0)$$

no parameters in $\hat{N}$

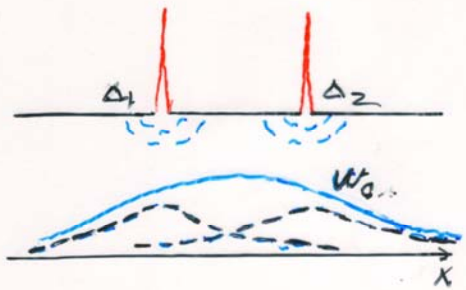
$$g(x,t) = \delta(x - x_0) \xrightarrow[\Delta t \to 0]{} const(x)$$



Line Plot of VAR1
Spreadsheet5 1v*10c
VAR1
4,5
4,0
3,5
3,0
2,5
2,0
1,5
1,0
0,5
0,0
Red
Blue
UV
photon energy

2 slits experiments

$w_0 \simeq \delta(x) \xrightarrow{t} \pm\infty$



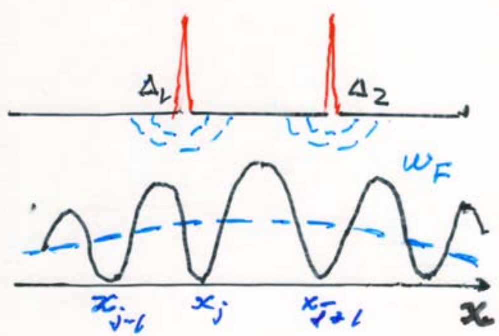
Stochastic mixture

$w_i(\vec{r} - \vec{r}_i, t)$

$L_c := m \in \Delta_1 .OR. m \in \Delta_2 \leftrightarrow q = q_1 .OR. q = q_2$

$w_c(x,t) = w_1(x,t) + w_2(x,t)$

$\forall x, \; w_c(x,t) > 0$



Topological structure of $\{m_0\}$

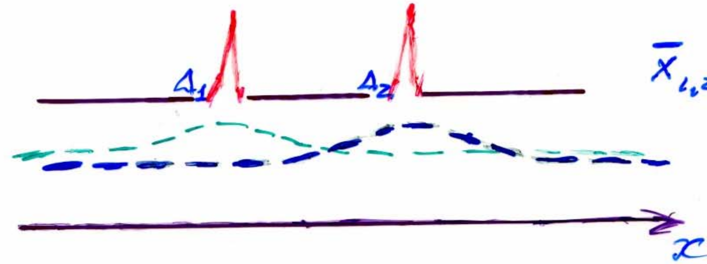
$L_F := m \in \Delta_1 .and. m \in \Delta_2 \to L_F \cap L_c = \emptyset$

$w_F(x,t) = w_n(x,t) + K(w_1(x,t) + w_2(x,t)) \to K \simeq 0$

$\exists x_j, \; w_F(x,t) = 0$

$n=2$: $w_{1,2}(x,t)$ - Schwartz distributions

$w_{1,2}(x,t) \neq 0$
$x \to \pm\infty$

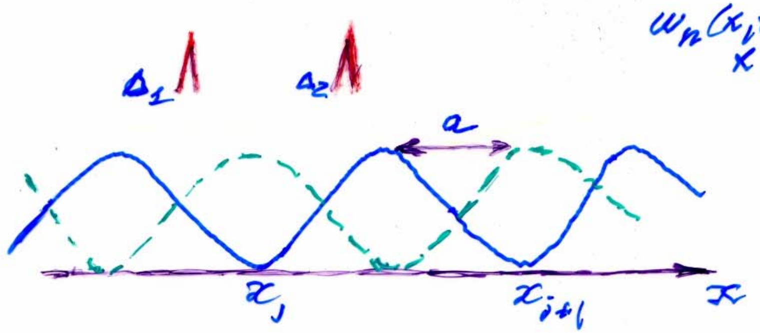


$\bar{x}_{1,2}$ - undefined

$m \in \Delta_1$ and $m \in \Delta_2$;

so $w_s(x,t) = w_n(x,t)$; $\to \exists x_j$; $w_n(x_j,t)=0$

$w_n(x,t) \neq 0$
$x \to \pm\infty$



if $w_s(x,t)$ is solution, $\bar{x}$ - undefined
then $w_s'(x,t) = w_s(x+a,t)$ is also
solution $\forall a$, $-\infty \leq a \leq \infty$

Fuzzy mechanics – space symmetry restoration

as the general low of free motion