Nambu-Poisson dynamics with some applications

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In the Universe, matter has manly two geometric structures, homogeneous, [Weinberg,1972] and hierarchical, [Okun, 1982] .
The homogeneous structures are naturally described by real numbers with an infinite number of digits in the fractional part and usual archimedean metrics. The hierarchical structures are described with p-adic numbers with an infinite number of digits in the integer part and non-archimedean metrics, [Koblitz, 1977].

A discrete, finite, regularized, version of the homogenous structures are homogeneous lattices with constant steps and distance rising as arithmetic progression. The discrete version of the hierarchical structures is hierarchical lattice-tree with scale rising in geometric progression.

There is an opinion that present day theoretical physics needs (almost) all mathematics, and the progress of modern mathematics is stimulated by fundamental problems of theoretical physics.

The Hamiltonian mechanics (HM) is in the fundamentals of mathematical description of the physical theories, [Faddeev,Takhtajan]. But HM is in a sense blind; e.g., it does not makes a difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) [Sinai,1996] and (super)integrable Hamiltonian systems (with maximal number of the integrals of motion).

Nabu mechanics (NM) [Nambu, 1973] is a proper generalization of the HM, which makes the difference between dynamical systems with different numbers of integrals of motion explicit.

Let us consider a general dynamical system described by the following system of the ordinary differential equations [Arnold,1969]

$$\dot{x}_n = f_n(x),\ 1 \leq n \leq N, \tag{1}$$

and $\dot{x}_n$ stands for the total derivative with respect to the parameter t.
When the number of the degrees of freedom is even, $1 \leq n, m \leq 2M$, and

$$f_n(x) = \varepsilon_{nm} \frac{\partial H_0}{\partial x_m},\ 1 \leq n, m \leq 2M, \tag{2}$$

the system (1) is Hamiltonian one and can be put in the form

$$\dot{x}_n = \{x_n, H_0\}_0, \tag{3}$$

where the Poisson bracket is defined as

$$\{A, B\}_0 = \varepsilon_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = A \frac{\overleftarrow{\partial}}{\partial x_n} \varepsilon_{nm} \frac{\overrightarrow{\partial}}{\partial x_m} B, \tag{4}$$

and summation rule under repeated indices has been used.

Let us consider the following Lagrangian

$$L = (\dot{x}_n - f_n(x))\psi_n \tag{5}$$

and the corresponding equations of motion

$$\begin{aligned} \dot{x}_n &= f_n(x), \\ \dot{\psi}_n &= -\frac{\partial f_m}{\partial x_n}\psi_m. \end{aligned} \tag{6}$$

The system (6) extends the general system (1) by linear equation for the variables ψ. The extended system can be put in the Hamiltonian form [Makhaldiani,Voskresenskaya,1997]

$$\begin{aligned} \dot{x}_n &= \{x_n, H_1\}_1, \\ \dot{\psi}_n &= \{\psi_n, H_1\}_1, \end{aligned} \tag{7}$$

where first level (order) Hamiltonian is

$$H_1 = f_n(x)\psi_n \tag{8}$$

and (first level) bracket is defined as

$$\{A, B\}_1 = A(\frac{\overleftarrow{\partial}}{\partial x_n}\frac{\overrightarrow{\partial}}{\partial \psi_n} - \frac{\overleftarrow{\partial}}{\partial \psi_n}\frac{\overrightarrow{\partial}}{\partial x_n})B. \tag{9}$$

Note that when the Grassmann grading [Berezin,1987] of the conjugated variables x_n and ψ_n

$$\{x_n, \psi_m\}_1 = \delta_{nm} \tag{10}$$

are different, the bracket (9) is known as Buttin bracket[Buttin,1996].

In the Faddeev-Jackiw formalism [Faddeev,Jackiw,1988] for the Hamiltonian treatment of systems defined by first-order Lagrangians, i.e. by a Lagrangian of the form

$$L = f_n(x)\dot{x}_n - H(x), \tag{11}$$

motion equations

$$f_{mn}\dot{x}_n = \frac{\partial H}{\partial x_m}, \tag{12}$$

for the regular structure function f_{mn}, can be put in the explicit hamiltonian (Poisson; Dirac) form

$$\dot{x}_n = f^{-1}_{nm}\frac{\partial H}{\partial x_m} = \{x_n, x_m\}\frac{\partial H}{\partial x_m} = \{x_n, H\}, \tag{13}$$

where the fundamental Poisson (Dirac) bracket is

$$\{x_n, x_m\} = f^{-1}_{nm},\ f_{mn} = \partial_m f_n - \partial_n f_m. \tag{14}$$

In Dirack's formalism of constrained systems [Dirac,1951], we have the following constraints and brackets

$$\begin{aligned} &\varphi_n = p_n - f_n(x) = 0,\ \{\varphi_n, \varphi_m\} = \partial_n f_m - \partial_m f_n = f_{n,m},\\ &\{A, B\}_D = \{A, B\} - \{A, \varphi_n\} f^{-1}_{n,m}\{\varphi_m, B\},\\ &\dot{x}_n = f^{-1}_{nm}\frac{\partial H}{\partial x_m} = \{x_n, H\}_D,\ \{x_n, x_m\}_D = f^{-1}_{n,m}. \end{aligned} \tag{15}$$

The system (6) is an important example of the first order hamiltonian systems. Indeed, in the new variables,

$$y_n^1 = x_n, y_n^2 = \psi_n, \tag{16}$$

lagrangian (5) takes the following first order form

$$\begin{aligned} L &= (\dot{x}_n - v_n(x))\psi_n \Rightarrow \frac{1}{2}(\dot{x}_n\psi_n - \dot{\psi}_n x_n) - v_n(x)\psi_n = \frac{1}{2}y_n^a \varepsilon^{ab}\dot{y}_n^b - H(y) \\ &= f_n^a(y)\dot{y}_n^a - H(y), f_n^a = \frac{1}{2}y_n^b\varepsilon^{ba}, H = y_n^2 v_n(y^1), \\ f_{nm}^{ab} &= \frac{\partial f_m^b}{\partial y_n^a} - \frac{\partial f_n^a}{\partial y_m^b} = \varepsilon^{ab}\delta_{nm}; \end{aligned} \tag{17}$$

corresponding motion equations are

$$\dot{y}_n^a = \varepsilon^{ab}\delta_{nm}\frac{\partial H}{y_m^b}, \tag{18}$$

the fundamental Poisson bracket is

$$\{y_n^a, y_m^b\} = \varepsilon^{ab}\delta_{nm}. \tag{19}$$

To the canonical quantization of this system corresponds

$$[\hat{y}_n^a, \hat{y}_m^b] = i\hbar\varepsilon^{ab}\delta_{nm},\ \hat{y}_n^1 = y_n^1,\ \hat{y}_n^2 = -i\hbar\frac{\partial}{\partial y_n^1} \tag{20}$$

In this quantum theory, classical part, motion equations for y_n^1, remain classical

Point vortex dynamics (PVD)

PVD can dy defined (see e.g. [Aref,1983, Meleshko,Konstantinov,1993]) as the following first order system

$$\dot{z}_n = i \sum_{m \neq n}^{N} \frac{\gamma_m}{z_n^* - z_m^*}, \; z_n = x_n + iy_n, \; 1 \leq n \leq N. \tag{21}$$

Corresponding first order lagrangian, hamiltonian, momenta, Poisson brackets and commutators are

$$\begin{aligned}
L &= \sum_n \frac{i}{2}\gamma_n(z_n \dot{z}_n^* - \dot{z}_n z_n^*) - \sum_{n \neq m} \gamma_n \gamma_m ln|z_n - z_m| \\
H &= \sum_{n \neq m} \gamma_n \gamma_m \ln|z_n - z_m| \\
&= \frac{1}{2} \sum_{n \neq m} \gamma_n \gamma_m (\ln(z_n - z_m) + \ln(p_n - p_m)), \\
p_n &= \frac{\partial L}{\partial \dot{z}_n} = -\frac{i}{2}\gamma_n z_n^*, \; p_n^* = \frac{\partial L}{\partial \dot{z}_n^*} = \frac{i}{2}\gamma_n z_n, \\
\{p_n, z_m\} &= \delta_{nm}, \; \{p_n^*, z_m^*\} = \delta_{nm}, \\
[p_n, z_m] &= -i\hbar\delta_{nm} \Rightarrow [x_n, y_m] = -i\frac{\hbar}{\gamma_n}\delta_{nm}
\end{aligned} \tag{22}$$

So, quantum vortex dynamics corresponds to the noncommutative space. It is natural to assume

$$\gamma_n = \frac{\hbar}{a^2} n, \; n = \pm 1, \pm 2, ... \tag{23}$$

and a is characteristic (fundamental) length.

In this section we consider the following dynamical system [Baleanu,Makhaldiani,1998]

$$\begin{aligned}
&\dot{x}_n = \gamma_n \sum_{m=1}^{p} (e^{x_{n+m}} - e^{x_{n-m}}),\\
&1 \le n \le N,\ 1 \le p \le [(N-1)/2],\ 3 \le N,\\
&x_{n+N} = x_n,
\end{aligned} \tag{24}$$

where γ_n are real numbers, and $[a]$ means the integer part of a.
The system (24) for $\gamma_n = 1$, $p = 1$ and $x_n = ln v_n$, becomes Volterra system [Volterra,1931]

$$\dot{v}_n = v_n(v_{n+1} - v_{n-1}). \tag{25}$$

It is also related to the Toda lattice system, [Toda,1981]

$$\dot{y}_n = e^{y_{n+1} - y_n} + e^{y_n - y_{n-1}}.$$

Indeed, if

$$x_n = y_n - y_{n-1},$$

then

$$\dot{x}_n = e^{x_{n+1}} - e^{x_{n-1}}.$$

If $\gamma_n = 1$ and $p \ge 1$, system (24) reduces to the Bogoiavlensky lattice system, [Bogoyavlensky,1988]

$$\dot{v}_n = v_n \sum_{m=1}^{p} (v_{n+m} - v_{n-m}). \tag{26}$$

System of three vortexes

For $N = 3$, $p = 1$ and arbitrary γ_n, (24) is related to the system of three vortexes of two-dimensional ideal hydrodynamics, [Aref,1983, Makhaldiani,1997,2]. The system of N vortexes can be described by the following system of differential equations, [Aref,1983, Meleshko,Konstantinov,1993]

$$\dot{z}_n = i \sum_{m \neq n}^{N} \frac{\gamma_m}{z_n^* - z_m^*}, \quad 1 \leq n \leq N, \tag{27}$$

where $z_n = x_n + iy_n$ are complex coordinate of the centre of n-th vortex.
For $N = 3$, it is easy to verify that the quantities

$$\begin{aligned} x_1 &= ln|z_2 - z_3|^2, \\ x_2 &= ln|z_3 - z_1|^2, \\ x_3 &= ln|z_1 - z_2|^2 \end{aligned} \tag{28}$$

satisfy the following system

$$\begin{aligned} \dot{x}_1 &= \gamma_1(e^{x_2} - e^{x_3}), \\ \dot{x}_2 &= \gamma_2(e^{x_3} - e^{x_1}), \\ \dot{x}_3 &= \gamma_3(e^{x_1} - e^{x_2}), \end{aligned} \tag{29}$$

after change of the time parameter as

$$dt = \frac{e^{(x_1+x_2+x_3)}}{4S} d\tau = e^{(x_1+x_2+x_3)/2} R d\tau, \tag{30}$$

where S is the area of the triangle with vertexes in the centres of the vortexes and R is the radius of the circle with the vortexes on it.

The system (29) has two integrals of motion

$$H_1 = \sum_{i=1}^{3} \frac{e^{x_i}}{\gamma_i}, \tag{31}$$
$$H_2 = \sum_{i=1}^{3} \frac{x_i}{\gamma_i}$$

and can be presented in the Nambu–Poisson form, [Makhaldiani,1997,2]

$$\dot{x}_i = \omega_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k} \tag{32}$$
$$= \{x_i, H_1, H_2\} = \omega_{ijk} \frac{e^{x_j}}{\gamma_j} \frac{1}{\gamma_k},$$

where

$$\omega_{ijk} = \epsilon_{ijk}\rho, \tag{33}$$
$$\rho = \gamma_1\gamma_2\gamma_3$$

and the Nambu–Poisson bracket of the functions A, B, C on the three-dimensional phase space is

$$\{A, B, C\} = \omega_{ijk} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} \frac{\partial C}{\partial x_k}. \tag{34}$$

The fundamental bracket is

$$\{x_1, x_2, x_3\} = \omega_{ijk}. \tag{35}$$

Then we can again change the time parameter as

$$du = \rho d\tau \tag{36}$$

and obtain Nambu mechanics, [Makhaldiani,1997,2]

$$\dot{x}_i = \epsilon_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k}.$$

Note that this system is superintegrable; for $N = 3$ degrees of freedom, we have maximal number of the integrals of motion $N - 1 = 2$.

Now we define x_3 from H_2,

$$x_3 = \gamma_3(H_2 - \frac{x_1}{\gamma_1} - \frac{x_2}{\gamma_2}), \tag{37}$$

insert it into H_1, find x_2 as an implicit function of x_1

$$\frac{e^{x_2}}{\gamma_2} + e^{\gamma_3(H_2 - \frac{x_1}{\gamma_1})} \frac{e^{-\frac{\gamma_3}{\gamma_2}x_2}}{\gamma_3} = H_1 - \frac{e^{x_1}}{\gamma_1}, \tag{38}$$

and integrate motion equation of x_1

$$\dot{x}_1 = \gamma_1(e^{x_2} - e^{x_3}) \equiv n(x_1), \quad \int_{x_{10}}^{x_1} \frac{dx}{n(x)} = \tau - \tau_0. \tag{39}$$

For

$$\frac{\gamma_3}{\gamma_2} = \pm 1; \pm 2; \pm 3; -4, \tag{40}$$

$n(x)$ is a superposition of elementary functions.

The next interesting case is $N = 4$ and $p = 1$,

$$\begin{aligned} \dot{x}_1 &= \gamma_1(e^{x_2} - e^{x_4}), \\ \dot{x}_2 &= \gamma_2(e^{x_3} - e^{x_1}), \\ \dot{x}_3 &= \gamma_3(e^{x_4} - e^{x_2}), \\ \dot{x}_4 &= \gamma_4(e^{x_1} - e^{x_3}). \end{aligned} \tag{41}$$

As in the $N = 3, p = 1$ case, for (41) we have two integrals of motion

$$H_1 = \frac{e^{x_1}}{\gamma_1} + \frac{e^{x_2}}{\gamma_2} + \frac{e^{x_3}}{\gamma_3} + \frac{e^{x_4}}{\gamma_4}, \tag{42}$$

$$H_2 = \frac{x_1}{\gamma_1} + \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3} + \frac{x_4}{\gamma_4}. \tag{43}$$

For the superintegrability of the system (41), we need one more integral of motion, H_3. To find that integral let us suppose Nambu's form of the system (41)

$$\dot{x}_n = \{x_n, H_1, H_2, H_3\} = \gamma_1\gamma_2\gamma_3\gamma_4\epsilon_{nmkl}\frac{\partial H_1}{\partial x_m}\frac{\partial H_2}{\partial x_k}\frac{\partial H_3}{\partial x_l}. \quad (44)$$

We found from (44) a solution for H_3

$$H_3 = -\frac{1}{2}(\frac{x_1}{\gamma_1} - \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3} - \frac{x_4}{\gamma_4}). \quad (45)$$

Now we have three integrals of motion and we can integrate the system (41). From (43) and (45) we get

$$x_4 = \gamma_4(\frac{H_2 + 2H_3}{2} - \frac{x_2}{\gamma_2}),$$
$$x_3 = \gamma_3(\frac{H_2 - 2H_3}{2} - \frac{x_1}{\gamma_1}) \quad (46)$$

and (42) gives us

$$\frac{e^{x_1}}{\gamma_1} + \frac{e^{x_2}}{\gamma_2} + \frac{e^{-\frac{\gamma_3}{\gamma_1}x_1}}{\gamma_3 e^{-\gamma_3(H_2/2 - H_3)}} + \frac{e^{-\frac{\gamma_4}{\gamma_2}x_2}}{\gamma_4 e^{-\gamma_4(H_2/2 + H_3)}} = H_1. \quad (47)$$

So x_2 is an implicit function of x_1,

$$x_2 = n_2(x_1, H_1, H_2, H_3), \tag{48}$$

when

$$\frac{\gamma_4}{\gamma_2} = \pm 1, \pm 2, \pm 3, -4, \tag{49}$$

the function n_2 reduces to the composition of the elementary functions.
Now we can solve the equation for x_1,

$$\dot{x}_1 = \gamma_1(e^{x_2} - e^{x_4}) \equiv n_1(x_1), \tag{50}$$

by one quadrature,

$$N(x_1) = \int_{x_{10}}^{x_1} \frac{dx}{n_1(x)} = t - t_0. \tag{51}$$

Note that, from the motion equations (41) or (46) it is easy to see that we have a separation of the odd and even degrees of freedom,

$$\frac{x_1}{\gamma_1} + \frac{x_3}{\gamma_3} = H_{13} = \frac{H_2}{2} - H_3, \quad \frac{x_2}{\gamma_2} + \frac{x_4}{\gamma_4} = H_{24} = \frac{H_2}{2} + H_3. \tag{52}$$

Now we can put the system in the form

$$\begin{aligned} \dot{x}_1 &= \gamma_1 (e^{x_2} - e^{\gamma_4 H_{24}} e^{-\frac{\gamma_4}{\gamma_2} x_2}) \equiv f_1(x_2) \\ \dot{x}_2 &= \gamma_2 (e^{\gamma_3 H_{13}} e^{-\frac{\gamma_3}{\gamma_1} x_1} - e^{x_1}) \equiv f_2(x_1) \end{aligned} \tag{53}$$

For numerical solution this system may be more convenient.

For the general case we have two integrals of motion for the system (24)

$$H_1 = \sum_{n=1}^{N} \frac{e^{x_n}}{\gamma_n}, \tag{54}$$

$$H_2 = \sum_{n=1}^{N} \frac{x_n}{\gamma_n}. \tag{55}$$

For even N, $N = 2M$, we know a third integral of motion

$$H_3 = \frac{1}{2} \sum_{n=1}^{2M} \frac{(-1)^n x_n}{\gamma_n}, \tag{56}$$

In this case, we have a separation of even and odd degrees of freedom,

$$\sum_{n=1}^{M} \frac{x_{2n-1}}{\gamma_{2n-1}} = H_{1m} = \frac{H_2}{2} - H_3,$$
$$\sum_{n=1}^{M} \frac{x_{2n}}{\gamma_{2n}} = H_{2m} = \frac{H_2}{2} + H_3. \tag{57}$$

When $N \geq 5$, for integrability, we need extra integrals of motion.

Simplectic reduction of three dimensional system to two dimensional one and quantization

NPD (32) reduces to two Hamiltonian-Poisson dynamics (HPD),

$$\dot{x}_n = \{x_n, H_1, H_2\} = \omega_{nmk} \frac{\partial H_1}{\partial x_m} \frac{\partial H_2}{\partial x_k}$$
$$= \{x_n, H_1\}_1 = \omega^1_{nm} \frac{\partial H_1}{\partial x_m}$$
$$= \{x_n, H_2\}_2 = \omega^2_{nm} \frac{\partial H_2}{\partial x_m}. \quad (58)$$

Corresponding Poisson structures, ω^1 and ω^2 are degenerate, because they are 3×3 antisymmetric tensors. They eigenvectors with vanishing eigenvalue are

$$(h_1)_n = \frac{\partial H_2}{\partial x_n} = \frac{1}{\gamma_n}, \quad (h_2)_n = \frac{\partial H_1}{\partial x_n} = \frac{e^{x_n}}{\gamma_n},$$
$$\omega^1 h_1 = \omega^2 h_2 = 0. \quad (59)$$

If we define e.g. x_3 from H_2, on the restricted "phase space" (x_1, x_2) with Hamiltonian $H(x_1, x_2) = H_1(x_1, x_2, x_3(x_1, x_2))$, we find the following regular symplectic dynamics

$$\dot{x}_n = \{x_n, H\} = \gamma_1\gamma_2\varepsilon_{nm}\frac{\partial H}{\partial x_n}, \quad n, m = 1, 2. \tag{60}$$

Indeed, e.g.

$$\dot{x}_1 = \gamma_1\gamma_2(\frac{e^{x_2}}{\gamma_2} + \frac{\partial x_3}{\partial x_2}\frac{e^{x_3}}{\gamma_3}) = \gamma_1(e^{x_2} - e^{x_3}). \tag{61}$$

Motion equations take canonical Hamiltonian form

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \tag{62}$$

after the change of variables as $x_1 = \gamma_1 x$, $x_2 = \gamma_2 p$. We can quantize this system introducing coordinate, momentum and Hamiltonian operators $\hat{x}, \hat{p}, \hat{H}$,

$$\begin{aligned} &\hat{x} = x, \quad \hat{p} = -i\hbar\frac{\partial}{\partial x}, \quad \hat{x}\hat{p} - \hat{p}\hat{x} \equiv [\hat{x}, \hat{p}] = i\hbar \\ &\hat{H} = \frac{e^{\gamma_1\hat{x}}}{\gamma_1} + \frac{e^{\gamma_2\hat{p}}}{\gamma_2} + \frac{e^{\gamma_3(H_2 - \hat{x} - \hat{p})}}{\gamma_3} \\ &= \frac{e^{\gamma_1\hat{x}}}{\gamma_1} + \frac{e^{\gamma_2\hat{p}}}{\gamma_2} + \frac{e^{\gamma_3 H_2 - \frac{i\hbar}{2}\gamma_3^2}}{\gamma_3}e^{-\gamma_3\hat{x}}e^{-\gamma_3\hat{p}} \end{aligned} \tag{63}$$

Simplectic reduction of four dimensional system to two dimensional one and quantization

It is easy to see that the motion equations on the restricted "phase space" (x_1, x_2) can be put in the regular simplectic form

$$x_n = \{x_n, H\} = \gamma_1\gamma_2\varepsilon_{nm}\frac{\partial H(x_1, x_2, x_3(x_1), x_4(x_2))}{\partial x_n}, \quad n, m = 1, 2. \tag{64}$$

Motion equations take canonical Hamiltonian form

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \tag{65}$$

after the change of variables as $x_1 = \gamma_1 x$, $x_2 = \gamma_2 p$. We can quantize this system introducing coordinate, momentum and Hamiltonian operators $\hat{x}, \hat{p}, \hat{H}$,

$$\begin{aligned}\hat{H} &= \frac{e^{\gamma_1\hat{x}}}{\gamma_1} + \frac{e^{\gamma_2\hat{p}}}{\gamma_2} + \frac{e^{\gamma_3(H_{13}-\hat{x})}}{\gamma_3} + \frac{e^{\gamma_4(H_{24}-\hat{p})}}{\gamma_4}, \\ &= \frac{e^{\gamma_1\hat{x}}}{\gamma_1} + \frac{e^{\gamma_3 H_{13}}}{\gamma_3}e^{-\gamma_3\hat{x}} + \frac{e^{\gamma_2\hat{p}}}{\gamma_2} + \frac{e^{\gamma_4 H_{24}}}{\gamma_4}e^{-\gamma_4\hat{p}} \\ &= K(\hat{p}) + V(x). \end{aligned} \tag{66}$$

The classical rotator has two integrals of motion

$$H_1 = H = H = \frac{1}{2}(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}),$$
$$H_2 = \frac{1}{2}M^2 = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2). \tag{67}$$

NP formulation of the dynamics is [Nambu, 1973],

$$\dot{x}_n = \{x_n, H_1, H_2\} = \varepsilon_{nmk}\frac{\partial H_1}{\partial x_m}\frac{\partial H_2}{\partial x_k}$$
$$= (I_m^{-1} - I_k^{-1})x_m x_k,\ x_n \equiv M_n,\ n, m, k = 1, 2, 3. \tag{68}$$

Deformed Nuclei and quantum rotator in NP formulation

The low-energy collective rotations and vibrations of atomic nuclei can be described by deformed quantum rotator model [Davidson,1965].The collective model of deformed nuclei is that of a rotating, deformed, almost rigid body with Hamiltonian

$$H = \frac{\hbar^2}{2}(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}) \quad (69)$$

inertia tensor $I = (I_1, I_2, I_3)$ in the principal axis system and body-fixed angular momentum operators $M = (M_1, M_2, M_3)$ satisfy

$$[M_1, M_2] = -iM_3, \quad (70)$$

cyclically. Defining an angular momentum representation $|l, m, k>$ diagonal in M and its projections on laboratory and principal Z axes by

$$\begin{aligned} M^2|l,m,k> &= l(l+1)|l,m,k>, \\ M_z|l,m,k> &= m|l,m,k>, \\ M_3|l,m,k> &= k|l,m,k>, \end{aligned} \quad (71)$$

in units of $\hbar$, the state functions for an asymmetric top designated by $|l, m>$ are

$$|l,m> = \sum_{k=-l}^{l} a_k |l,m,k>, \ \Delta k = 2, \quad (72)$$

the rigid rotator Hamiltonian only connects state components for which k is either even or odd. That is, the state functions of the most general top can be initially classified into two broad categories, one being a linear combination of functions for even k, the other for odd k.

As an example of the infinite dimensional Nambu-Poisson dynamics, let me conside the following extension of Schrödinger quantum mechanics [Makhaldiani,2000]

$$iV_t = \Delta V - \frac{V^2}{2}, \tag{73}$$

$$i\psi_t = -\Delta\psi + V\psi. \tag{74}$$

An interesting solution to the equation for the potential (73) is

$$V = \frac{4(4-d)}{r^2}, \tag{75}$$

where d is the dimension of the spase. In the case of $d = 1$, we have the potential of conformal quantum mechanics.

The variational formulation of the extended quantum theory, (73,74) is given by the following Lagrangian

$$L = (iV_t - \Delta V + \frac{1}{2}V^2)\psi. \tag{76}$$

The momentum variables are

$$\begin{aligned} P_v &= \frac{\partial L}{\partial V_t} = i\psi, \\ P_\psi &= 0. \end{aligned} \tag{77}$$

As Hamiltonians of the Nambu-theoretic formulation, we take the following integrals of motion

$$\begin{aligned} H_1 &= \int d^d x(\Delta V - \frac{1}{2}V^2)\psi, \\ H_2 &= \int d^d x(P_v - i\psi), \\ H_3 &= \int d^d x P_\psi. \end{aligned} \tag{78}$$

We invent unifying vector notation, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (\psi, P_\psi, V, P_v)$. Then it may be verified that the equations of the extended quantum theory can be put in the following Nambu-theoretic form

$$\begin{aligned} \phi_t(x) &= \{\phi(x), H_1, H_2, H_3\} \\ &= i \int \frac{\delta(\phi(x), H_1, H_2, H_3)}{\delta(\phi_1(y), \phi_2(y), \phi_3(y), \phi_4(y))} dy, \end{aligned} \tag{79}$$

where the bracket is defined as

$$\begin{aligned}\{A_1, A_2, A_3, A_4\} &= i\varepsilon_{ijkl} \int \frac{\delta A_1}{\delta \phi_i(y)} \frac{\delta A_2}{\delta \phi_j(y)} \frac{\delta A_3}{\delta \phi_k(y)} \frac{\delta A_4}{\delta \phi_l(y)} dy \\ &= i \int \frac{\delta(A_1, A_2, A_3, A_4)}{\delta(\phi_1(y), \phi_2(y), \phi_3(y), \phi_4(y))} dy \\ &= i det(\frac{\delta A_k}{\delta \phi_l}).\end{aligned} \quad (80)$$

Nambu-Poisson formulation of the hydrodynamics

The motion equations of an ideal (incompressible liquid) hydrodynamics (the Euler equations)

$$\begin{array}{l} V_t + (V\nabla)V + \nabla P = 0, \\ \nabla V = 0, \end{array} \tag{81}$$

are hamiltonian equations [Arnold,1969].
The hamiltonian structure can be introduced in terms of

$$\Omega = \nabla \times V, \tag{82}$$

with motion equation

$$\begin{array}{rcl} \Omega_t & = & \nabla \times [V, \Omega] \\ & = & \{\Omega, H\} \end{array} \tag{83}$$

where the square bracket denote vector product; the hamiltonian H is the energy of the system,

$$H = \frac{1}{2}\int d^3x V^2 \tag{84}$$

and the Poisson bracket for any two functional F and G is defined as

$$\{F, G\} = \int d^3x(\Omega, [\nabla \times \frac{\delta F}{\delta \Omega}, \nabla \times \frac{\delta G}{\delta \Omega}]). \tag{85}$$

Statement. The Poisson bracket is a reduction of the corresponding Nambu bracket,

$$\begin{aligned}
\{F,G\} &= \int d^3x(\frac{\delta\Gamma}{\delta V},[\frac{\delta F}{\delta V},\frac{\delta G}{\delta V}]) \\
&= \int d^3x\varepsilon_{kmn}\frac{\delta\Gamma}{\delta V_k}\frac{\delta F}{\delta V_m}\frac{\delta G}{\delta V_n} \\
&= \{\Gamma,F,G\},
\end{aligned} \quad (86)$$

where

$$\Gamma = \frac{1}{2}\int d^3x(V,\Omega), \quad (87)$$

is the integral of motion, which characterizes the liquid flow topology [Moffat,1969]. Now we put the hydrodynamic equation in the Nambu form [Makhaldiani,1997]

$$\Omega_t = \{\Gamma,\Omega,H\} = \{\Omega,H_1,H_2\}, \quad (88)$$

where,

$$H_1 = -\Gamma, H_2 = H. \quad (89)$$

The bi-Hamiltonian formulation of the integrable systems is based on the following expressions of a dynamical system

$$
\begin{aligned}
v_t &= \{v(x), H_1\}_1 = f_1 \frac{\delta H_1}{\delta v} \\
&= \{v(x), H_2\}_2 = f_2 \frac{\delta H_2}{\delta v} = F(v, v_x, v_{xx}, ...),
\end{aligned} \tag{90}
$$

in the case, e.g., of the Korteweg - de Vries(KdV) equation,

$$
\begin{aligned}
&\{v(x), v(y)\}_1 = f_1 \delta(x-y),\ f_1 = \partial,\ H_1 = \frac{1}{2}\int dx(vv_{xx} + 3v^3); \\
&\{v(x), v(y)\}_2 = f_2 \delta(x-y),\ f_2 = \partial^3 + 2\partial v + 2v\partial,\ H_2 = \frac{1}{2}\int dxv^2; \\
&F = v_{xxx} + 6vv_x
\end{aligned} \tag{91}
$$

Geodesic motion of the point particles and integrals of motion

Geodesic motion of the particles maybe described by the following action functional

$$S = \int_1^2 L(|\dot{x}|)ds, \tag{92}$$

where

$$|\dot{x}|^2 = g_{ab}\dot{x}^a\dot{x}^b \tag{93}$$

and g_{ab} is metric tensor. The corresponding Euler-Lagrange equation

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^a}) - \frac{\partial L}{\partial x^a} = 0 \tag{94}$$

gives the extremal trajectories of the variation of the action (92)

$$\delta S = \int_1^2 ds(\frac{\partial L}{\partial x^a} - \frac{d}{ds}(\frac{\partial L}{\partial \dot{x}^a}))\delta x^a + (\frac{\partial L}{\partial \dot{x}^a}\delta x^a)_1^2, \tag{95}$$

with fixed ends, $\delta x^a(1) = \delta x^a(2) = 0$, and have the form

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \tag{96}$$

where

$$\dot{x}^a = \frac{dx^a}{ds} \tag{97}$$

is the proper time derivative,

$$ds^2 = g_{ab} dx^a dx^b, \tag{98}$$

gives the geodesic interval and

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (g_{db,c} + g_{dc,b} - g_{bc,d}) \tag{99}$$

is the Chistoffel's symbols.
Usually considered forms of the Lagrangian are $L = |\dot{x}|$ or $\frac{1}{2}|\dot{x}|^2$. The first one gives the reparametrization invariant action, the second one is easy for Hamiltonian formulation [DeWitt, 1965]. In the following we restrict ourselves by the last form of the Lagrangian

$$L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b. \tag{100}$$

Corresponding Hamiltonian

$$H = p_a \dot{x}^a - L \tag{101}$$

is

$$H = \frac{1}{2} g^{ab} p_a p_b, \tag{102}$$

where the momentum is

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = g_{ab} \dot{x}^b \tag{103}$$

and g^{ab} is the inverse metric tensor,

$$g^{ac} g_{cb} = \delta^a_b. \tag{104}$$

The Hamilton's equations of motion are

$$\dot{x}^a = \{x^a, H\}_0 = g^{ab} p_b, \\ \dot{p}_a = \{p_a, H\}_0 = -\frac{1}{2}\frac{\partial g^{bc}}{\partial x^a} p_b p_c, \tag{105}$$

where the Poisson bracket is

$$\{A, B\}_0 = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} = A(\overleftarrow{\partial}_{x^a} \overrightarrow{\partial}_{p_a} - \overleftarrow{\partial}_{p_a} \overrightarrow{\partial}_{x^a})B \tag{106}$$

$$= A \overleftarrow{\partial}_{z_n} \varepsilon_{nm} \overrightarrow{\partial}_{z_m} B, \tag{107}$$

and with the unifying variables z_n

$$z_n = x^n,\ z_{n+N} = p_n,\ 1 \leq n \leq N \tag{108}$$

the Hamilton's equations of motion (105) takes the form (1).

Integrals of motion H fulfil the following equations

$$\begin{aligned}
\frac{d}{ds}H(x,\dot{x}) &= (\dot{x}^a\frac{\partial}{\partial x^a} + \ddot{x}^a\frac{\partial}{\partial \dot{x}^a})H \\
&= (\dot{x}^a\frac{\partial}{\partial x^a} - \Gamma^a_{bc}\dot{x}^b\dot{x}^c\frac{\partial}{\partial \dot{x}^a})H \\
&= \dot{x}^a\nabla_a H = 0, \\
\frac{d}{ds}H(x,p) &= (g^{ab}p_b\frac{\partial}{\partial x^a} - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c\frac{\partial}{\partial p_a})H \\
&= p_b\nabla^b H = 0,
\end{aligned} \tag{109}$$

where

$$\begin{aligned}
\frac{d}{ds} &= \dot{x}^a\nabla_a = p_b\nabla^b, \\
\nabla_a &= \frac{\partial}{\partial x^a} - \Gamma^b_{ac}\dot{x}^c\frac{\partial}{\partial \dot{x}^b}, \\
\nabla^b &= g^{ba}\nabla_a = g^{ba}\frac{\partial}{\partial x^a} - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_c\frac{\partial}{\partial p_a}.
\end{aligned} \tag{110}$$

For the linear in $\dot{x}$ integrals

$$H_1 = K_a(x)\dot{x}^a = K^a p_a \tag{111}$$

we have

$$\begin{aligned}\dot{H}_1 = \dot{x}^a \nabla_a H_1 &= \frac{\partial K_b}{\partial x_a}\dot{x}^a\dot{x}^b - K_c\Gamma^c_{ab}\dot{x}^a\dot{x}^b \\ &= (K_{a,b} - K_c\Gamma^c_{ab})\dot{x}^a\dot{x}^b \\ &= K_{a;b}\dot{x}^a\dot{x}^b = \frac{1}{2}(K_{a;b} + K_{b;a})\dot{x}^a\dot{x}^b \\ &= K_{(a;b)}\dot{x}^a\dot{x}^b = 0.\end{aligned} \tag{112}$$

So, from the expression (112), we see one-to-one correspondence between the expression of the first order integrals of the motion (111) and the nontrivial solutions of the following equation for the so-called Killing vector K_a

$$K_{(a;b)} = 0. \tag{113}$$

For quadratic in $\dot{x}$ integrals

$$H_2 = K_{ab}(x)\dot{x}^a\dot{x}^b \tag{114}$$

we have

$$\begin{aligned}\dot{H}_2 &= (K_{ab,c} - K_{db}\Gamma^d_{ac} - K_{ad}\Gamma^d_{bc})\dot{x}^a\dot{x}^b\dot{x}^c \\ &= K_{ab;c}\dot{x}^a\dot{x}^b\dot{x}^c = \frac{1}{3}(K_{ab;c} + K_{bc;a} + K_{ca;b})\dot{x}^a\dot{x}^b\dot{x}^c \\ &= K_{(ab;c)}\dot{x}^a\dot{x}^b\dot{x}^c = 0.\end{aligned} \tag{115}$$

So, we have one-to-one correspondence between the existence of the second order integrals of motion (114) and the nontrivial solutions of the following equation for the tensor K_{ab}

$$K_{(ab;c)} = 0. \tag{116}$$

Now we prove the following:
<u>Theorem 1</u>. A necessary and sufficient condition that the following polynomials

$$H_n = K_{a_1 a_2 \dots a_n}(x) \dot{x}^{a_1} \dot{x}^{a_2} \dots \dot{x}^{a_n} = K^{a_1 a_2 \dots a_n}(x) p_{a_1} p_{a_2} \dots p_{a_n} \tag{117}$$

are integrals of the geodesic motion, (96, 105) is that the symmetric tensors $K_{a_1 a_2 \dots a_n}$, fulfil the equation

$$K_{(a_1 a_2 \dots a_n ; a)} = 0. \tag{118}$$

In fact,

$$\begin{aligned} \dot{H}_n &= \dot{x}^a \nabla_a H_n = K_{(a_1 a_2 \dots a_n ; a)} \dot{x}^{a_1} \dot{x}^{a_2} \dots \dot{x}^{a_n} \dot{x}^a = 0, \\ &= p_a \nabla^a H_n = K^{(a_1 a_2 \dots a_n ; a)} p_{a_1} p_{a_2} \dots p_{a_n} p_a = 0, \end{aligned} \tag{119}$$

which proves the theorem, see [Sommers, 1973].

The symmetric tensors, which fulfil the equation (118), are known as Killing tensors.
Note that, as the metric tensor is covariantly constant, $g_{ab;c} = 0$, there is always the second order Killing tensor

$$K_{ab} = g_{ab} \tag{120}$$

and the corresponding integral of motion, Hamiltonian, H_0,

$$2H_0 = g_{ab}\dot{x}^a\dot{x}^b. \tag{121}$$

Let us define an interesting algebra on Killing tensors.
Theorem 2. The following symmetrized product of the Killing tensors K^n and K^m

$$K^{(a_1 a_2 \dots a_n} K^{a_{n+1} a_{n+2} \dots a_{n+m})} = K^{a_1 a_2 \dots a_{n+m}}, \tag{122}$$

is (reducible) Killing tensor.
In fact, let us multiply the corresponding integrals of motion

$$\begin{aligned} H_n H_m = K^{(a_1 a_2 \dots a_n} K^{a_{n+1} a_{n+2} \dots a_{n+m})} p_{a_1} p_{a_2} \dots p_{a_{n+m}} = \\ = K_{(a_1 a_2 \dots a_n} K_{a_{n+1} a_{n+2} \dots a_{n+m})} \dot{x}^{a_1} \dot{x}^{a_2} \dots \dot{x}^{a_{n+m}} = H_{n+m}, \end{aligned} \tag{123}$$

which, using the Theorem 1, proves this theorem.

We have the following bracket algebra of the integrals of motion

$$\{H_n, H_m\}_0 = H_{n+m-1}. \tag{124}$$

This algebra gives another method of the construction of the Killing tensors. As an example let us calculate the bracket for the integrals $H_1 = K^a p_a$ and $H_2 = K^{ab} p_a p_b$

$$\begin{aligned} &\{H_1, H_2\}_0 = K^a \{p_a, K^{bc}\} p_b p_c + K^{bc} \{K^a, p_b p_c\} p_a \\ &= (K^{ab} K^c{}_{,a} + K^{ac} K^b{}_{,a} - K^{bc}{}_{,a} K^a) p_b p_c \\ &= \tilde{K}^{ab} p_a p_b. \end{aligned} \tag{125}$$

Let us consider another, tensor, generalization of the scalar integral of motion (111)

$$\begin{aligned} H_{a_1 a_2 \ldots a_{m-1}} &= A_{a_1 a_2 \ldots a_m}(x) \dot{x}^{a_m}, \\ H^{a_1 a_2 \cdots a_{m-1}} &= A^{a_1 a_2 \cdots a_m}(x) p_{a_m}, \end{aligned} \tag{126}$$

where the tensors $A_{a_1 a_2 \ldots a_m}(x)$ and $A^{a_1 a_2 \cdots a_m}(x)$ are skew-symmetric.

We have the following:

Theorem 3[1]. A necessary and sufficient condition that the tensors (126) are (covariantly) constant (parallel) along any geodesic $x^a(s)$ is that the covariant derivative of the skew-symmetric tensor $A_{a_1 a_2 \dots a_m}(x)$ is also skew-symmetric

$$A_{a_1 a_2 \dots a_m; a_{m+1}} + A_{a_1 a_2 \dots a_{m+1}; a_m} = 0. \tag{127}$$

In fact, as $x^a(s)$ is geodesic, we have

$$\begin{aligned} \frac{D\dot{x}^a}{Ds} &= \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \\ \frac{Dp_a}{Ds} &= \dot{p}_a + \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c = g_{ab}\frac{D\dot{x}^b}{Ds} = 0 \end{aligned} \tag{128}$$

and

$$\begin{aligned} &\frac{D}{Ds}(A_{a_1 a_2 \dots a_m}(x)\dot{x}^{a_m}) = A_{a_1 a_2 \dots a_m; a_{m+1}}\dot{x}^{a_m}\dot{x}^{a_{m+1}} \\ &= \frac{1}{2}(A_{a_1 a_2 \dots a_m; a_{m+1}} + A_{a_1 a_2 \dots a_{m+1}; a_m})\dot{x}^{a_m}\dot{x}^{a_{m+1}} = 0, \\ &\frac{D}{Ds}(A^{a_1 a_2 \dots a_m}(x)p_{a_m}) = A^{a_1 a_2 \dots a_m; a_{m+1}}p_{a_m}p_{a_{m+1}} \\ &= \frac{1}{2}(A^{a_1 a_2 \dots a_m; a_{m+1}} + A^{a_1 a_2 \dots a_{m+1}; a_m})p_{a_m}p_{a_{m+1}} = 0, \end{aligned} \tag{129}$$

which proves the theorem.

[1] This theorem is slight modification of the corresponding theorem from [Yano, 1952]

From the tensor integrals (126) we can construct the second rank Killing tensor.
Theorem 4. The following (symmetric) product of the tensors A^n and B^n gives a second rank Killing tensors

$$A^{a_1 a_2 \dots a_n (a} B_{a_1 a_2 \dots a_n}{}^{b)} = K^{ab}. \tag{130}$$

In fact, if we multiply the integrals

$$\begin{aligned} A^n &= A^{a_1 a_2 \dots a_n} = A^{a_1 a_2 \dots a_n a} p_a, \\ B_n &= B^{a_1 a_2 \dots a_n} = B_{a_1 a_2 \dots a_n}{}^{b} p_b, \end{aligned} \tag{131}$$

we obtain again integral

$$\begin{aligned} A^n B_n &= A^{a_1 a_2 \dots a_n (a} B_{a_1 a_2 \dots a_n}{}^{b)} p_a p_b \\ &= K^{ab} p_a p_b = H_2, \end{aligned} \tag{132}$$

and using the Theorem 1, we prove the theorem 4.
So, if we have a nontrivial solution of the equations (127) , we can construct second integral of motion H_2

$$H_2 = K^{ab} p_a p_b, \tag{133}$$

and with original Hamiltonian (102)

$$H_1 = 2H = g^{ab} p_a p_b, \tag{134}$$

we will have bi-Hamiltonian system.

Than, we can apply the general method of the integration of the bi-Hamiltonian systems [Magri, 1978],[Okubo,Das,1988]. Also we can construct the Nambu-Poisson formulation [Baleanu,Makhaldiani,1998] of this system

$$\begin{aligned} \dot{x}_n &= \{x_n, H_1, H_2\} \\ &= \omega_{nmk}(x)\frac{\partial H_1}{\partial x_m}\frac{\partial H_2}{\partial x_k}, \end{aligned} \tag{135}$$

where the Nambu-Poisson structure tensor ω_{nmk} we identify by comparision of the system (135) with the original system (105).
Note, that the skew-symmetric tensors, $A_{a_1 a_2 \dots a_n}$, which have the property, that its covariant derivative is also skew-symmetric, were considered by Bochner [Bochner, 1948] (see [Yano, 1952]) and/but in literature are known as the Killing-Yano tensors [Gibbons, Rietdijk, van Holten, 1993].

Now we return to our extended system (6) and formulate conditions for the integrals of motion $H(x, \psi)$

$$H = H_0(x) + H_1 + ... + H_N, \tag{136}$$

where

$$H_n = A_{k_1 k_2 \dots k_n}(x)\psi_{k_1}\psi_{k_2}...\psi_{k_N}, \ 1 \le n \le N, \tag{137}$$

we are assuming Grassmann valued ψ_n and the tensor $A_{k_1 k_2 \dots k_n}$ is skew-symmetric. For integrals (136) we have

$$\dot{H} = \{\sum_{n=0}^{N} H_n, H_1\} = \sum_{n=0}^{N}\{H_n, H_1\} = \sum_{n=0}^{N} \dot{H}_n = 0. \tag{138}$$

Now we see, that each term in the sum (136) must be conserved separately.
In particular for Hamiltonian systems (2), zeroth, H_0 and first level H_1, (8), Hamiltonians are integrals of motion. For $n = 0$

$$\dot{H}_0 = H_{0,k} f_k = 0, \tag{139}$$

which reduce to the condition (109), in the case of the geodesic motion of the particle (105) and defines corresponding modifications of the polynomial integrals of motion (117).

For $1 \leq n \leq N$ we have

$$\dot{H}_n = \dot{A}_{k_1k_2...k_n}\psi_{k_1}\psi_{k_2}...\psi_{k_N} + A_{k_1k_2...k_n}\dot{\psi}_{k_1}\psi_{k_2}...\psi_{k_N} + ... + A_{k_1k_2...k_n}\psi_{k_1}\psi_{k_2}...\dot{\psi}_{k_N}$$
$$= (A_{k_1k_2...k_n,k}f_k - A_{kk_2...k_n}f_{k_1,k} - ... - A_{k_1...k_{n-1}k}f_{k_n,k})\psi_{k_1}\psi_{k_2}...\psi_{k_N} = 0, \quad (140)$$

and there is one-to-one correspondence between the existence of the integrals (137) and the existence of the nontrivial solutions of the following equations

$$\frac{D}{Dt}A_{k_1k_2...k_n} = \{\dot{A}_{k_1k_2...k_n} - f_{k_1,k}A_{kk_2...k_n} - ... - f_{k_n,k}A_{k_1...k_{n-1}k}\}$$
$$= \{A_{k_1k_2...k_n,k}f_k - A_{kk_2...k_n}f_{k_1,k} - ... - A_{k_1...k_{n-1}k}f_{k_n,k}\} = 0, \quad (141)$$

where under the bracket operation, $\{B_{k_1,...,k_N}\} = \{B\}$ we understand complete anti-symmetrization with respect to the free indexes.

For $n = 1$ the system (141) gives

$$A_{k_1,k} f_k - A_k f_{k_1,k} = 0 \tag{142}$$

and this equation has at list one solution, $A_k = f_k$. If we have two (or more) independent first order integrals

$$H_1^{(1)} = A_k^1 \Psi_k;\ H_1^{(2)} = A_k^2 \Psi_k, ... \tag{143}$$

we can construct corresponding reducible second (or higher)order MBKY tensor(s)

$$\begin{aligned} &H_2 = H_1^{(1)} H_1^{(2)} = A_k^1 A_l^2 \Psi_k \Psi_l = A_{kl} \Psi_k \Psi_l; \\ &H_M = H_1^{(1)} ... H_M^{(M)} = A_{k_1...k_M} \Psi_{k_1} ... \Psi_{k_M}, \\ &A_{k_1...k_M} = \{A_{k_1}^{(1)} ... A_{k_M}^{(M)}\},\ 2 \le M \le N \end{aligned} \tag{144}$$

The system (141) defines a Generalization of the Bochner-Killing-Yano structures (118, 127), of the geodesic motion of the point particle, for the case of the general (24) (and extended (6)) dynamical systems. Having $A_M, 2 \le M \le N$ independent MBKY structures, we can construct corresponding second order Killing tensors and Nambu-Poisson dynamics. In the superintegrable case, we have maximal number of the motion integrals, N-1.
The structures defined by the system (141) we will call the Modified Bochner-Killing-Yano structures or MBKY structures for short, [Makhaldiani,1999].

Extended geodesic motion of the point particles and Grassmann valued integrals of motion

Let us take the following Lagrangian

$$\begin{aligned} L &= (\dot{x}^a - g^{ab}p_b)\phi_a - (\dot{p}_a + \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c)\psi^a \\ &= (\dot{x}^a - g^{ab}p_b)\phi_a + (\dot{\psi}^a - \frac{1}{2}\frac{\partial g^{ab}}{\partial x^c}\psi^c p_b)p_a + \frac{d}{ds}(\psi^a p_a) \\ &= L_1 + \frac{d}{ds}(\psi^a p_a). \end{aligned} \tag{145}$$

New momentum variables are

$$\frac{\partial L_1}{\partial \dot{x}^a} = \phi_a, \quad \frac{\partial L_1}{\partial \dot{\psi}^a} = p_a, \tag{146}$$

(fundamental) brackets are

$$\begin{aligned} &\{x^a, \phi_b\}_1 = \delta^a_b, \\ &\{\psi^a, p_b\}_1 = \delta^b_a, \\ &\{A, B\}_1 = A(\overleftarrow{\partial}_{x^a}\overrightarrow{\partial}_{\phi_a} + \overleftarrow{\partial}_{\psi^a}\overrightarrow{\partial}_{p_a} - \overleftarrow{\partial}_{\phi_a}\overrightarrow{\partial}_{x^a} - \overleftarrow{\partial}_{p_a}\overrightarrow{\partial}_{\psi^a})B \\ &\qquad = A\overleftarrow{\partial}_{Z_n}\varepsilon_{nm}\overrightarrow{\partial}_{Z_m}B. \end{aligned} \tag{147}$$

The Hamiltonian is

$$H_1 = g^{ab}\phi_a p_b + \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}\psi^a p_b p_c, \tag{148}$$

the equations of motion are

$$\begin{aligned}
\dot{x}^a &= g^{ab}p_b, \\
\dot{p}_a &= -\frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c, \\
\dot{\phi}_a &= -\frac{\partial g^{bc}}{\partial x^a}p_b\phi_c - \frac{1}{2}\frac{\partial^2 g^{bc}}{\partial x^a \partial x^e}p_b p_c \psi^e, \\
\dot{\psi}^a &= \frac{\partial g^{ab}}{\partial x^c}p_b\psi^c + g^{ab}\phi_b.
\end{aligned} \tag{149}$$

Note that the extended system (149) and Hamiltonian (148) can be obtained from the system (105) and Hamiltonian (102) by the following simple shift of the variables

$$\begin{aligned}
x^a &\Rightarrow x^a + \theta\psi^a, \\
p_a &\Rightarrow p_a + \theta\phi_a,
\end{aligned} \tag{150}$$

where θ- Grassmann parameter, $\theta^2 = 0$.
In fact,

$$\begin{aligned}
H_0 = \frac{1}{2}g^{ab}p_a p_b &\Rightarrow \frac{1}{2}g^{ab}p_a p_b + \theta(\frac{1}{2}g^{ab}_{,c}p_a p_b\psi^c + g^{ab}\phi_a p_b) \\
&= H_0 + \theta H_1.
\end{aligned} \tag{151}$$

The Lagrangian L_1 (145) can be obtained by the shift (150) from the following first order Lagrangian

$$L = -\frac{1}{2} g^{ab} p_a p_b + \dot{x}^a p_a, \tag{152}$$

which is equivalent to the Lagrangian (100).
In fact, under the shift (150) we have

$$\begin{aligned} L &= -\frac{1}{2} g^{ab} p_a p_b + \dot{x}^a p_a \\ &\Rightarrow -\frac{1}{2}(g^{ab} + g^{ab},c\theta\psi^c)(p_a + \theta\phi_a)(p_b + \theta\phi_b) + (\dot{x}^a + \theta\dot{\psi}^a)(p_a + \theta\phi_a) \\ &= L_0 + \theta L_1. \end{aligned} \tag{153}$$

Let us define (extending the zeroth level bracket (106)) the Grassmann even bracket [Berezin,1987]

$$\begin{aligned} \{x^a, p_b\}_0 &= \delta^a_b, \\ \{\psi^a, \phi_b\}_0 &= \delta^a_b, \\ \{A, B\}_0 &= \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} + \frac{\partial A}{\partial \psi^a}\frac{\partial B}{\partial \phi_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} - \frac{\partial A}{\partial \phi_a}\frac{\partial B}{\partial \psi^a} \\ &= A(\overleftarrow{\partial}_{x^a}\overrightarrow{\partial}_{p_a} + \overleftarrow{\partial}_{\psi^a}\overrightarrow{\partial}_{\phi_a} - \overleftarrow{\partial}_{p_a}\overrightarrow{\partial}_{x^a} - \overleftarrow{\partial}_{\phi_a}\overrightarrow{\partial}_{\psi^a})B \\ &= A\,\overleftarrow{\partial}_{Z_n}\,\varepsilon_{nm}\,\overrightarrow{\partial}_{Z_m}\,B. \end{aligned} \tag{154}$$

An interesting problem is to construct an even Hamiltonian.
General form $H(x, \psi, p, \phi)$ of the integrals of motion of the extended system (149) fulfils the following equation

$$\dot{H} = (\dot{x}^a \frac{\partial}{\partial x^a} + \dot{\psi}^a \frac{\partial}{\partial \psi^a} + \dot{p}_a \frac{\partial}{\partial p_a} + \dot{\phi}_a \frac{\partial}{\partial \phi_a})$$
$$= (p_b \nabla^b + \phi_b \nabla_1^b + \psi^c \nabla_{2c}) H = 0, \tag{155}$$

where

$$\nabla^b = g^{ba} \frac{\partial}{\partial x^a} - \frac{1}{2} \frac{\partial g^{bc}}{\partial x^a} p_c \frac{\partial}{\partial p_a},$$
$$\nabla_1^b = g^{ba} \frac{\partial}{\partial \psi^a} - \frac{\partial g^{bc}}{\partial x^a} p_c \frac{\partial}{\partial \phi_a},$$
$$\nabla_{2c} = \frac{\partial g^{ab}}{\partial x^c} p_b \frac{\partial}{\partial \psi^a} - \frac{1}{2} \frac{\partial^2 g^{be}}{\partial x^c \partial x^a} p_b p_e \frac{\partial}{\partial \phi_a}. \tag{156}$$

Spacetime forms of various degrees generate symmetries in particle and string supersymmetric worldvolume actions. In string theory, invariance of the worldsheet action requires that these forms are parallel with respect to a suitable connection leading to special holonomy manifolds. The conditions for the invariance of the action of supersymmetric particle under symmetries generated by spacetime forms are somewhat different. To describe these symmetries, let X a superfield which is a map from the worldline supermanifold $m(1|1)$, with coordinates (t, θ), into the spacetime M. The transformation generated by a spacetime (1 + 1)-form f is

$$\delta X^n = a_l f^n_{k_1,k_2,...,k_l} DX^{k_1} DX^{k_2}...DX^{k_l}, \tag{157}$$

where the index is raised using the spacetime metric g and a_l is an infinitesimal parameter; D is the worldline superspace derivative $D^2 = i\partial_t$. Requiring that the worldline action written in superfields,

$$I = -\frac{i}{2}\int dtd\theta g_{nm} DX^n \partial_t X^m, \tag{158}$$

to be invariant under (157), one finds that the covariant derivative of the form f coincides with the exterior derivative [Gibbons, Rietdijk, van Holten, 1993],

$$\begin{aligned} &\nabla f = (l+2)^{-1} df, \\ &\nabla_{n_1} f_{n_2...n_{l+2}} = \nabla_{[n_1} f_{n_1,n_2...n_{l+2}]} \end{aligned} \tag{159}$$

The canonical quantization of locally-supersymmetric $O(N)$-extended spinning particle models

$$A = \int dt(p_\mu \dot{x^\mu} + \frac{i}{2}\psi_{n\mu}\dot{\psi_n^\mu} - eH - i\chi_n Q_n - \frac{i}{2}a_{nm}J_{nm}),$$
$$H = \frac{1}{2}p_\mu p^\mu,\ Q_n = p_\mu \psi_n^\mu,\ J_{nm} = \psi_{n\mu}\psi_m^\mu, \tag{160}$$

yields equations of motions for spin-$\frac{N}{2}$ fields (wave functions) in terms of the corresponding linearized curvatures.
Note that we can obtain this extended hamiltonian structure from scalar particle hamiltonian by the following shift

$$p_\mu \to p_\mu + \frac{i}{e}\chi_n\psi_{n\mu} \Rightarrow eH = e\frac{1}{2}p_\mu p^\mu \to eH + i\chi_n Q_n + \frac{i}{2}a_{nm}J_{nm},$$
$$a_{nm} = \frac{i}{e}\chi_n\chi_m,\ n = 1, 2, ..., N. \tag{161}$$

In QFT existence of a given theory means, that we can control its behavior at some scales (short or large distances) by renormalization theory [Collins, 1984].
If the theory exists, than we want to solve it, which means to determine what happens on other (large or short) scales. This is the problem (and content) of Renormdynamics.
The result of the Renormdynamics, the solution of its discrete or continual motion equations, is the effective QFT on a given scale (different from the initial one).
We can invent scale variable λ and consider QFT on $D+1+1$ dimensional space-time-scale. For the scale variable $\lambda \in (0,1]$ it is natural to consider q-discretization, $0 < q < 1,\ \lambda_n = q^n,\ n = 0, 1, 2,$ ... and p - adic, nonarchimedian metric, with $q^{-1} = p$ - prime integer number.
The field variable $\varphi(x, t, \lambda)$ is complex function of the real, x, t, and p - adic, λ, variables. The solution of the UV renormdynamic problem means, to find evolution from finite to small scales with respect to the scale time $\tau = \ln \lambda/\lambda_0 \in (0, -\infty)$. Solution of the IR renormdynamic problem means to find evolution from finite to the large scales, $\tau = \ln \lambda/\lambda_0 \in (0, \infty)$.

This evolution is determined by Renormdynamic motion equations with respect to the scale-time.
As a concrete model, we take a relativistic scalar field model with lagrangian (see e.g. [Makhaldiani, 1980])

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{g}{n}\varphi^n, \ \mu = 0, 1, ..., D-1 \quad (162)$$

The mass dimension of the coupling constant is

$$[g] = d_g = D - n\frac{D-2}{2} = D + n - \frac{nD}{2}. \quad (163)$$

In the case

$$n = \frac{2D}{D-2} = 2 + \frac{4}{D-2} = 2 + \epsilon(D)$$
$$D = \frac{2n}{n-2} = 2 + \frac{4}{n-2} = 2 + \epsilon(n) \quad (164)$$

the coupling constant g is dimensionless, and the model is renormalizable.
We take the euklidean form of the QFT which unifies quantum and statistical physics problems. In the case of the QFT, we can return (in)to minkowsky space by transformation: $p_D = ip_0, \ x_D = -ix_0$.

The main objects of the theory are Green functions - correlation functions - correlators,

$$G_m(x_1, x_2, ..., x_m) =< \varphi(x_1)\varphi(x_2)...\varphi(x_m) >$$
$$= Z_0^{-1} \int d\varphi(x)\varphi(x_1)\varphi(x_2)...\varphi(x_m)e^{-S(\varphi)} \quad (165)$$

where $d\varphi$ is an invariant measure,

$$d(\varphi + a) = d\varphi. \quad (166)$$

For gaussian actions,

$$S = S_2 = \frac{1}{2} \int dxdy\phi(x)A(x, y)\phi(y) = \varphi \cdot A \cdot \varphi \quad (167)$$

the QFT is solvable,

$$G_m(x_1, ..., x_m) = \frac{\delta^m}{\delta J(x_1)...J(x_m)} lnZ_J|_{J=0},$$
$$Z_J = \int d\varphi e^{-S_2 + J\cdot\varphi} = \exp(\frac{1}{2} \int dxdyJ(x)A^{-1}(x, y)J(y))$$
$$= \exp(\frac{1}{2} J \cdot A^{-1} \cdot J) \quad (168)$$

Nontrivial problem is to calculate correlators for non gaussian QFT.

Effective action

Generating functional for connected correlators is

$$F(J) = \ln Z_J, \ \frac{\delta F(J)}{\delta J(x)} = \frac{1}{Z_J}\frac{\delta Z_J}{\delta J(x)} \equiv < \varphi(x) >_J \equiv \phi(x) - \tag{169}$$

is observable value of the field, generated by source J. We have

$$\frac{\delta}{\delta J}(F(J) - J\cdot\phi)|_{\phi=const} = 0, \tag{170}$$

so

$$\begin{aligned} & J\cdot\phi - F(J) = S_q(\phi) = S(\phi) + R(\phi) \\ & = \sum_{n\geq 1}\frac{1}{n!}\int dx_1 dx_2 ... dx_n \Gamma_n(x_1, x_2, ..., x_n)\phi(x_1)\phi(x_2)...\phi(x_n), \\ & \frac{\delta S_q}{\delta\phi(x)} = J(x); \ \frac{\delta^2 S_q}{\delta\phi(x_1)\delta\phi(x_2)} = \frac{\delta J(x_2)}{\delta\phi(x_1)} = \frac{\delta J(x_1)}{\delta\phi(x_2)} = \Gamma_2(x_1, x_2) \end{aligned} \tag{171}$$

$R(\phi)$ - is quantum corrections to the classical action.
The connected part of the two point correlator - propagator, is

$$\begin{aligned} & < \varphi(x_1)\varphi(x_2) >_c = < \varphi(x_1)\varphi(x_2) > - < \varphi(x_1) >< \varphi(x_2) > \\ & = \frac{1}{Z(J)}\frac{\delta^2 Z(J)}{\delta J(x_1)\delta J(x_2)} - \frac{1}{Z(J)}\frac{\delta Z(J)}{\delta J(x_1)}\frac{1}{Z(J)}\frac{\delta Z(J)}{\delta J(x_2)} = \Gamma_2(x_1, x_2) \end{aligned} \tag{172}$$

Perturbative series have the following qualitative form

$$f(g) = f_0 + f_1 g + ... + f_n g^n + ..., \; f_n = n!P(n)$$
$$f(x) = \sum_{n \geq 0} P(n) n! x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \; \delta = x\frac{d}{dx} \quad (173)$$

In usual sense these series are divergent, but with proper nomalization of the expansion parametre g, the coefficients of the series are rational numbers and if experimental dates indicates for some rational value for g, e.g. in QED

$$g = \frac{e^2}{4\pi} = \frac{1}{137.0...} \quad (174)$$

then we can take corresponding prime number and consider p-adic convergence of the series. In the case of QED, we have

$$f(g) = \sum f_n p^{-n}, \; f_n = n!P(n), \; p = 137,$$
$$|f|_p \leq \sum |f_n|_p p^n \quad (175)$$

In the Youkava theory of strong interections (see e.g. [Bogoliubov,1959]), we take $g = 13$,

$$f(g) = \sum f_n p^n,\ f_n = n!P(n),\ p = 13,$$
$$|f|_p \leq \sum |f_n|_p p^{-n} < \frac{1}{1 - p^{-1}} \quad (176)$$

So, the series is convergent. If the limit is rational number, we consider it as an observable value of the corresponding physical quantity. Note also, that the inverse coupling expansions, e.g. in lattice(gauge) theories,

$$f(\beta) = \sum r_n \beta^n, \quad (177)$$

are also p-adically convergent for $\beta = p^k$. We can take the following scenery. We fix coupling constants and masses, e.g in QED or QCD, in low order perturbative expansions. Than put the models on lattice and calculate observable quantities as inverse coupling expansions, e.g.

$$f(\alpha) = \sum r_n \alpha^{-n},$$
$$\alpha_{QED}(0) = 1/137;\ \alpha_{QCD}(m_Z) = 0.11... = 1/3^2 \quad (178)$$

Every (good) school boy/girl knows what is

$$\frac{d^n}{dx^n} = \partial^n = (\partial)^n, \tag{179}$$

but what is its following extension

$$\frac{d^\alpha}{dx^\alpha} = \partial^\alpha \ , \quad \alpha \in \Re \ ? \tag{180}$$

Let us consider the integer derivatives of the monomials

$$\begin{aligned}\frac{d^n}{dx^n}x^m &= m(m-1)...(m-(n-1))x^{m-n}, \quad n \le m, \\ &= \frac{\Gamma(m+1)}{\Gamma(m+1-n)}x^{m-n}.\end{aligned} \tag{181}$$

L.Euler (1707 - 1783) invented the following definition of the fractal derivatives,

$$\frac{d^\alpha}{dx^\alpha}x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}. \tag{182}$$

J.Liouville (1809-1882) takes exponents as a base functions,

$$\frac{d^\alpha}{dx^\alpha}e^{ax} = a^\alpha e^{ax}. \tag{183}$$

The following Cauchy formula

$$I^n_{0,x}f = \int_0^x dx_n \int_0^{x_{n-1}} dx_{n-2} ... \int_0^{x_2} dx_1 f(x_1) = \frac{1}{\Gamma(n)} \int_0^x dy(x-y)^{n-1} f(y) \tag{184}$$

permits analytic extension from integer n to complex α,

$$I^\alpha_{0,x}f = \frac{1}{\Gamma(\alpha)} \int_0^x dy(x-y)^{\alpha-1} f(y) \tag{185}$$

J.H. Holmgren invented (in 1863) the following integral transformation,

$$D^{-\alpha}_{c,x} f = \frac{1}{\Gamma(\alpha)} \int\limits_c^x |x-t|^{\alpha-1} f(t) dt. \tag{186}$$

It is easy to show that

$$D_{c,x}^{-\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)}(x^{m+\alpha} - c^{m+\alpha}),$$
$$D_{c,x}^{-\alpha} e^{ax} = a^{-\alpha}(e^{ax} - e^{ac}), \quad (187)$$

so, $c = 0$, when $m + \alpha \geq 0$, in Holmgren's definition of the fractal calculus, corresponds to the Euler's definition, and $c = -\infty$, when $a > 0$, corresponds to the Liouville's definition. Holmgren's definition of the fractal calculus reduce to the Euler's definition for finite c, and to the Liouvill's definition for $c = \infty$,

$$D_{c,x}^{-\alpha} f = D_{0,x}^{-\alpha} f - D_{0,c}^{-\alpha} f,$$
$$D_{\infty,x}^{-\alpha} f = D_{-\infty,x}^{-\alpha} f - D_{-\infty,\infty}^{-\alpha} f. \quad (188)$$

We considered the following modification of the $c = 0$ case [Makhaldiani,2003],

$$D_{0,x}^{-\alpha} f = \frac{|x|^\alpha}{\Gamma(\alpha)} \int\limits_0^1 |1-t|^{\alpha-1} f(xt)dt, = \frac{|x|^\alpha}{\Gamma(\alpha)} B(\alpha,\ \partial x) f(x)$$
$$= |x|^\alpha \frac{\Gamma(\partial x)}{\Gamma(\alpha+\partial x)} f(x),\ f(xt) = t^{x\frac{d}{dx}} f(x). \qquad (189)$$

As an example, consider Euler B-function,

$$B(\alpha,\beta) = \int_0^1 dx |1-x|^{\alpha-1}|x|^{\beta-1} = \Gamma(\alpha)\Gamma(\beta) D_{01}^{-\alpha} D_{0x}^{1-\beta} 1 = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (190)$$

We can define also FC as

$$D^\alpha f = (D^{-\alpha})^{-1} f = \frac{\Gamma(\partial x+\alpha)}{\Gamma(\partial x)}(|x|^{-\alpha} f),\ \partial x = \delta+1,\ \delta = x\partial \qquad (191)$$

For the Liouville's case,

$$D_{-\infty,x}^\alpha f = (D_{-\infty,x})^\alpha f = (\partial_x)^\alpha f, \qquad (192)$$

$$\partial_x^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} e^{-t\partial_x} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt t^{\alpha-1} f(x-t)$$
$$= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x dt (x-t)^{\alpha-1} f(t) = D_{-\infty,x}^{-\alpha} f. \qquad (193)$$

The integrals can be calculated as

$$D^{-n}f = (D^{-1})^n f, \tag{194}$$

where

$$D^{-1}f = x\frac{\Gamma(\partial x)}{\Gamma(1+\partial x)}f = x\frac{1}{\partial x}f = x(\partial x)^{-1}f = (\partial)^{-1}f = \int_0^x dt f(t). \tag{195}$$

Let us consider Weierstrass C.T.W. (1815 - 1897) fractal function

$$f(t) = \sum_{n\geq 0} a^n e^{i(b^n t+\varphi_n)}, \quad a<1, \quad ab>1. \tag{196}$$

For fractals we have no integer derivatives,

$$f^{(1)}(t) = i\sum (ab)^n e^{i(b^n t+\varphi_n)} = \infty, \tag{197}$$

but the fractal derivative,

$$f^{(\alpha)}(t) = \sum (ab^\alpha)^n e^{i(b^n t+\pi\alpha/2+\varphi_n)}, \tag{198}$$

when $ab^\alpha = a' < 1$, is another fractal (196).

p-adic analog of the fractal calculus (186) ,

$$D_x^{-\alpha} f = \frac{1}{\Gamma_p(\alpha)} \int\limits_{Q_p} |x - t|_p^{\alpha - 1} f(t) dt, \tag{199}$$

where $f(x)$ is a complex function of the p-adic variable x, with p-adic Γ–function

$$\Gamma_p(\alpha) = \int\limits_{Q_p} dt |t|_p^{\alpha-1} \chi(t) = \frac{1 - p^{\alpha-1}}{1 - p^{-\alpha}}, \tag{200}$$

was considered by V.S. Vladimirov [Vladimirov,1988].
The following modification of p-adic FC is given in [Makhaldiani,2003]

$$\begin{aligned} D_x^{-\alpha} f &= \frac{|x|_p^{\alpha}}{\Gamma_p(\alpha)} \int\limits_{Q_p} |1 - t|_p^{\alpha-1} f(xt) dt \\ &= |x|_p^{\alpha} \frac{\Gamma_p(\partial|x|)}{\Gamma_p(\alpha + \partial|x|)} f(x). \end{aligned} \tag{201}$$

Fractal qalculus

The basic object of q-calculus [Gasper,Rahman,1990] is q-derivative

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} = \frac{1 - q^{x\partial}}{(1-q)x} f(x), \quad (202)$$

where either $0 < q < 1$ or $1 < q < \infty$. In the limit $q \to 1, D_q \to \partial_x$.
Now we define the fractal q-calculus,

$$\begin{aligned} D_q^\alpha f(x) &= (D_q)^\alpha f(x) \\ &= ((1-q)x)^{-\alpha}(f(x) + \sum_{n\geq 1}(-1)^n \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} f(q^n x)). \end{aligned} \quad (203)$$

For the case $\alpha = -1$, we obtain the integral

$$D_q^{-1} f(x) = (1-q)x(1-q^{x\partial})^{-1} f(x) = (1-q)x \sum_{n\geq 0} f(q^n x). \quad (204)$$

In the case of $1 < q < \infty$, we can give a good analytic sense to these expressions for prime numbers $q = p = 2, 3, 5, ..., 29, ..., 137, ...$ This is an *algebra-analytic quantization* of the q-calculus and corresponding physical models. Note also, that p-adic calculus is the natural tool for the physical models defined on the fractal(space)s like Bete lattice (or Brua-Tits trees, in mathematical literature).
Note also symmetric a definition of the calculus

$$D_{qs} f(x) = \frac{f(q^{-1}x) - f(qx)}{(q^{-1} - q)x} f(x). \quad (205)$$

Usual finite difference calculus is based on the following (left) derivative operator

$$D_- f(x) = \frac{f(x) - f(x-h)}{h} = (\frac{1 - e^{-h\partial}}{h}) f(x). \tag{206}$$

We define corresponding fractal calculus as

$$D_-^{\alpha} f(x) = (D_-)^{\alpha} f(x). \tag{207}$$

In the case of $\alpha = -1$, we have usual finite difference sum as regularization of the Riemann integral

$$D_-^{-1} f(x) = h(f(x) + f(x-h) + f(x-2h) + ...). \tag{208}$$

(I believe that) the fractal calculus (and geometry) are the proper language for the quantume (field) theories, and discrete versions of the fractal calculus are proper regularizations of the fractal calculus and field theories.

Hypergeometric functions

A hypergeometric series, in the most general sense, is a power series in which the ratio of successive coefficients indexed by n is a rational function of n,

$$f(x) = \sum_{n\geq 0} a_n x^n,\ a_{n+1} = R(n)a_n,\ R(n) = \frac{P(\alpha, n)}{Q(\beta, n)} \tag{209}$$

so

$$\begin{aligned} &P(\alpha, \delta)f(x) = Q(\beta, \delta)(f(x) - f(0))/x, \\ &f(x) - f(0) = xR(\delta)f(x), f(x) = (1 - xR(\delta))^{-1} f(0), \delta = x\partial_x \end{aligned} \tag{210}$$

Hypergeometric functions have many particular special functions as special cases, including many elementary functions, the Bessel functions, the incomplete gamma function, the error function, the elliptic integrals and the classical orthogonal polynomials, because the hypergeometric functions are solutions to the hypergeometric differential equation, which is a fairly general second-order ordinary differential equation.

In a generalization given by Eduard Heine in the late nineteenth century, the ratio of successive terms, instead of being a rational function of n, are considered to be a rational function of q^n

$$
\begin{aligned}
&f(x) = \sum_{n\geq 0} a_n x^n,\ a_{n+1} = R(q^n)a_n,\ R(n) = \frac{P(\alpha, q^n)}{Q(\beta, q^n)},\\
&P(\alpha, q^\delta)f(x) = Q(\beta, q^\delta)(f(x) - f(0))/x,\\
&f(x) - f(0) = xR(q^\delta)f(x), f(x) = (1 - xR(q^\delta))^{-1}f(0), \delta = x\partial_x
\end{aligned} \tag{211}
$$

Another generalization, the elliptic hypergeometric series, are those series where the ratio of terms is an elliptic function (a doubly periodic meromorphic function) of n.
There are a number of new definitions of hypergeometric series, by Aomoto, Gelfand and others; and applications for example to the combinatorics of arranging a number of hyperplanes in complex N-space.

Hypergeometric field theory (HFT)

Formal solutions for the the hypergeometric functions (210,211), we put in the fieldtheoretic form,

$$
\begin{aligned}
&f(x) = G(x)f(0),\\
&G(x) =< \psi(x)\phi(0) >= \frac{\delta^2 \ln Z}{\delta J(x)\delta I(0)} = (1 - xR)^{-1},\\
&Z = \int d\psi d\phi e^{-S+I\phi+J\psi} = e^{I(1-xR)^{-1}J},\\
&S = \int \psi(1 - xR)\phi = \int \psi(Q - xP)\varphi,\ \phi = Q\varphi.
\end{aligned}
\qquad (212)
$$

When we invent interaction terms, we obtain nontrivial HFT. In terms of the fundamental fields, ψ, φ, we have local field model.

For LFs (see, e.g. [Miller,1977]), we find the following formulas [Makhaldiani,2011]

$$
\begin{aligned}
&F_A(a;b_1,...,b_n;c_1,...,c_n;z_1,...,z_n)=\frac{(a)_{\delta_1+...+\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c_1)_{\delta_1}...(c_n)_{\delta_n}}\ e^{z_1+...+z_n}\\
&=\frac{(a)_{\delta_1+...+\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n)\\
&=T^{-1}(a)F^n=\Sigma_{m\geq 0}\frac{(a)_{m_1+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c_1)_{m_1}...(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\ |z_1|+...+|z_n|<1;\\
&F_B(a_1,...,a_n;b_1,...,b_n;c;z_1,...,z_n)=\frac{(a_1)_{\delta_1}...(a_n)_{\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c)_{\delta_1+...+\delta_n}}\ e^{z_1+...+z_n}\\
&=\frac{(c_1)_{\delta_1}...(c_n)_{\delta_n}}{(c)_{\delta_1+...+\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n)=T(c)F^n\\
&=\Sigma_{m\geq 0}\frac{(a_1)_{m_1}...(a_n)_{m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\ |z_1|<1,...,|z_n|<1; \qquad (213)
\end{aligned}
$$

$$
\begin{aligned}
&F_C(a;b;c_1,...,c_n;z_1,...,z_n) = \frac{(a)_{\delta_1+...+\delta_n}(b)_{\delta_1+...+\delta_n}}{(c_1)_{\delta_1}...(c_n)_{\delta_n}}\ e^{z_1+...+z_n}\\
&= \frac{(a)_{\delta_1+...+\delta_n}(b)_{\delta_1+...+\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n)\\
&= T^{-1}(a)T^{-1}(b)F^n = T^{-1}(b)F_A\\
&= \Sigma_{m\geq 0}\frac{(a)_{m_1+...+m_n}(b)_{m_1+...+m_n}}{(c_1)_{m_1}...(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\ |z_1|^{1/2}+...+|z_n|^{1/2}<1;\\
&F_D(a;b_1,...,b_n;c;z_1,...,z_n) = \frac{(a)_{\delta_1+...+\delta_n}(b_1)_{\delta_1}...(b_n)_{\delta_n}}{(c)_{\delta_1+...\delta_n}}\ e^{z_1+...+z_n}\\
&= \frac{(a)_{\delta_1+...+\delta_n}(c_1)_{\delta_1}...(c_n)_{\delta_n}}{(a_1)_{\delta_1}...(a_n)_{\delta_n}(c)_{\delta_1+...\delta_n}}\ F(a_1,b_1;c_1;z_1)...F(a_n,b_n;c_n;z_n)\\
&= T^{-1}(a)T(c)F^n = T(c)F_A = T^{-1}(a)F_B\\
&= \Sigma_{m\geq 0}\frac{(a)_{m_1+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c_1)_{m_1}...(c_n)_{m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\ |z_1|<1,...,|z_n|<1.
\end{aligned}
\tag{214}
$$

In the paper ([Lomidze, 1994]) the following formula were proposed

$$B_n(r_0, r_1, ..., r_n) = det[x_j^{i-1} \int_{x_{j-1/x_j}}^{1} u^{i-1}(1-u)^{r_j-1} \prod_{k=0,k\neq j}^{n} (\frac{x_j u - x_k}{x_j - x_k})^{r_k-1} du]/det[x_j^{i-1}]$$
$$= \frac{\Gamma(r_0)\Gamma(r_1)...\Gamma(r_n)}{\Gamma(r_o + r_1 + ... + r_n)},\ 0 = x_0 < x_1 < x_2 < ... < x_n,\ n \geq 1. \quad (215)$$

Let us put the formula in the following factorized form

$$LB_n(x, r) \equiv det[x_j^{i-1} \int_{x_{j-1/x_j}}^{1} du u^{i+r_0-2}(1-u)^{r_j-1} \prod_{k=1,k\neq j}^{n} (\frac{x_j u - x_k}{x_j - x_k})^{r_k-1}]$$
$$= det V_n(x) B_n(r),\ V_n(x) = [x_j^{i-1}],\ B_n(r) = \frac{\Gamma(r_0)\Gamma(r_1)...\Gamma(r_n)}{\Gamma(r_0 + r_1 + ... + r_n)} \quad (216)$$

Now, it is enough to proof this formula for general values of x_i and particular values of r_i, e.g., $r_i = 1$, and for general values of r_i and particular values of x_i, e.g. $x_i = p^i,\ 1 \leq i \leq n$. In the case of $r_i = 1$, right hand side of the formula is equal to the Vandermonde determinant divided by $n!$ The left hand side is the determinant of the matrix with elements
$A_{ij} = x_j^{i-1}(1 - (x_{j-1}/x_j)^i)/i$
When we calculate determinant of this matrix, from the row i, we factorize $1/i, 2 \leq i \leq n$ which gives the $1/n!$ the rest matrix we calculate transforming the matrix to the form of the Vandermonde matrix.

This is the half way of the proof. Let us take the concrete values of $x_i = p^i,\ 1 \le i \le n$, where p is positive integer and general complex values for $r_i, 0 \le i \le n$, and calculate both sides of the equality. For Vandermonde determinant we find for high values of p the following asymptotic

$$detV = p^N,\ N = \sum_{k=2}^{n} k(k-1) = \frac{n(n^2-1)}{3} \tag{217}$$

The matrix elements are

$$\begin{aligned}
B_{ij} &= x_j^{i-1} \int_{x_{j-1}/x_j}^{1} u^{i+r_0-2}(1-u)^{r_j-1} \prod_{k=1,k\neq j}^{n} (\frac{x_j u - x_k}{x_j - x_k})^{r_k-1} du] \\
&= x_j^{i-1}(\prod_{1\le k<j} (\frac{x_j}{x_j - x_k})^{r_k-1} \prod_{j<k\le n} (\frac{x_k}{x_k - x_j})^{r_k-1} \int_{x_{j-1}/x_j}^{1} u^{i+r_0-2}(1-u)^{r_j-1} \\
&\cdot \prod_{1\le k<j} (u - x_k/x_j)^{r_k-1} \prod_{j<k\le n} (1 - x_j/x_k u)^{r_k-1} du \\
&= p^{(i-1)j}(\int_0^1 u^{i+r_0-2+\sum_{k=1}^{j-1}(r_k-1)}(1-u)^{r_j-1} du \\
&= p^{(i-1)j} B(i + \sum_{k=0}^{j-1}(r_k - 1), r_j)
\end{aligned} \tag{218}$$

For $n = 2$ we have

$$
\begin{aligned}
B_{11} &= \int_0^1 u^{r_0-1}(1-u)^{r_1-1}du = \frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0+r_1)}, \\
B_{22} &= p^2\int_0^1 u^{r_0+r_1-1}(1-u)^{r_2-1}du = \frac{\Gamma(r_0+r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)}, \\
LB_2/V_2 &= B_{11}B_{22}/p^2 = \frac{\Gamma(r_0)\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)}
\end{aligned}
\tag{219}
$$

For $n = 3$,

$$
\begin{aligned}
B_{11} &= \int_0^1 u^{r_0-1}(1-u)^{r_1-1} = \frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0+r_1)} = B(r_0, r_1), \\
B_{22} &= p^2\int_0^1 u^{r_0+r_1-1}(1-u)^{r_2-1} = p^2\frac{\Gamma(r_0+r_1)\Gamma(r_2)}{\Gamma(r_0+r_1+r_2)}, \\
B_{33} &= p^6\int_0^1 u^{r_0+r_1+r_2-1}(1-u)^{r_3-1} = p^6\frac{\Gamma(r_0+r_1+r_2)\Gamma(r_3)}{\Gamma(r_0+r_1+r_2+r_3)} \\
LB_3/V_3 &= B_{11}B_{22}B_{33}/p^8 = \frac{\Gamma(r_0)\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)}{\Gamma(r_0+r_1+r_2+r_3)}
\end{aligned}
\tag{220}
$$

Now it is obvious the last step of the proof [Makhaldiani,2011]

$$
\begin{aligned}
&LB_n(x,r) = detV_n(x)B(r_0,r_1)...B(r_0+r_1+...+r_{n-1},r_n) \\
&= detV_n(x)B_n(r) \\
&V_n(x) = [x_j^{i-1}],\ B_n(r) = \frac{\Gamma(r_0)\Gamma(r_1)..\Gamma(r_n)}{\Gamma(r_0+r_1+...+r_n)}
\end{aligned}
\tag{221}
$$

Let us consider the following action

$$S = \frac{1}{2}\int\limits_{Q_v} dx\Phi(x)D_x^\alpha\Phi,\ \ v = 1, 2, 3, 5, ..., 29, ..., 137, ... \qquad (222)$$

Q_1 is real number field, Q_p, p - prime, are p-adic number fields. In the momentum representation

$$S = \frac{1}{2}\int\limits_{Q_v} du\tilde{\Phi}(-u)|u|_v^\alpha\tilde{\Phi}(u),\ \ \Phi(x) = \int\limits_{Q_v} du\chi_v(ux)\tilde{\Phi}(u),$$
$$D^{-\alpha}\chi_v(ux) = |u|_v^{-\alpha}\chi_v(ux). \qquad (223)$$

The statistical sum of the corresponding quantum theory is

$$Z_v = \int d\Phi e^{-\frac{1}{2}\int \Phi D^\alpha \Phi} = det^{-1/2}D^\alpha = (\prod_u |u|_v)^{-\alpha/2}. \qquad (224)$$

For (symmetrized, 4-tachyon) Veneziano amplitude we have (see, e.g. [Kaku,2000])

$$B_s(\alpha,\beta) = B(\alpha,\beta) + B(\beta,\gamma) + B(\gamma,\alpha) = \int_{-\infty}^{\infty} dx|1-x|^{\alpha-1}|x|^{\beta-1}, \quad \alpha+\beta+\gamma = 1 \tag{225}$$

For the p-adic Veneziano amplitude we take

$$B_p(\alpha,\beta) = \int_{Q_p} dx|1-x|_p^{\alpha-1}|x|_p^{\beta-1} = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{226}$$

Now we obtain the N-tachyon amplitude using fractal calculus. We consider the dynamics of particle given by multicomponent generalization of the action (222), $\Phi \to x^\mu$.

For the closed trajectory of the particle passing through N points, we have

$$A(x_1, x_2, ..., x_N) = \int dt \int dt_1 ... \int dt_N \delta(t - \Sigma t_n)$$
$$v(x_1, t_1; x_2, t_2) v(x_2, t_2; x_3, t_3) ... v(x_N, t_N; x_1, t_1)$$
$$= \int dx(t) \Pi(\int dt_n \delta(x^\mu(t_n) - x_n^\mu)) exp(-S[x(t)])$$
$$= \int \Pi(dk_n^\mu \chi(k_n x_n)) \tilde{A}(k), \tag{227}$$

where

$$\tilde{A}(k) = \int dx V(k_1) V(k_2) ... V(k_N) exp(-S),$$
$$V(k_n) = \int dt \chi(-k_n x(t)) \tag{228}$$

is vertex function.
Motion equation

$$D^\alpha x^\mu - i\Sigma k_n^\mu \delta(t - t_n) = 0, \tag{229}$$

in the momentum representation

$$|u|^\alpha \tilde{x}^\mu(u) - i\Sigma_n k_n^\mu \chi(-u t_n) = 0 \tag{230}$$

have the solution

$$\tilde{x}^\mu(u) = i\Sigma k_n^\mu \frac{\chi(-u t_n)}{|u|^\alpha}, \quad u \neq 0, \tag{231}$$

the constraint

$$\Sigma_n k_n = 0, \tag{232}$$

and the zero mod $\tilde{x}_n^\mu(0)$, which is arbitrary. Integration in (227) with respect to this zero mod gives the constraint (232). On the solution of the equation (229)

$$x^\mu(t) = iD_t^{-\alpha}\Sigma_n k_n^\mu \delta(t - t_n) = \frac{i}{\Gamma(\alpha)}\Sigma_n k_n^\mu |t - t_n|^{\alpha-1}, \tag{233}$$

the action (222) takes value

$$S = -\frac{1}{\Gamma(\alpha)}\Sigma_{n<m} k_n k_m |t_n - t_m|^{\alpha-1},$$
$$\tilde{A}(k) = \int \Pi_{n=1}^N dt_n exp(-S) \tag{234}$$

In the limit, $\alpha \to 1$, for p-adic case we obtain

$$x^\mu(t) = -i\frac{p-1}{p\ lnp}\Sigma_n k_n^\mu ln|t - t_n|,$$
$$S[x(t)] = \frac{p-1}{p\ lnp}\Sigma_{n<m} k_n k_m\ ln|t_n - t_m|,$$
$$\tilde{A}(k) = \int \Pi_{n=1}^N dt_n \Pi_{n<m}|t_n - t_m|^{\frac{p-1}{p\ lnp}k_n k_m}. \tag{235}$$

Now in the limit $p = q^{-1} \to 1$ we obtain the proper expressions of the real case

$$x^{\mu}(t) = -i\Sigma_n k_n^{\mu} ln|t - t_n|,$$
$$S[x(t)] = \Sigma_{n<m} k_n k_m \ ln|t_n - t_m|,$$
$$\tilde{A}(k) = \int \Pi_{n=1}^{N} dt_n \Pi_{n<m} |t_n - t_m|^{k_n k_m}. \quad (236)$$

By fractal calculus and vector generalization of the model (222), fundamental string amplitudes were obtained in [Makhaldiani,1988].

The RD equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The beta function and the quarks mass anomalous dimension are among the most prominent objects for QCD RD equations. The calculation of the one-loop β-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions [Gross,Wilczek,1973, Politzer,1973, 't Hooft,1972].
The MS-scheme ['t Hooft,1972] belongs to the class of massless schemes where the β-function does not depend on masses of the theory and the first two coefficients of the β-function are scheme-independent.

The Lagrangian of QCD with massive quarks in the covariant gauge

$$
\begin{aligned}
& L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{q}_n(i\gamma D - m_n)q_n \\
& -\frac{1}{2\xi}(\partial A)^2 + \partial^\mu \bar{c}^a(\partial_\mu c^a + gf^{abc}A^b_\mu c^c) \\
& F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \\
& (D_\mu)_{kl} = \delta_{kl}\partial_\mu - igt^a_{kl}A^a_\mu,
\end{aligned} \tag{237}
$$

$A^a_\mu, a = 1, ..., N_c^2 - 1$ are gluon; $q_n, n = 1, ..., n_f$ are quark; c^a are ghost fields; ξ is gauge parameter; t^a are generators of fundamental representation and f^{abc} are structure constants of the Lie algebra

$$
[t^a, t^b] = if^{abc}t^c, \tag{238}
$$

we will consider an arbitrary compact semi-simple Lie group G. For QCD, $G = SU(N_c), N_c = 3$.

The RD equation for the coupling constant is

$$\dot{a} = \beta(a) = -\beta_2 a^2 - \beta_3 a^3 - \beta_4 a^4 - \beta_5 a^5 + O(a^6),$$
$$a = \alpha_s/\pi = \frac{g^2}{4\pi^2},$$
$$\int_{a_0}^{a} \frac{da}{\beta(a)} = t - t_0 = \ln\frac{\mu}{\mu_0}, \tag{239}$$

μ is the 't Hooft unit of mass, the renormalization point in the MS-scheme. To calculate the β-function we need to calculate the renormalization constant Z of the coupling constant, $a_b = Za$, where a_b is the bare (unrenormalized) charge.

The expression of the β-function can be obtained in the following way

$$0 = d(a_b\mu^{2\varepsilon})/dt = \mu^{2\varepsilon}(\varepsilon Za + \frac{\partial(Za)}{\partial a}\frac{da}{dt})$$
$$\Rightarrow \frac{da}{dt} = \beta(a,\varepsilon) = \frac{-\varepsilon Za}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),$$
$$\beta(a) = a\frac{d}{da}(aZ_1) \qquad (240)$$

where

$$\beta(a,\varepsilon) = \frac{D-4}{2}a + \beta(a) \qquad (241)$$

is $D-$dimensional $\beta-$function and Z_1 is the residue of the first pole in ε expansion

$$Z(a,\varepsilon) = 1 + Z_1\varepsilon^{-1} + ... + Z_n\varepsilon^{-n} + ... \qquad (242)$$

Since Z does not depend explicitly on μ, the β-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ.

For quark anomalous dimension, RD equation is

$$\dot{b} = \gamma(a) = -\gamma_1 a - \gamma_2 a^2 - \gamma_3 a^3 - \gamma_4 a^4 + O(a^5),$$
$$b = \ln m_q,$$
$$b(t) = b_0 + \int_{t_0}^{t} dt\gamma(a(t)) = b_0 + \int_{a_0}^{a} da\gamma(a)/\beta(a). \quad (243)$$

To calculate the quark mass anomalous dimension $\gamma(g)$ we need to calculate the renormalization constant Z_m of the quark mass $m_b = Z_m m,\ m_b$ is the bare (unrenormalized) quark mass. Than we find the function $\gamma(g)$ in the following way

$$0 = \dot{m}_b = \dot{Z}_m m + Z_m \dot{m} = Z_m m((\ln Z_m)^{\cdot} + (\ln m)^{\cdot})$$
$$\Rightarrow \gamma(a) = -\frac{d\ln Z_m}{dt}$$
$$= -\frac{d\ln Z_m}{da}\frac{da}{dt} = -\frac{d\ln Z_m}{da}(-\varepsilon a + \beta(a)) = a\frac{dZ_{m1}}{da}, \quad (244)$$

where RD equation in $D-$dimension is

$$\dot{a} = -\varepsilon a + \beta(a) = \beta_1 a + \beta_2 a^2 + ... \quad (245)$$

and Z_{m1} is the coefficient of the first pole in the $\varepsilon-$expantion of the Z_m in MS-scheme

$$Z_m(\varepsilon, g) = 1 + \frac{Z_{m1}(g)}{\varepsilon} + \frac{Z_{m2}(g)}{\varepsilon^2} + ... \quad (246)$$

Since Z_m does not depend explicitly on μ and m, the γ_m-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ.

RD equation,

$$\dot{a} = \beta_1 a + \beta_2 a^2 + ... \quad (247)$$

can be reparametrized,

$$a(t) = f(A(t)) = A + f_2 A^2 + ... + f_n A^n + ... = \sum_{n\geq 1} f_n A^n,$$
$$\dot{A} = b_1 A + b_2 A^2 + ... = \sum_{n\geq 1} b_n A^n, \quad (248)$$

$$\begin{aligned}
\dot{a} &= \dot{A} f'(A) = (b_1 A + b_2 A^2 + ...)(1 + 2f_2 A + ... + n f_n A^{n-1} + ...) \\
&= \beta_1 (A + f_2 A^2 + ... + f_n A^n + ...) + \beta_2 (A^2 + 2 f_2 A^3 + ...) + ... \\
&+ \beta_n (A^n + n f_2 A^{n+1} + ...) + ... \\
&= \beta_1 A + (\beta_2 + \beta_1 f_2) A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3) A^3 + \\
&... + (\beta_n + (n-1)\beta_{n-1} f_2 + ... + \beta_1 f_n) A^n + ... \\
&= \sum_{n, n_1, n_2 \geq 1} A^n b_{n_1} n_2 f_{n_2} \delta_{n, n_1 + n_2 - 1} \\
&= \sum_{n, m \geq 1; m_1, ..., m_k \geq 0} A^n \beta_m f_1^{m_1} ... f_k^{m_k} f(n, m, m_1, ..., m_k),
\end{aligned}$$
$$f(n, m, m_1, ..., m_k) = \frac{m!}{m_1! ... m_k!} \delta_{n, m_1 + 2m_2 + ... + k m_k} \delta_{m, m_1 + m_2 + ... + m_k}, \quad (249)$$

$$
\begin{aligned}
&b_1 = \beta_1,\ b_2 = \beta_2 + f_2\beta_1 - 2f_2b_1 = \beta_2 - f_2\beta_1,\\
&b_3 = \beta_3 + 2f_2\beta_2 + f_3\beta_1 - 2f_2b_2 - 3f_3b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1,\\
&b_4 = \beta_4 + 3f_2\beta_3 + f_2^2\beta_2 + 2f_3\beta_2 - 3f_4b_1 - 3f_3b_2 - 2f_2b_3, ...\\
&b_n = \beta_n + ... + \beta_1 f_n - 2f_2b_{n-1} - ... - nf_nb_1, ...
\end{aligned} \tag{250}
$$

so, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1.
We can fix any higher coefficient with zero value, if we take

$$
f_2 = \frac{\beta_2}{\beta_1},\ f_3 = \frac{\beta_3}{2\beta_1} + f_2^2,\ ...,f_n = \frac{\beta_n + ...}{(n-1)\beta_1}, ... \tag{251}
$$

In this case we have exact classical dynamics in the (external) space-time and simple scale dynamics,

$$
\begin{aligned}
&g = (\mu/\mu_0)^{-2\varepsilon} g_0 = e^{-2\varepsilon\tau} g_0;\\
&\varphi(\tau, t, x) = e^{-(D-2)/2\tau}\varphi_0(t, x),\\
&\psi(\tau, t, x) = e^{-(D-1)/2\tau}\psi_0(t, x)
\end{aligned} \tag{252}
$$

We will consider in applications the case when only one of higher coefficient is nonzero.
In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but β_2 and β_3.

From the relations (250), in the critical dimenshion ($\beta_1 = 0$), we find that, we can define the minimal form of the RD equation

$$\dot{A} = \beta_2 A^2 + \beta_3 A^3, \tag{253}$$

e.g. $b_4 = 0$ when

$$f_3 = \frac{\beta_4}{\beta_2} + \frac{\beta_3}{\beta_2} f_2 + f_2^2, \tag{254}$$

f_2 remains arbitrary and we can make choice $f_2 = 0$. We can solve (253) as implicit function,

$$u^{\beta_3/\beta_2} e^{-u} = ce^{\beta_2 t},\ u = \frac{1}{A} + \frac{\beta_3}{\beta_2} \tag{255}$$

than, as in the noncritical case, explicit solution will be given by reparametrization representation.

If we know somehow the coefficients β_n, e.g. for first several exact and for others asymptotic values (see e.g. [Kazakov,Shirkov,1980]) than we can construct reparametrization function (248) and find the dynamics of the running coupling constant. This is similar to the action-angular canonical transformation of the analytic mechanics (see e.g. [Faddeev,Takhtajan]).

Renormdynamic functions (RDF)

We will call RDF functions $g_n = f_n(t)$, which are solutions of the RD motion equations

$$\dot{g}_n = \beta_n(g), 1 \leq n \leq N. \tag{256}$$

In the simplest case of one coupling constant, the function $g = f(t)$, is constant $g = g_c$ when $\beta(g_c) = 0$, or is invertible (monotone). Indeed,

$$\dot{g} = f'(t) = f'(f^{-1}(g)) = \beta(g). \tag{257}$$

Each monotone interval ends by UV and IR fixed points and describes corresponding phase of the system.
Based on real experiments and computer simulations, quantum gauge theory in four dimensions is believed to have a mass gap. This is one of the most fundamental facts that makes the Universe the way it is. In the lattice (gauge) theory approach to the renormdynamics (see, e.g. [Makhaldiani,1986]), recently running coupling constant dynamics were calculated for $SU(2)$ Yang-Mills model [Bogolubsky et al,2009]. The result is in agreement with perturbative calculations at small scales; at an intermediate scale the coupling constant reaches its maximum($\simeq 1.25$); than decrease. So, at the maximum, we may have nontrivial zero of the $\beta-$function, which corresponds to the conformal invariance of the gluodynamics at this point. Beyond this point we have another phase, strong coupling phase with decreasing coupling constant similar (identical?!) to the abelian (monopole?) theory.

Note that, in the case of the two coupling constants,

$$\dot{g}_1 = \beta_1(g_1, g_2),$$
$$\dot{g}_2 = \beta_2(g_1, g_2), \tag{258}$$

we can reformulate RD as

$$g_1 \equiv g; g_2 = f_2(t) \equiv \tau,$$
$$\frac{dg_1}{dg_2} = \frac{dg}{d\tau} \equiv \dot{g} = \beta(g, \tau) = \frac{\beta_1(g, \tau)}{\beta_2(g, \tau)} \tag{259}$$

and RDF must fulfil corresponding restrictions. E.g. if

$$g_1 = f_1(t) = g = f(\tau) = f(f_2(t)), g_2 = f_2(t) = \tau \tag{260}$$

So, if we approximate the form of the curve near maximum as

$$a(t) = a_c - b|t - t_c|^n, \tag{261}$$

for the $\beta-$function we obtain

$$\dot{a} = \beta(a, t) = sign(t_c - t)bn(\frac{a_c - a}{b})^{\frac{n-1}{n}}. \tag{262}$$

Of course this is not usual $\beta-$function, function of a only. It depends also on t. For $t > t_c$ we have perturbative phase. For $n > 1$, $\beta(a_c, t) = 0$. Explicit dependence on time variable in one coupling case indicates on implicit two coupling case.

So, at the critical point we may have low energy unification of the two abelian couplings, weak-electromagnetic and strong-monopole couplings. According to the Dirac quantization rule, for the electron-e and monopole-g charges we have

$$eg = \frac{n}{2}, \ n = \pm 1, \pm 2, ... \tag{263}$$

so, at the selfdual, critical, point, according to the computational results, we have prediction:

$$\alpha_e = \alpha_g = \frac{n}{8\pi} \simeq \frac{5}{4} \Rightarrow n \simeq 10\pi \simeq 31, \tag{264}$$

this low energy unification prime number 31 is a twin of the grandunification point prime number 29, [Makhaldiani,2011]. If we take low energy value for the electromagnetic fine structure constant, $\alpha = 1/137$, we can predict corresponding value for magnetic "fine structure" constant

$$\alpha_g = \frac{g^2}{4\pi} = (\frac{31}{2})^2 \frac{137}{(4\pi)^2} \simeq 208 = 16 \times 13; \ \simeq 209 = 11 \times 19. \tag{265}$$

If we have a solution $x_n = x_{0n}$ (a state) of the following system of motion equations (of the corresponding dynamical system)

$$\dot{x}_n = f_n(x),\ 1 \leq n \leq N, \tag{266}$$

we can consider the question of stability of the solution, the existence of the solutions of the type $x_n = x_{0n} + g_n$, for small values of g_n. If there are solutions with rising g_n, of the corresponding motion equations

$$\begin{aligned} &\dot{g}_n = \beta_n(g), \\ &\beta_n(g) = f_n(x_0 + g) - f_n(x_0) = \beta_{1nm} g_m + \beta_{2nmk} g_m g_k + ..., \\ &\beta_{kn...m} = f^{(n...m)}(x_0) \end{aligned} \tag{267}$$

we say that the solution x_{0n} is not stable.
The linear approximation, we transform into diagonal form,

$$\begin{aligned} &\dot{g}_n = \beta_{1nm} g_m,\ h_n = A_{nm} g_m, \\ &\dot{h}_n = \lambda_n h_n,\ \lambda_n \delta_{nm} = (A \beta_1 A^{-1})_{nm}, \end{aligned} \tag{268}$$

if all of the λ_n are purely imaginary $\lambda_n = i\omega_n$, we have stable solution (in the linear approximation): small deviations remain small. If real parts of all λ_n are negative, we have asymptotic stability: deviations decrease. If some λ_n are zero, we have undefined case. In regular case, when the matrix β_1 has inverse, by reparametrization trick we can construct the formal solution of the nonlinear equation for g_n, and try to investigate its convergence properties.

In the case of several integrals of motion, $H_n,\ 1 \leq n \leq N$, we can formulate Renormdynamics as Nambu - Poisson dynamics (see e.g. [Makhaldiani, 2007])

$$\dot{\varphi}(x) = [\varphi(x), H_1, H_2, ..., H_N], \tag{269}$$

where φ is an observable as a function of the coupling constants $x_m,\ 1 \leq m \leq M$.
In the case of Standard model [Weinberg,1995], we have three coupling constants, $M = 3$.

The renormdynamic motion equations

$$\dot{g}_n = \beta_n(g), \ 1 \leq n \leq N \tag{270}$$

where $g_n, \ 1 \leq n \leq N$, are coupling constants, can be presented as nonlinear part of a hamiltonian system with linear part

$$\dot{\Psi}_n = -\frac{\partial \beta_m}{\partial g_n}\Psi_m, \tag{271}$$

hamiltonian and canonical Poisson bracket as

$$H = \sum_{n=1}^{N} \beta(g)_n \Psi_n, \ \{g_n, \Psi_m\} = \delta_{nm} \tag{272}$$

In this extended version, we can define optimal control theory approach [Pontryagin, 1983] to the unified field theories. We can start from the unified value of the coupling constant, e.g. $\alpha^{-1}(M) = 29.0...$ at the scale of unification M, put the aim to reach the SM scale with values of the coupling constants measured in experiments, and find optimal threshold corrections to the RD coefficients [Makhaldiani,2010].

For connected vertex functions Γ_n, (171)

$$\Gamma_n(x_1, x_2, ..., x_n; g, m, \mu) = Z^{n/2}(\mu)\Gamma_{0n}(x_1, x_2, ..., x_n; g_0, m_0),$$
$$(D - \frac{n}{2}\gamma)\Gamma_n(x; g, m, \mu) = 0; \quad (273)$$

For effective action S_q,

$$(D - \frac{1}{2}\gamma \int dx\phi(x)\frac{\delta}{\delta\phi(x)})S_q(\phi) = 0,$$
$$(D - \frac{1}{2}\gamma\phi\frac{\partial}{\partial\phi})V(\phi) = 0, \ V(\phi) = S_q(\phi(x))|_{\phi(x)=\phi=const}, \quad (274)$$

where $V(\phi)$ is effective potential. For the effective potential in the RD (conformal) fixed point, $\gamma(g) = \gamma(g_c) \equiv \gamma_c$ we have the following wave equation and corresponding (auto model) solution

$$(\partial_t - \frac{\gamma_c}{2}\partial_z)V = 0,$$
$$V(\phi, \mu) = f(z + vt) = F(\frac{\phi}{\mu^v}), \ t = \ln\frac{\mu}{\mu_0}, z = \ln\frac{\phi}{\phi_0}, v = \frac{\gamma_c}{2}. \quad (275)$$

The fundamental quark and gluon degrees of freedom are the relevant ones at high temperatures and/or densities. Since these degrees of freedom are confined in the low temperature and density regime there must be a quark and/or gluon (de)confinement phase transition.
It is difficult to describe the phase transition because there is not known a local parameter which can be linked to confinement. We consider the fractal dimension of the hadronic/quark-gluon space as order parameter of (de)confinement phase transition. It has value less than 3 in the abelian, hadronic, phase, and more than 3, in nonabelian, quark-gluon, phase.

Let us consider $l-$particle semi-inclusive distribution

$$F_l(n,q) = \frac{d^l\sigma_n}{\bar{d}q_1...\bar{d}q_l} = \frac{1}{n!}\int\prod_{i=1}^{n}\bar{d}q_i'\delta(p_1+p_2-\Sigma_{i=1}^{l}q_i-\Sigma_{i=1}^{n}q_i')$$
$$\cdot|M_{n+l+2}(p_1,p_2,q_1,...,q_l,q_1',...,q_n';g(\mu),m(\mu)),\mu)|^2,$$
$$\bar{d}p \equiv \frac{d^3p}{E(p)},\ E(p) = \sqrt{p^2+m^2}. \qquad (276)$$

From the renormdynamic equation

$$DM_{n+l+2} = \frac{\gamma}{2}(n+l+2)M_{n+l+2}, \tag{277}$$

we obtain

$$\begin{aligned}
&DF_l(n,q) = \gamma(n+l+2)F_l(n,q),\\
&DF_l(q) = \gamma(< n > + l + 2)F_l(q),\\
&D < n^k(q) >= \gamma(< n^{k+1}(q) > - < n^k(q) >< n(q) >),\\
&DC_k = \gamma < n(q) > (C_{k+1} - C_k(1 + k(C_2 - 1)))\\
&F_l(q) \equiv \frac{d^l\sigma}{\bar{d}q_1 ... \bar{d}q_l} = \sum_n \frac{d^l\sigma_n}{\bar{d}q_1 ... \bar{d}q_l},\ < n^k(q) >= \frac{\sum_n n^k d^l\sigma_n/\bar{d}q^l}{\sum_n d^l\sigma_n/\bar{d}q^l}\\
&C_k = \frac{< n^k(q) >}{< n(q) >^k}
\end{aligned} \tag{278}$$

From dimensional considerations, the following combination of cross sections [Koba et al, 1972] must be universal function

$$< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \qquad (279)$$

Corresponding relation for the inclusive cross sections is [Matveev et al, 1976].

$$< n(p) > \frac{d\sigma_n}{\bar{d}p} / \frac{d\sigma}{\bar{d}p} = \Psi(\frac{n}{< n(p) >}). \qquad (280)$$

Indeed, let us define $n-$dimension of observables [Makhaldiani, 1980]

$$[n] = 1, \ [\sigma_n] = -1, \sigma = \Sigma_n \sigma_n, \ [\sigma] = 0, \ [< n >] = 1. \qquad (281)$$

The following expression does not depend on any dimensional quantities and must have a corresponding universal form

$$P_n =< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \qquad (282)$$

Let us find an explicit form of the universal functions using renormdynamic equations.

From the definition of the moments we have

$$C_k = \int_0^\infty dx x^k \Psi(x), \tag{283}$$

so they are universal parameters,

$$\begin{aligned} &DC_k = 0 \Rightarrow C_{k+1} = (1 + k(C_2 - 1))C_k \Rightarrow \\ &C_k = (1 + (k-1)(C_2 - 1))...(1 + 2(C_2 - 1))C_2. \end{aligned} \tag{284}$$

Now we can invert momentum transform and find (see [Makhaldiani, 1980] and appendix) universal functions [Ernst, Schmit, 1976], [Darbaidze et al, 1978].

$$\begin{aligned} &\Psi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n = \frac{c^c}{\Gamma(c)} z^{c-1} e^{-cz}, \\ &C_2 = 1 + \frac{1}{c} \end{aligned} \tag{285}$$

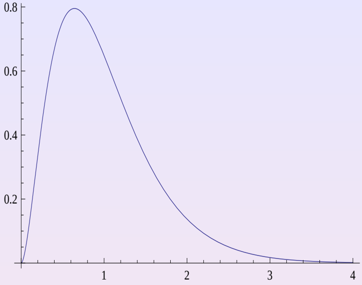

Figure: KNO distribution (285), $\Psi(z)$, with $c = 2.8$

The value of the parameter c can be measured from the dispersion low,

$$D = \sqrt{< n^2 > - < n >^2} = \sqrt{C_2 - 1} < n >= A < n >,$$
$$A = \frac{1}{\sqrt{c}} \simeq 0.6, \ c = 2.8;$$
$$(c = 3, \ A = 0.58) \tag{286}$$

which is in accordance with $n-$dimension counting.

We can calculate also $1/<n>$ correction to the scaling function (see appendix)

$$<n>\frac{\sigma_n}{\sigma}=\Psi=\Psi_0\left(\frac{n}{<n>}\right)+\frac{1}{<n>}\Psi_1\left(\frac{n}{<n>}\right),$$
$$C_k=C_k^0+\frac{1}{<n>}C_k^1,$$
$$C_k^0=\int_0^\infty dx x^k\Psi_0(x),\ C_k^1=\int_0^\infty dx x^k\Psi_1(x),$$
$$\Psi_1(z)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dn z^{-n-1}C_n^1=\frac{C_2^1c^2}{2}\left(z-2+\frac{c-1}{cz}\right)\Psi_0 \quad (287)$$

The characteristic function we define as

$$\Phi(t) = \int_0^\infty dx e^{tx} \Psi(x) = (1 - t/c)^{-c},\ Re(t) < c \tag{288}$$

For the moments of the distribution, we have

$$\Phi^{(k)}(0) = C_k = (-c)(-c-1)...(-c-k+1)(-1/c)^k = \frac{\Gamma(c+k)}{\Gamma(c)c^k} \tag{289}$$

Note that it is an infinitely divisible characteristic function, i.e.

$$\Phi(t) = (\Phi_n(t))^n,\ \Phi_n(t) = (1 - t/c)^{-c/n} \tag{290}$$

If we calculate observable(mean) value of x, we find

$$\begin{aligned} &< x >= \Phi'(0) = n\Phi(0)_n{}' = n < x >_n, \\ &< x >_n = \frac{< x >}{n} \end{aligned} \tag{291}$$

For the second moment and dispersion, we have

$$
\begin{aligned}
&< x^2 >= \Phi^{(2)}(0) = n < x^2 >_n + n(n-1) < x >_n^2, \\
&\mathrm{D}^2 = < x^2 > - < x >^2 = n(< x^2 >_n - < x >_n^2) = n\mathrm{D}_n^2 \\
&\mathrm{D}_n^2 = \frac{\mathrm{D}^2}{n} = \frac{\mathrm{D}^2}{< x >} < x >_n
\end{aligned} \tag{292}
$$

In a sense, any Hamiltonian quantum (and classical) system can be described by infinitely divisible distributions, because in the functional integral formulation, we use the following step

$$U(t) = e^{-itH} = (e^{-i\frac{t}{N}H})^N \tag{293}$$

In the case of our scalar field theory (162),

$$\begin{aligned} L(\varphi) &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{g}{n}\varphi^n \\ &= g^{\frac{2}{2-n}}(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{1}{n}\phi^n) \end{aligned} \tag{294}$$

so, to the constituent field ϕ_N corresponds higher value of the coupling constant,

$$g_N = gN^{\frac{n-2}{2}} \tag{295}$$

For weak nonlinearity, $n = 2 + 2\varepsilon, d = 2/\varepsilon + 2, g_N = g(1 + \varepsilon \ln N + O(\varepsilon^2))$

Closed equation of renormdynamics for the generating function of the observables

Let us consider a generating function of the topological crossections

$$F(h,g,m,\mu)=\Sigma_{n\geq 2}h^n\sigma_n,$$
$$\sigma_n=\frac{1}{n!}\frac{d^n}{dh^n}F|_{h=0},$$
$$\sigma=F|_{h=1},\quad <n>=\frac{d}{dh}\ln F|_{h=1},... \tag{296}$$

It is natural that for the generating function we have closed renormdynamic equation [Makhaldiani, 1980]

$$(D-\gamma(\frac{h\partial}{\partial h}+2))F=0,$$
$$F(h(\mu),g(\mu),m(\mu),\mu)=F(h(\bar\mu),g(\bar\mu),m(\bar\mu),\bar\mu)\exp(2\int_{\bar\mu}^{\mu}\frac{d\rho}{\rho}\gamma(g(\rho))),$$
$$\bar h=\bar h(\bar\mu)=h(\mu)\exp(\int_{\mu}^{\bar\mu}\frac{d\rho}{\rho}\gamma(g(\rho))),$$
$$\bar m=\bar m(\bar\mu)=m(\mu)\exp(\int_{\mu}^{\bar\mu}\frac{d\rho}{\rho}\eta(g(\rho))),\ \int_g^{\bar g}\frac{dg}{\beta(g)}=\ln\frac{\bar\mu}{\mu} \tag{297}$$

Let us find generating function in the case of KNO scaling. From the definition of Generating function and using topological cross section from KNO, we find

$$F(h) = \sum_n h^n \frac{\sigma}{< n >} \Psi(\frac{n}{< n >}) = \frac{\sigma}{< n >} \sum \Psi(\frac{n}{< n >}) h^n$$
$$= \frac{\sigma}{< n >} \Psi(\frac{\delta}{< n >}) \frac{h^2}{1-h}, \ \delta \equiv h\frac{d}{dh}, \ q^\delta f(h) = f(qh), \tag{298}$$

Now we can find more concrete form of the generating function, with the explicit form of KNO function,

$$(\frac{\delta}{< n >})^{c-1} \exp(-c\frac{\delta}{< n >}) \frac{h^2}{1-h} = (\frac{\delta}{< n >})^{c-1} \frac{q^2h^2}{1-qh}$$
$$= \frac{1}{< n >^{c-1}} \frac{1}{\Gamma(1-c)} \int_0^\infty \frac{dt}{t^c} \frac{q^2h^2e^{-2t}}{1-qhe^{-t}}, \tag{299}$$

so

$$F(h)_{KNO} = \frac{c^c}{\Gamma(c)} \frac{\sigma}{< n >^c} \frac{1}{\Gamma(1-c)} \int_0^\infty \frac{dt}{t^c} \frac{q^2h^2e^{-2t}}{1-qhe^{-t}},$$
$$q = \exp(-\frac{c}{< n >}) \tag{300}$$

Indeed, if we expend and than integrate under this formula, we hind

$$F(h) = \frac{c^c}{\Gamma(c)} \frac{\sigma}{< n >^c} \sum_{n \geq 2} h^n n^{c-1} \exp(-\frac{c}{< n >} n) \tag{301}$$

which corresponds to the considered explicit form of the KNO function.

Negative binomial distribution (NBD) is defined as

$$P(n) = \frac{\Gamma(n+r)}{n!\Gamma(r)} p^n (1-p)^r, \ \sum_{n \geq 0} P(n) = 1, \quad (302)$$

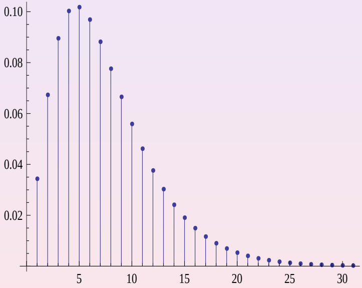

Figure: $P(n), (302), r = 2.8, p = 0.3, < n >= 6$

NBD provides a very good parametrization for multiplicity distributions in e^+e^- annihilation; in deep inelastic lepton scattering; in proton-proton collisions; in proton-nucleus scattering.

Hadronic collisions at high energies (LHC) lead to charged multiplicity distributions whose shapes are well fitted by a single NBD in fixed intervals of central (pseudo)rapidity η [ALICE,2010].

It is interesting to understand how NBD fits such a different reactions?

Let us consider NBD for normed topological cross sections

$$\frac{\sigma_n}{\sigma} = P(n) = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(1+\frac{k}{<n>})^{-n}(1+\frac{<n>}{k})^{-k}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{<n>}{<n>+k})^n(\frac{k}{k+<n>})^k,$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\frac{(\frac{k}{<n>})^k}{(1+\frac{k}{<n>})^{k+n}},$$
$$r = k > 0,\ p = \frac{<n>}{<n>+k}. \tag{303}$$

The generating function for NBD is

$$F(h) = (1+\frac{<n>}{k}(1-h))^{-k} = (1+\frac{<n>}{k})^{-k}(1-ah))^{-k},$$
$$a = p = \frac{<n>}{<n>+k}. \tag{304}$$

Indeed,

$$
\begin{aligned}
(1-ah))^{-k} &= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1} e^{-t(1-ah)} \\
&= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1} e^{-t}\sum_0^\infty \frac{(tah)^n}{n!} \\
&= \sum_0^\infty \frac{\Gamma(n+k)a^n}{\Gamma(k)n!}h^n, \\
P(n) &= (1+\frac{<n>}{k})^{-k}\frac{\Gamma(n+k)}{\Gamma(k)n!}(\frac{<n>}{<n>+k})^n \\
&= \frac{k^k\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)}(<n>+k)^{-(n+k)} <n>^n \\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)}
\end{aligned} \tag{305}
$$

Note that KNO characteristic function (288) coincides with the NBD generating function (304) when $t =< n > (h-1),\ c = k$.
The Bose-Einstein distribution is a special case of NBD with $k = 1$.

If k is negative, the NBD becomes a positive binomial distribution, narrower than Poisson (corresponding to negative correlations).
For negative (integer) values of $k = -N$, we have Binomial GF

$$F_{bd} = (1 + \frac{< n >}{N}(h-1))^N = (a + bh)^N,\ a = 1 - \frac{< n >}{N}, b = \frac{< n >}{N},$$
$$P_{bd}(n) = C_N^n(\frac{< n >}{N})^n(1 - \frac{< n >}{N})^{N-n} \tag{306}$$

(In a sense) we have a (quantum) spectrum for the parameter k, which contains any (positive) real values and (with finite number of states) the negative integer values, $(0 \leq n \leq N)$

From the generating function we have

$$< n^2 >= (\frac{hd}{dh})^2 F(h)|_{h=1} = \frac{k+1}{k} < n >^2 + < n >, \tag{307}$$

for dispersion we obtain

$$\begin{aligned} D &= \sqrt{< n^2 > - < n >^2} = \frac{1}{\sqrt{k}} < n > (1 + \frac{k}{< n >})^{1/2} \\ &= \frac{1}{\sqrt{k}} < n > + \frac{\sqrt{k}}{2} + O(1/ < n >), \end{aligned} \tag{308}$$

so the dispersion low for KNO and NBD distributions are the same, with $c = k$, for high values of the mean multiplicity.
The factorial moments of NBD,

$$F_m = (\frac{d}{dh})^m F(h)|_{h=1} = \frac{< n(n-1)...(n-m+1) >}{< n >^m} = \frac{\Gamma(m+k)}{\Gamma(m)k^m}, \tag{309}$$

and usual normalized moments of KNO (289) coincides.

Using fractal calculus (see e.g. [Makhaldiani,2003]),

$$D_{0,x}^{-\alpha} f = \frac{|x|^\alpha}{\Gamma(\alpha)} \int_0^1 |1-t|^{\alpha-1} f(xt)dt, = \frac{|x|^\alpha}{\Gamma(\alpha)} B(\alpha,\ \partial x) f(x)$$
$$= |x|^\alpha \frac{\Gamma(\partial x)}{\Gamma(\alpha+\partial x)} f(x),\ f(xt) = t^{x\frac{d}{dx}} f(x). \quad (310)$$

we can define factorial and cumulant moments for not only negative integer values of q, but for any complex indexes,

$$F_{-q} = <n>^q\ D_{0,x}^{-q} G_{NBD}(x)|_{x=0} = \frac{k^q \Gamma(k-q)}{\Gamma(k)},$$
$$K_{-q} = <n>^q\ D_{0,x}^{-q} \ln G_{NBD}(x)|_{x=0} = k^{q+1}\Gamma(-q),$$
$$H_{-q} = \frac{\Gamma(k+1)\Gamma(-q)}{\Gamma(k-q)} \quad (311)$$

Let us show that NBD is a discrete distribution corresponding to the KNO scaling,

$$\lim_{<n>\to\infty} <n> P_n|_{\frac{n}{<n>}=z} = \Psi(z) \tag{312}$$

Indeed, using the following asymptotic formula

$$\Gamma(x+1) = x^x e^{-x}\sqrt{2\pi x}(1 + \frac{1}{12x} + O(x^{-2})), \tag{313}$$

we find

$$<n> P_n = <n> \frac{(n+k-1)^{n+k-1}e^{-(n+k-1)}}{\Gamma(k)n^n e^{-n}} \frac{k^k}{n^k} <n> z^k e^{-k\frac{n+k}{<n>}}$$
$$= \frac{k^k}{\Gamma(k)} z^{k-1}e^{-kz} + O(1/<n>) \tag{314}$$

We can calculate also $1/<n>$ correction term to the KNO from the NBD. The answer is

$$\Psi = \frac{k^k}{\Gamma(k)} z^{k-1} e^{-kz}(1 + \frac{k^2}{2}(z - 2 + \frac{k-1}{kz})\frac{1}{<n>}) \tag{315}$$

This form coincides with the corrected KNO (287) for $c = k$ and $C_2^1 = 1$.

We have seen that KNO characteristic function (288) and NBD GF (304) have almost same form. This relation become in coincidence if

$$c = k,\ t = (h-1)\frac{< n >}{k} \tag{316}$$

Now the definition of the characteristic function (288) can be read as

$$\int_0^\infty e^{-<n>z(1-h)}\Psi(z)dz = (1 + \frac{< n >}{k}(1-h))^{-k} \tag{317}$$

which means that Poisson GF weighted by KNO distribution gives NBD GF. Because of this, the NBD is the gamma-Poisson (mixture) distribution.

Fore high values of $x_2 = k$ the NBD distribution reduces to the Poisson distribution

$$F(x_1, x_2, h) = (1 + \frac{x_1}{x_2}(1-h))^{-x_2} \Rightarrow e^{-x_1(1-h)} = e^{-<n>}e^{h<n>}$$
$$= \sum P(n)h^n,$$
$$P(n) = e^{-<n>}\frac{<n>^n}{n!} \qquad (318)$$

For the Poisson distribution

$$\frac{d^2F(h)}{dh^2}|_{h=1} =< n(n-1) >=< n >^2,$$
$$D^2 =< n^2 > - < n >^2 =< n > . \qquad (319)$$

In the case of NBD, we had the following dispersion low

$$D^2 = \frac{1}{k} < n >^2 + < n >, \qquad (320)$$

which coincides withe previous expression for high values of k.
Poisson GF belongs to the class of the infinitely divisible distributions,

$$F(h, < n >) = (F(h, < n > /k))^k \qquad (321)$$

For high values of $< n >$, the Poisson distribution reduces to the Gauss distribution

$$P(n) = e^{-<n>} \frac{< n >^n}{n!} \Rightarrow \frac{1}{\sqrt{2\pi < n >}} \exp(-\frac{(n- < n >)^2}{2 < n >}) \qquad (322)$$

For high values of k in the integral relation (317), in the KNO function dominates the value $z_c = 1$ and both sides of the relation reduce to the Poisson GF.

An useful property of the negative binomial distribution with parameters

$$< n >, k$$

is that it is (also) the distribution of a sum of k independent random variables drawn from a Bose-Einstein distribution[2] with mean $< n > /k$,

$$\begin{aligned}
&P_n = \frac{1}{< n > +1}(\frac{< n >}{< n > +1})^n \\
&= (e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2})e^{-\beta\hbar\omega(n+1/2)},\ T = \frac{\hbar\omega}{\ln\frac{< n >+1}{< n >}} \\
&\sum_{n\geq 0} P_n = 1,\ \sum nP_n =< n >= \frac{1}{e^{\beta\hbar\omega-1}},\ T \simeq \hbar\omega < n >,\ < n >\gg 1, \\
&P(x) = \sum_n x^n P_n = (1+ < n > (1-x))^{-1}.
\end{aligned} \tag{323}$$

[2] A Bose-Einstein, or geometrical, distribution is a thermal distribution for single state systems.

This is easily seen from the generating function in (304), remembering that the generating function of a sum of independent random variables is the product of their generating functions.
Indeed, for

$$n = n_1 + n_2 + ... + n_k, \tag{324}$$

with n_i independent of each other, the probability distribution of n is

$$P_n = \sum_{n_1,...,n_k} \delta(n - \sum n_i) p_{n_1} ... p_{n_k},$$
$$P(x) = \sum_n x^n P_n = p(x)^k \tag{325}$$

This has a consequence that an incoherent superposition of N emitters that have a negative binomial distribution with parameters $k, < n >$ produces a negative binomial distribution with parameters $Nk, N < n >$.

So, for the GF of NBD we have (N=2)

$$F(k,<n>)F(k,<n>)=F(2k,2<n>) \tag{326}$$

And more general formula (N=m) is

$$F(k,<n>)^m=F(mk,m<n>) \tag{327}$$

We can put this equation in the closed nonlocal form

$$Q_qF=F^q, \tag{328}$$

where

$$Q_q=q^D, \quad D=\frac{kd}{dk}+\frac{<n>d}{d<n>}=\frac{x_1d}{dx_1}+\frac{x_2d}{dx_2} \tag{329}$$

Note that temperature defined in (323) gives an estimation of the Glukvar temperature when it radiates hadrons. If we take $\hbar\omega=100MeV$, to $T\simeq T_c\simeq 200MeV$ corresponds $<n>\simeq 1.5$ If we take $\hbar\omega=10MeV$, to $T\simeq T_c\simeq 200MeV$ corresponds $<n>\simeq 20$.
We see that universality of NBD in hadron-production is similar to the universality of black body radiation.

p-adic string amplitudes can be obtained as tree amplitudes of the field theory with the following lagrangian and motion equation (see e.g. [Brekke, Freund, 1993])

$$L = \frac{1}{2}\Phi Q_p \Phi - \frac{1}{p+1}\Phi^{p+1},$$
$$Q_p\Phi = \Phi^p,\ Q_p = p^D \quad (330)$$
$$D = -\frac{1}{2}\triangle,\ \triangle = -\partial^2_{x_0} + \partial^2_{x_1} + ... + \partial^2_{x_{n-1}}, \quad (331)$$

Φ - is real scalar field on D-dimensional space-time with coordinates $x = (x_0, x_1, ..., x_{D-1})$. We have trivial, $\Phi = 0$ and $\Phi = 1$, and following nontrivial solutions of the equation (330)

$$\Phi(x_0, x_1, ..., x_{D-1}) = p^{\frac{D}{2(p-1)}} e^{\frac{1-p^{-1}}{2\ln p}(x_0^2 - x_1^2 - x_2^2 - ... - x_{D-1}^2)} \quad (332)$$

The equation (330) permits factorization of its solutions $\Phi(x) = \Phi(x_0)\Phi(x_1)...\Phi(x_{D-1})$, every factor of which fulfils one dimensional equation

$$p^{\varepsilon\partial_x^2}\Phi(x) = \Phi(x)^p, \ \varepsilon = \pm\frac{1}{2} \qquad (333)$$

The trivial solution of the equations are $\Phi = 0$ and $\Phi = 1$. For nontrivial solution of (333), we have

$$p^{\varepsilon\partial_x^2}\Phi(x) = e^{a\partial^2}\Phi(x) = \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{\infty} dy e^{-\frac{1}{4a}y^2 + y\partial}\Phi(x)$$
$$= \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{\infty} dy e^{-\frac{1}{4a}y^2}\Phi(x+y) = \Phi(x)^p, \ a = \varepsilon \ln p \qquad (334)$$

If we (de quantize) put, $p = q$, and take (classical) limit, $q \to 1$, the motion equation reduce to

$$\varepsilon\partial_x^2\Phi = \Phi \ln \Phi, \qquad (335)$$

with solution

$$\Phi(x) = e^{\frac{1}{2}} e^{\frac{x^2}{4\varepsilon}}. \qquad (336)$$

It is obvious that the anzac

$$\Phi = Ae^{bx^2}, \tag{337}$$

can pass the equation (334). Indeed, the solution is

$$\begin{aligned}
&\Phi(x) = p^{\frac{1}{2(p-1)}} e^{\frac{1-p^{-1}}{4\varepsilon \ln p} x^2}, \\
&\Phi(x_0, x_1, ..., x_{D-1}) = p^{\frac{D}{2(p-1)}} e^{\frac{1-p^{-1}}{2 \ln p} (x_0^2 - x_1^2 - x_2^2 - ... - x_{D-1}^2)}
\end{aligned} \tag{338}$$

Now, we can define the following class of motion equations

$$Q_q F = F^q, \tag{339}$$

where

$$Q_q = q^D, \quad D = D_1(x_1) + ... + D_l(x_l), \tag{340}$$

$D_k(x)$ is some (differential) operator depending on x. In the case of the NBD GF,

$$D_k(x) = \frac{xd}{dx}. \tag{341}$$

For this (Qlike) class of equations, we have factorization property

$$\begin{aligned} &F = F(x_1, ..., x_l) = F_1(x_1)...F_l(x_l), \\ &q^{D_k(x)} F_k(x) = c_k F_k(x)^q, 1 \leq k \leq l, \ c_1 c_2 ... c_l = 1. \end{aligned} \tag{342}$$

For NBD distribution we have corresponding multiplication(convolution)formulas

$$(P \star P)_n \equiv \sum_{m=0}^{n} P_m(k, <n>) P_{n-m}(k, <n>)$$
$$= P_n(2k, 2<n>) = Q_2 P_n(k, <n>), ... \quad (343)$$

So, we can say, that star-product on the distributions of NBD corresponds ordinary product for GF.
It will be nice to have similar things for string field theory(SFT) [Kaku,2000].
SFT motion equation is

$$Q\Phi = \Phi \star \Phi \quad (344)$$

For stringfield GF F we may have

$$QF = F^2. \quad (345)$$

By construction we know the solution of the nice equation (328) as GF of NBD, F. We obtain corresponding differential equations, if we consider $q = 1 + \varepsilon$, for small ε,

$$(D(D-1)...(D-m+1) - (lnF)^m)\Psi = 0,$$
$$(\frac{\Gamma(D+1)}{\Gamma(D+1-m)} - (\ln F)^m)\Psi = 0,$$
$$(D_m - \Phi^m)\Psi = 0, m = 1, 2, 3, ...$$
$$D_m = \frac{\Gamma(D+1)}{\Gamma(D+1-m)}, \Phi = \ln F, \quad (346)$$

with the solution $\Psi = F = \exp(\Phi)$. In the case of the NBD and p-adic string, we have correspondingly

$$D = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2};$$
$$D = -\frac{1}{2}\triangle, \ \triangle = -\partial^2_{x_0} + \partial^2_{x_1} + ... + \partial^2_{x_{n-1}}. \quad (347)$$

These equations have meaning not only for integer m.

For high mean multiplicities we have corresponding equations for KNO

$$\begin{aligned} Q_2\Psi(z) &= \Psi \star \Psi \equiv \int_0^z \Psi(t)\Psi(z-t)dt \\ &= z\int_0^1 dt t^{\delta_1}(1-t)^{\delta_2}\Psi(z_1)\Psi(z_2)|_{z_1=z_2=z} \\ &= z\frac{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}{\Gamma(\delta_1+\delta_2+2)}\Psi(z_1)\Psi(z_2)|_{z_1=z_2=z} \end{aligned} \quad (348)$$

Due to the explicit form of the operator D, these equations and corresponding solutions have the symmetry under the change of the variables

$$k \to ak,\ <n> \to b<n>\ . \quad (349)$$

When

$$a = \frac{<n>}{k},\ b = \frac{k}{<n>}, \quad (350)$$

we obtain the symmetry with respect to the transformations $k \leftrightarrow <n>, x_1 \leftrightarrow x_2$.

The Riemann zeta function $\zeta(s)$ is defined for complex $s = \sigma + it$ and $\sigma > 1$ by the expansion

$$\zeta(s) = \sum_{n \geq 1} n^{-s}, \ Res > 1. \tag{351}$$

All complex zeros, $s = \alpha + i\beta$, of $\zeta(\sigma + it)$ function lie in the critical stripe $0 < \sigma < 1$, symmetrically with respect to the real axe and critical line $\sigma = 1/2$. So it is enough to investigate zeros with $\alpha \leq 1/2$ and $\beta > 0$. These zeros are of three type, with small, intermediate and big ordinates.

The Riemann hypothesis [Titchmarsh,1986] states that the (non-trivial) complex zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$.
At the beginning of the XX century Polya and Hilbert made a conjecture that the imaginary part of the Riemann zeros could be the oscillation frequencies of a physical system (ζ - (mem)brane).
After the advent of Quantum Mechanics, the Polya-Hilbert conjecture was formulated as the existence of a self-adjoint operator whose spectrum contains the imaginary part of the Riemann zeros.
The Riemann hypothesis (RH) is a central problem in Pure Mathematics due to its connection with Number theory and other branches of Mathematics and Physics.

The functional equation is (see e.g. [Titchmarsh,1986])

$$\zeta(1-s)=\frac{2\Gamma(s)}{(2\pi)^s}\cos(\frac{\pi s}{2})\zeta(s) \tag{352}$$

From this equation we see the real (trivial) zeros of zeta function:

$$\zeta(-2n)=0,\ n=1,2,... \tag{353}$$

Also, at s=1, zeta has pole with reside 1.
From Field theory and statistical physics point of view, the functional equation (352) is duality relation, with self dual (or critical) line in the complex plane, at $s=1/2+i\beta$,

$$\zeta(\frac{1}{2}-i\beta)=\frac{2\Gamma(s)}{(2\pi)^s}\cos(\frac{\pi s}{2})\zeta(\frac{1}{2}+i\beta), \tag{354}$$

we see that complex zeros lie symmetrically with respect to the real axe.
On the critical line, (nontrivial) zeros of zeta corresponds to the infinite value of the free energy,

$$F=-T\ln\zeta. \tag{355}$$

At the point with $\beta = 14.134725...$ is located the first zero. In the interval $10 < \beta < 100$, zeta has 29 zeros. The first few million zeros have been computed and all lie on the critical line. It has been proved that uncountably many zeros lie on critical line.
The first relation of zeta function with prime numbers is given by the following formula,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \; Res > 1. \tag{356}$$

Another formula, which can be used on critical line, is

$$\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{n \geq 1} (-1)^{n+1} n^{-s}, \; Res > 0. \tag{357}$$

Let us consider the values $q = n, n = 1, 2, 3, ...$ and take sum of the corresponding equations (339), we find

$$\zeta(-D)F = \frac{F}{1-F} \tag{358}$$

In the case of the NBD we know the solutions of this equation.
Now we invent a Hamiltonian H with spectrum corresponding to the set of nontrivial zeros of the zeta function, in correspondence with Riemann hypothesis,

$$\begin{aligned} &-D_n = \frac{n}{2} + iH_n,\ H_n = i(\frac{n}{2} + D_n), \\ &D_n = x_1\partial_1 + x_2\partial_2 + ... + x_n\partial_n,\ H_n^+ = H_n = \sum_{m=1}^{n} H_1(x_m), \\ &H_1 = i(\frac{1}{2} + x\partial_x) = -\frac{1}{2}(x\hat{p} + \hat{p}x),\ \hat{p} = -i\partial_x \end{aligned} \tag{359}$$

The Hamiltonian $H = H_n$ is hermitian, its spectrum is real. The case $n = 1$ corresponds to the Riemann hypothesis.

The case $n = 2$, corresponds to NBD,

$$\zeta(1 + iH_2)F = \frac{F}{1-F},\ \zeta(1 + iH_2)|_F = \frac{1}{1-F},$$
$$F(x_1, x_2; h) = (1 + \frac{x_1}{x_2}(1-h))^{-x_2} \tag{360}$$

Let us scale $x_2 \to \lambda x_2$ and take $\lambda \to \infty$ in (360), we obtain

$$\zeta(\frac{1}{2} + iH_1(x))e^{-(1-h)x} = \frac{1}{e^{(1-h)x} - 1},$$
$$\frac{1}{\zeta(\frac{1}{2} + iH(x))}\frac{1}{e^{\varepsilon x} - 1} = e^{-\varepsilon x},$$
$$H(x) = i(\frac{1}{2} + x\partial_x) = -\frac{1}{2}(x\hat{p} + \hat{p}x),\ H^+ = H, \varepsilon = 1 - h. \tag{361}$$

Now we scale $x \to xy$, multiply the equation by y^{s-1} and integrate

$$\frac{1}{\zeta(\frac{1}{2}+iH(x))}\int_0^\infty dy\frac{y^{s-1}}{e^{\varepsilon xy}-1} = \int_0^\infty dy e^{-\varepsilon xy}y^{s-1} = \frac{1}{(\varepsilon x)^s}\Gamma(s),$$

$$\frac{1}{\zeta(\frac{1}{2}+iH(x))}\int_0^\infty dy\frac{y^{s-1}}{e^{\varepsilon xy}-1} = \frac{1}{\zeta(\frac{1}{2}+iH(x))}x^{-s}\varepsilon^{-s}\Gamma(s)\zeta(s), \quad (362)$$

so

$$\zeta(\frac{1}{2}+iH(x))x^{-s} = \zeta(s)x^{-s} \Rightarrow H(x)\psi_E = E\psi_E,$$
$$\psi_E = cx^{-s},\ s = \frac{1}{2}+iE, \quad (363)$$

we have correct way and can return to the previous step (361) and take the following transformation

$$\frac{1}{e^{\varepsilon xy}-1}=\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic}dEx^{-iE-1/2}\varphi(E,\varepsilon y),$$
$$\varphi(E,\varepsilon y)=\int_0^\infty dx\frac{x^{iE-\frac{1}{2}}}{e^{\varepsilon xy}-1}=\frac{\Gamma(\frac{1}{2}+iE)}{(\varepsilon y)^{iE+1/2}}\zeta(\frac{1}{2}+iE),$$
$$\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic}dEx^{-iE-1/2}\varphi(E,\varepsilon y)\frac{1}{\zeta(1/2+iE)}=e^{-\varepsilon xy} \qquad (364)$$

If we take the following formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}dt}{e^t - 1}, \tag{365}$$

which says that ζ function is the Mellin transformation, we can find

$$\Gamma(1 + iH_2)\frac{F}{1 - F} = \int_0^\infty \frac{dt/t}{e^t - 1} F^{1/t}, \tag{366}$$

or

$$\Gamma(1 + iH_2)\Phi = \int_0^\infty \frac{dt/t}{e^t - 1} (\frac{\Phi}{1 + \Phi})^{1/t},$$
$$\Phi = \frac{F}{1 - F} = \frac{1}{(1 + \frac{x_1}{x_2}(1 - h))^{x_2} - 1} \tag{367}$$

We can obtain also the following equation with argument of ζ_N on critical axis

$$\zeta_N(\frac{1}{2} + iH_1(x_2))F(x_1, x_2, h) = \sum_{n=1}^{N} \frac{1}{(1 + \frac{x_1}{nx_2}(1-h))^{nx_2}}$$

$$= \sum_{n=1}^{N} F(x_1, nx_2, h),$$

$$\zeta_N(\frac{1}{2} + iH_1(x_2))F(\lambda x_1, x_2, h) = \sum_{n=1}^{N} \frac{1}{(1 + \frac{\lambda x_1}{nx_2}(1-h))^{nx_2}}$$

$$= \sum_{n=1}^{N} F(\lambda x_1, nx_2, h) \simeq Ne^{-\lambda(1-h)x_1}, N \gg 1. \quad (368)$$

Let us calculate next therm in the $1/\lambda$ expansion in the (360)

$$F(x_1, \lambda x_2, h) = (1 + \frac{\varepsilon x_1}{\lambda x_2})^{-\lambda x_2} = e^{-\lambda x_2 \ln(1+\varepsilon \frac{x_1}{\lambda x_2})}$$
$$= e^{-\varepsilon x_1} e^{\frac{(\varepsilon x_1)^2}{2\lambda x_2} + O(\lambda^{-2})} = e^{-\varepsilon x_1}(1 + \frac{(\varepsilon x_1)^2}{2\lambda x_2} + O(\lambda^{-2})),$$
$$(F^{-1} - 1)^{-1} = (e^{\lambda x_2 \ln(1+\varepsilon \frac{x_1}{\lambda x_2})} - 1)^{-1}$$
$$= \frac{1}{e^{\varepsilon x_1} - 1}(1 + \frac{e^{\varepsilon x_1}}{e^{\varepsilon x_1} - 1} \frac{(\varepsilon x_1)^2}{2\lambda x_2} + O(\lambda^{-2})) \quad (369)$$

The zero order term, λ^0 we already considered. The next, λ^{-1} order term gives the following relations

$$\zeta(-\delta_1 - \delta_2)\frac{x_1^2}{x_2} e^{-\varepsilon x_1} = \frac{1}{x_2}\zeta(1 - \delta_1)x_1^2 e^{-\varepsilon x_1} = \frac{x_1^2 e^{\varepsilon x_1}}{x_2(e^{\varepsilon x_1} - 1)^2},$$
$$\zeta(1 - \delta)x^2 e^{-\varepsilon x} = \frac{x^2 e^{\varepsilon x}}{(e^{\varepsilon x} - 1)^2} = x^2 e^{-\varepsilon x} + O(e^{-2\varepsilon x})$$
$$\zeta(1 - \delta)\Psi = E\Psi + O(e^{-2\varepsilon x}), \Psi = x^2 e^{-\varepsilon x}, E = 1. \quad (370)$$

There have been a number of approaches to understanding the Riemann hypothesis based on physics (for a comprehensive list see [Watkins]) According to the idea of Berry and Keating, [Berry,Keating,1997] the real solutions E_n of

$$\zeta(\frac{1}{2} + iE_n) = 0, \tag{371}$$

are energy levels, eigenvalues of a quantum Hermitian operator (the Riemann operator) associated with the one-dimensional classical hyperbolic Hamiltonian

$$H_c = xp, \tag{372}$$

where x and p are the conjugate coordinate and momentum.

They suggest a quantization condition generating Riemann zeros. This Hamiltonian breaks time-reversal invariance since $(x, p) \to (x, -p) \Rightarrow H \to -H$. The classical Hamiltonian $H = xp$ of linear dilation, i.e. multiplication in x and contraction in p, gives the Hamiltonian equations:

$$\begin{aligned} \dot{x} &= x, \\ \dot{p} &= -p, \end{aligned} \tag{373}$$

with the solution

$$\begin{aligned} x(t) &= x_0 e^t, \\ p(t) &= p_0 e^{-t} \end{aligned} \tag{374}$$

for any nonzero $E = x_0 p_0 = x(t)p(t)$ is hyperbola in phase space.

The system is quantized by considering the dilation operator in the x space

$$H = \frac{1}{2}(xp + px) = -i\hbar(\frac{1}{2} + x\partial_x), \quad (375)$$

which is the simplest formally Hermitian operator corresponding to the classical Hamiltonian. The eigenvalue equation

$$H\psi_E = E\psi_E, \quad (376)$$

is satisfied by the eigenfunctions

$$\psi_E(x) = cx^{-\frac{1}{2}+\frac{i}{\hbar}E}, \quad (377)$$

where the complex constant c is arbitrary, since the solutions are not square-integrable. To the normalization

$$\int_0^\infty dx\psi_E(x)^*\psi_{E'}(x) = \delta(E - E'), \quad (378)$$

corresponds $c = 1/\sqrt{2\pi}$.

We have seen that

$$\zeta(\frac{1}{2}+iH)e^{-\varepsilon x} = \frac{1}{e^{\varepsilon x}-1},$$
$$H = -i(\frac{1}{2}+x\partial_x) = x^{1/2}px^{1/2}, p = -i\partial_x, \quad (379)$$

than

$$e^{-\varepsilon x} = \int dE x^{-1/2+iE}\varphi(E,\varepsilon), \varphi(E,\varepsilon) = \frac{1}{2\pi}\int_0^\infty dx x^{-1/2-iE}e^{-\varepsilon x}$$
$$= \frac{\varepsilon^{-1/2+iE}}{2\pi}\Gamma(1/2+iE);$$
$$\zeta(\frac{1}{2}+iE)\varphi(E,\varepsilon) = \frac{1}{2\pi}\int_0^\infty dx \frac{x^{-1/2-iE}}{e^{\varepsilon x}-1}$$
$$= \frac{\varepsilon^{-1/2+iE}}{2\pi}\Gamma(1/2+iE)\zeta(\frac{1}{2}+iE). \quad (380)$$

From the equation (361) we have

$$\zeta(\frac{1}{2}+iH_1(x))e^{-\varepsilon x}=\frac{1}{e^{\varepsilon x}-1},\ H_1=i(\frac{1}{2}+x\partial_x),$$
$$\zeta(-x\partial_x)(1-\varepsilon x+\frac{(\varepsilon x)^2}{2}+...)=\frac{1}{\varepsilon x}(1-(\frac{\varepsilon x}{2}+\frac{(\varepsilon x)^2}{6}+...)+$$
$$+(\frac{\varepsilon x}{2}+...)^2+...), \tag{381}$$

so

$$\zeta(0)=-\frac{1}{2},\ \zeta(-1)=-\frac{1}{12},... \tag{382}$$

Note that, a little calculation shows that, the $(\varepsilon x)^2$ terms cancels on the r.h.s, in accordance with $\zeta(-2)=0$.

More curious question concerns with the term $1/\varepsilon x$ on the r.h.s. To it corresponds the term with actual infinitesimal coefficient on the l.h.s.

$$\frac{1}{\zeta(1)}\frac{1}{\varepsilon x}, \tag{383}$$

in the spirit of the nonstandard analysis (see, e.g. [Davis,1977]), we can imagine that such a terms always present but on the r.h.s we may not note them.

For other values of zeta function we will use the following expansion

$$\frac{1}{e^x-1} = \frac{1}{x+\frac{x^2}{2}+\frac{x^3}{3!}+...} = \frac{1}{x} - \frac{1}{2} + \sum_{k\geq 1} \frac{B_{2k}x^{2k-1}}{(2k)!},$$
$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}, ... \tag{384}$$

and obtain

$$\zeta(1-2n) = -\frac{B_{2n}}{2n},\ n \geq 1. \tag{385}$$

Let us imagine space-time development of the the multiparticle process and try to describe it by some (phenomenological) dynamical equation. We start to find the equation for the Poisson distribution and than naturally extend them for the NBD case.
Let us define an integer valued variable $n(t)$ as a number of events (produced particles) at the time $t,\ n(0)=0$. The probability of event $n(t),\ P(t,n)$, is defined from the following motion equation

$$\begin{array}{l} P_t \equiv \dfrac{\partial P(t,n)}{\partial t} = r(P(t,n-1) - P(t,n)),\ n \geq 1 \\ P_t(t,0)) = -rP(t,0), \\ P(t,n) = 0,\ n < 0, \end{array} \tag{386}$$

so

$$\begin{array}{l} P(t,0) \equiv P_0(t) = e^{-rt}, \\ P(t,n) = Q(t,n)P_0(t), \\ Q_t(t,n) = rQ(t,n-1),\ Q(t,0) = 1. \end{array} \tag{387}$$

To solve the equation for Q, we invent its generating function

$$F(t,h) = \sum_{n\geq 0} h^n Q(t,n), \tag{388}$$

and solve corresponding equation

$$F_t = rhF,\ F(t,h) = e^{rth} = \sum h^n \frac{(rt)^n}{n!},\ Q(t,n) = \frac{(rt)^n}{n!}, \tag{389}$$

so

$$P(t,n) = e^{-rt}\frac{(rt)^n}{n!} \tag{390}$$

is the Poisson distribution.
If we compare this distribution with (322), we identify $< n >= rt$, as if we have a free particle motion with velocity r and the distance is the mean multiplicity. This way we have a connection between n-dimension of the multiplicity and the usual dimension of trajectory.

As the equation gives right solution, its generalization may give more general distribution, so we will generalize the equation (386). For this, we put the equation in the closed form

$$P_t(t,n) = r(e^{-\partial_n} - 1)P(t,n) = \sum_{k\geq 1} D_k \partial^k P(t,n), \; D_k = (-1)^k \frac{r}{k!}, \tag{391}$$

where the $D_k, \; k \geq 1$, are generalized diffusion coefficients.
For other values of the coefficients, we will have other distributions.

For mean square deviation of the trajectory we have

$$< (x-\bar{x})^2 >=< x^2 > - < x >^2 \equiv D(x)^2 \sim t^{2/d_f}, \tag{392}$$

where d_f is fractal dimension. For smooth classical trajectory of particles we have $d_f = 1$; for free stochastic, Brownian, trajectory, all diffusion coefficients are zero but D_2, we have $d_f = 2$. In the case of Poisson process we have,

$$D(n)^2 =< n^2 > - < n >^2 \sim t,\ d_f = 2. \tag{393}$$

In the case of the NBD and KNO distributions

$$D(n)^2 \sim t^2,\ d_f = 1. \tag{394}$$

As we have seen, rasing k, KNO reduce to the Poisson, so we have a dimensional (phase) transition from the phase with dimension 1 to the phase with dimension 2. It is interesting, if somehow this phase transition is connected to the other phase transitions in strong interaction processes.
For the Poisson distribution GF is solution of the following equation,

$$\dot{F} = -r(1-h)F, \tag{395}$$

For the NBD corresponding equation is

$$\dot{F} = \frac{-r(1-h)}{1+\frac{rt}{k}(1-h)}F = -R(t)F,\ R(t) = \frac{r(1-h)}{1+\frac{rt}{k}(1-h)}. \tag{396}$$

If we change the time variable as $t = T^{d_f}$, we reduce the dispersion low from general fractal to the NBD like case. Corresponding transformation for the evolution equation is

$$F_T = -d_f T^{d_f - 1} R(T^{d_F})F, \tag{397}$$

we ask that this equation coincides with NBD motion equation, and define rate function $R(T)$

$$d_f T^{d_f - 1} R(T^{d_F}) = \frac{r(1-h)}{1+\frac{rT}{k}(1-h)}, \tag{398}$$

now the following equation defines a production processes with fractal dimension d_F

$$F_t = -R(t)F,\ R(t) = \frac{r(1-h)}{d_F t^{\frac{d_F - 1}{d_F}}(1+\frac{rt^{1/d_F}}{k}(1-h))} \tag{399}$$

Now we would like to consider a model of multiparticle production based on the d-dimensional sphere, and (try to) motivate the values of the NBD parameter k. The volum of the d-dimensional sphere with radius r, in units of hadron size r_h is

$$v(d,r) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}(\frac{r}{r_h})^d \tag{400}$$

Note that,

$$v(0,r) = 1,\ v(1,r) = 2\frac{r}{r_h},$$
$$v(-1,r) = \frac{1}{\pi}\frac{r_h}{r} \tag{401}$$

If we identify this dimensionless quantity with corresponding coulomb energy formula,

$$\frac{1}{\pi} = \frac{e^2}{4\pi}, \tag{402}$$

we find $e = \pm 2$.

For less then -1 even integer values of d, and $r \neq 0,\ v = 0$. For negative odd integer $d = -2n + 1$

$$v(-2n+1, r) = \frac{\pi^{-n+1/2}}{\Gamma(-n+3/2)}(\frac{r_h}{r})^{2n-1},\ n \geq 1, \tag{403}$$

$$v(-3, r) = -\frac{1}{2\pi^2}(\frac{r_h}{r})^3,\ v(-5, r) = \frac{3}{4\pi^3}(\frac{r_h}{r})^5 \tag{404}$$

Note that,

$$v(2, r)v(3, r)v(-5, r) = \frac{1}{\pi},\ v(1, r)v(2, r)v(-3, r) = -\frac{1}{\pi} \tag{405}$$

We postulate that after collision,it appear intermediate state with almost spherical form and constant energy density. Than the radius of the sphere rise, dimension decrease, volume remains constant. At the last moment of the expansion, when the crossection of the one dimensional sphere - string become of order of hadron size, hadronic string divide in k independent sectors which start to radiate hadrons with geometric (Boze-Einstein) distribution, so all of the string final state radiate according to the NBD distribution.

So, from the volume of the hadronic string,

$$v = \pi(\frac{r}{r_h})^2 \frac{l}{r_h} = \pi k, \tag{406}$$

we find the NBD parameter k,

$$k = \frac{\pi^{d/2-1}}{\Gamma(d/2+1)}(\frac{r}{r_h})^d \tag{407}$$

Knowing, from experimental date, the parameter k, we can restrict the region of the values of the parameters d and r of the primordial sphere (PS),

$$r(d) = (\frac{\Gamma(d/2+1)}{\pi^{d/2-1}}k)^{1/d} r_h,$$
$$r(3) = (\frac{3}{4}k)^{1/3} r_h,\ r(2) = k^{1/2} r_h,\ r(1) = \frac{\pi}{2} k r_h \tag{408}$$

If the value of $r(d)$ will be a few r_h, the matter in the PS will be in the hadronic phase. If the value of r will be of order $10 r_h$, we can speak about deconfined, quark-gluon, Glukvar, phase. From the formula (408), we see, that to have for the r, the value of order $10 r_h$, in $d = 3$ dimension, we need the value for k of order 1000, which is not realistic.

So in our model, we need to consider the lower than one, fractal, dimensions. It is consistent with the following intuitive picture. Confined matter have point-like geometry, withe dimension zero. Primordial sphere of Glukvar have nonzero fractal dimension, which is less than one,

$$k = 3,\ r(0.7395)/r_h = 10.00,$$
$$k = 4,\ r(0.8384)/r_h = 10.00 \qquad (409)$$

From the experimental data we find the parameter k of the NBD as a function of energy, $k = k(s)$. Then, by our spherical model, we construct fractal dimension of the Glukvar as a function of $k(s)$.
If we suppose that radius of the primordial sphere r is of order (or less) of r_h. Than we will have higher dimensional PS, e.g.

d	r/r_h	k
3	1.3104	3.0002
4	1.1756	3.0003
6	1.1053	2.9994
8	1.1517	3.9990

With extra dimensions gravitation interactions may become strong at the LHC energies,

$$V(r) = \frac{m_1 m_2}{m^{2+d}} \frac{1}{r^{1+d}} \tag{410}$$

If the extra dimensions are compactified with(in) size R, at $r >> R$,

$$V(r) \simeq \frac{m_1 m_2}{m^2 (mR)^d} \frac{1}{r} = \frac{m_1 m_2}{M_{Pl}^2} \frac{1}{r}, \tag{411}$$

where (4-dimensional) Planck mass is given by

$$M_{Pl}^2 = m^{2+d} R^d, \tag{412}$$

so the scale of extra dimensions is given as

$$R = \frac{1}{m} (\frac{M_{Pl}}{m})^{\frac{2}{d}} \tag{413}$$

If we take $m = 1TeV, (GeV^{-1} = 0.2fm)$

$$R(d) = 2 \cdot 10^{-17} \cdot (\frac{M_{Pl}}{1TeV})^{\frac{2}{d}} \cdot cm,$$
$$R(1) = 2 \cdot 10^{15} cm,$$
$$R(2) = 0.2\ cm\ !$$
$$R(3) = 10^{-7} cm\ !$$
$$R(4) = 2 \cdot 10^{-9} cm,$$
$$R(6) \sim 10^{-11} cm \qquad (414)$$

Note that lab measurements of $G_N (= 1/M_{Pl}^2, M_{Pl} = 1.2 \cdot 10^{19} GeV)$ have been made only on scales of about 1 cm to 1 m; 1 astronomical unit(AU) (mean distance between Sun and Earth) is $1.5 \cdot 10^{13} cm$; the scale of the periodic structure of the Universe, $L = 128 Mps \simeq 4 \cdot 10^{26} cm$. It is curious which (small) value of the extra dimension corresponds to L?

$$\begin{aligned} d &= 2\frac{\ln \frac{M_{Pl}}{m}}{\ln(mL)} = 0.74,\ m = 1TeV, \\ &= 0.81,\ m = 100GeV, \\ &= 0.07,\ m = 10^{17} GeV. \end{aligned} \qquad (415)$$

Motion equations of physics (applied mathematics in general) connect different observable quantities and reduce the number of independently measurable quantities. More fundamental equation contains less number of independent quantities. When (before) we solve the equations, we invent dimensionless invariant variables, than one solution can describe all of the class of phenomena.
In the z - Scaling (zS) approach to the inclusive multiparticle distributions (MPD) (see, e.g. [Tokarev, Zborovsky, 2007a]), different inclusive distributions depending on the variables $x_1, ...x_n$, are described by universal function $\Psi(z)$ of fractal variable z,

$$z = x_1^{-\alpha_1} ... x_n^{-\alpha_n}. \tag{416}$$

It is interesting to find a dynamical system which generates this distributions and describes corresponding MPD.

We can find a good function if we know its derivative. Let us consider the following RD like equation

$$z\frac{d}{dz}\Psi = V(\Psi),$$
$$\int_{\Psi(z_0)}^{\Psi(z)} \frac{dx}{V(x)} = \ln\frac{z}{z_0} \qquad (417)$$

In $x-$representation,

$$\ln z = -\sum_{k=1}^{n} \alpha_k \ln x_k,\ \ \delta_z = z\frac{d}{dz} = -\sum_k \frac{\delta_k}{n_h\alpha_k},$$
$$\sum_{k=1}^{n} \frac{x_k}{n_h\alpha_k}\frac{\partial}{\partial x_k}\Psi(x_1,...,x_n) + V(\Psi) = 0, \qquad (418)$$

e.g.

$$z = \delta_z z = -\sum_{k=1}^{n} \frac{x_k}{n_h\alpha_k}\frac{\partial}{\partial x_k} x_1^{-\alpha_1}...x_n^{-\alpha_n} = z,\ \ n_h = n. \qquad (419)$$

In the case of NBD GF (328), we have

$$n = 2,\ x_1 = k,\ x_2 =< n >,\ \alpha_1 = \alpha_2 = 1,\ n_h = 1,$$
$$\Psi = F,\ V(\Psi) = -\Psi \ln \Psi. \quad (420)$$

In the case of the $z-$scaling, [Tokarev, Zborovsky, 2007a],

$$n = 4,\ x_3 = y_a,\ x_4 = y_b,$$
$$\alpha_1 = \delta_1,\ \alpha_2 = \delta_2,\ \alpha_3 = \varepsilon_a,\ \alpha_4 = \varepsilon_b, n_h = 4, \quad (421)$$

for infinite resolution, $\alpha_n = 1,\ n = 1, 2, 3, 4$. In z variable the equation for Ψ has universal form. In the case of $n = 2, \alpha_1 = \alpha_2 = 1,\ n_h = 1$, we find that $V(\Psi) = -\Psi \ln \Psi$, so if this form is applicable also in the case of n=4,

$$z\frac{d}{dz}\Psi(z) = -\Psi \ln \Psi,$$
$$\Psi(z) = e^{c/z} = (\Psi(z_0)^{z_0})^{\frac{1}{z}} = \Psi(z_0)^{\frac{z_0}{z}},$$
$$c = z_0 \ln \Psi(z_0) < 0,\ z \in (0, \infty),\ \Psi(z) \in (0, 1). \quad (422)$$

Note that the fundamental equation is invariant with respect to the scale transformation $z \to \lambda z$, but the solution is not, the scale transformation transforms one solution into another solution. This is an example of the spontaneous breaking of the (scale) symmetry by the states of the system.

As a dimensionless physical quantity $\Psi(z)$ may depend only on the running coupling constant $g(\tau),\ \tau = \ln z/z_0$

$$z\frac{d}{dz}\Psi = \dot{\Psi} = \frac{d\Psi}{dg}\beta(g) = U(g) = U(f^{-1}(\Psi)) = V(\Psi),$$
$$\Psi(\tau) = f(g(\tau)),\ g = f^{-1}(\Psi(\tau)) \quad (423)$$

According to the paper [Tokarev, Zborovsky, 2007a], for high values of z, $\Psi(z) \sim z^{-\beta}$; for small z, $\Psi(z) \sim$ const.
So, for high z,

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = -\beta\Psi(z); \tag{424}$$

for smaller values of z, $\Psi(z)$ rise and we expect nonlinear terms in $V(\Psi)$,

$$V(\Psi) = -\beta\Psi + \gamma\Psi^2. \tag{425}$$

With this function, we can solve the equation for Ψ(see appendix) and find

$$\Psi(z) = \frac{1}{\frac{\gamma}{\beta} + cz^{\beta}}. \tag{426}$$

RD equation of the z-Scaling,

$$z\frac{d}{dz}\Psi(z) = V(\Psi),\ V(\Psi) = V_1\Psi + V_2\Psi^2 + ... + V_n\Psi^n + ... \qquad (427)$$

can be reparametrized,

$$\begin{aligned}
&\Psi(z) = f(\psi(z)) = \psi(z) + f_2\psi^2 + ... + f_n\psi^n + ...\\
&z\frac{d}{dz}\psi(z) = v(z) = v_1\psi(z) + v_2\psi^2 + ... + v_n\psi^n + ...\\
&(v_1\psi(z) + v_2\psi^2 + ... + v_n\psi^n + ...)(1 + 2f_2\psi + ... + nf_n\psi^{n-1} + ...)\\
&= V_1(\psi + f_2\psi^2 + ... + f_n\psi^n + ...)\\
&+V_2(\psi^2 + 2f_2\psi^3 + ...) + ... + V_n(\psi^n + nf_2\psi^{n+1} + ...) + ...\\
&= V_1\psi + (V_2 + V_1f_2)\psi^2 + (V_3 + 2V_2f_2 + V_1f_3)\psi^3 +\\
&... + (V_n + (n-1)V_{n-1}f_2 + ... + V_1f_n)\psi^n + ...\\
&v_1 = V_1,\\
&v_2 = V_2 - f_2V_1,\\
&v_3 = V_3 + 2V_2f_2 + V_1f_3 - 2f_2v_2 - 3f_3v_1 = V_3 + 2(f_2^2 - f_3)V_1, ...\\
&v_n = V_n + (n-1)V_{n-1}f_2 + ... + V_1f_n - 2f_2v_{n-1} - ... - nf_nv_1, \qquad (428)
\end{aligned}$$

so, by reparametrization, we can change any coefficient of potential V but V_1.

We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{V_2}{V_1},\ f_3 = \frac{V_3}{2V_1} + f_2^2 = \frac{V_3}{2V_1} + (\frac{V_2}{V_1}^2),\ ...$$
$$f_n = \frac{V_n + (n-1)V_{n-1}f_2 + ... + 2V_2 f_{n-1}}{(n-1)V_1}, ... \tag{429}$$

We will consider the case when only one of higher coefficient is nonzero and give explicit form of the solution Ψ.

More general solution for Ψ

Let us consider more general potential V

$$z\frac{d}{dz}\Psi = V(\Psi) = -\beta\Psi(z) + \gamma\Psi(z)^{1+n} \quad (430)$$

Corresponding solution for Ψ is

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz^{n\beta})^{\frac{1}{n}}} \quad (431)$$

More general solution contains three parameters and may better describe the data of inclusive distributions.

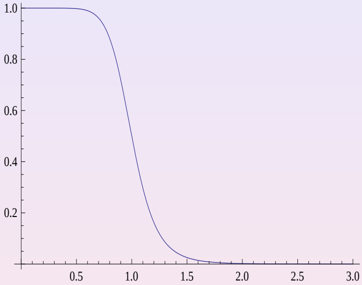

Figure: z-scaling distribution (431), $\Psi(z, 9, 9, 1, 1)$

In the case of $n = 1$ we reproduce the previous solution.

Another "natural" case is $n = 1/\beta$,

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz)^{\beta}} \tag{432}$$

In this case, we can absorb (interpret) the combined parameter by shift and scaling

$$z \to \frac{1}{c}(z - \frac{\gamma}{\beta}) \tag{433}$$

Another interesting point of view is to predict the value of β

$$\beta = \frac{1}{n} = 0.5; 0.33; 0.25; 0.2; ..., \ \ n = 2, 3, 4, 5, ... \tag{434}$$

For experimentally suggested value $\beta \simeq 9, n = 0.11$

In the case of $n=-\varepsilon,\ \beta=\gamma=1/\varepsilon,\ c=\varepsilon k$, we will have

$$V(\Psi)=-\Psi\ln\Psi,\ \Psi(z)=e^{\frac{k}{z}} \tag{435}$$

This form of $\Psi-$function interpolates between asymptotic values of Ψ and predicts its behavior in the intermediate region.
The three parameter function is restricted by the normalization condition

$$\int_0^\infty \Psi(z)dz=1,$$
$$B(\frac{\beta-1}{\beta n},\frac{1}{\beta n})=(\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}}\frac{\beta n}{c^{\beta n}}, \tag{436}$$

so remains only two free parameter. When $\beta n=1$, we have

$$c=(\beta-1)(\frac{\beta}{\gamma})^{\beta-1} \tag{437}$$

If $\beta n=1$ and $\beta=\gamma$, than $c=\beta-1$.
In general

$$c^{\beta n}=(\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}}\frac{\beta n}{B(\frac{\beta-1}{\beta n},\frac{1}{\beta n})} \tag{438}$$

RD equation of the z-scaling (430), after substitution,

$$\Psi(z) = (\varphi(z))^{\frac{1}{n}}, \tag{439}$$

reduce to the $n = 1$ case with scaled parameters,

$$\dot{\varphi} = -\beta n\varphi + \gamma n\varphi^2, \tag{440}$$

this substitution could be motivated also by the structure of the solution (431),

$$\Psi(z, \beta, \gamma, n, c) = \Psi(z, \beta n, \gamma n, 1, c)^{\frac{1}{n}} = \Psi(z, \beta, \gamma, \beta n, c)^{\beta}. \tag{441}$$

General RD equation takes form

$$\dot{\varphi} = nv_1\varphi + nv_2\varphi^{1+\frac{1}{n}} + nv_3\varphi^{1+\frac{2}{n}} + ... + nv_n\varphi^2 + nv_{n+1}\varphi^{2+\frac{1}{n}} + ... \tag{442}$$

The dimension of the space(-time) is the model dependent concept. E.g. for the fundamental bosonic string model (in flat space-time) the dimension is 26; for superstring model the dimension is 10 [Kaku,2000].
Let us imagine that we have some action-functional formulation with the fundamental motion equation

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = V(\Psi) = -\beta\Psi + \gamma\Psi^{1+n}. \tag{443}$$

Than, the corresponding Lagrangian contains the following mass and interaction parts

$$-\frac{\beta}{2}\Psi^2 + \frac{\gamma}{2+n}\Psi^{2+n} \tag{444}$$

The action gives renormalizable (effective quantum field theory) model when

$$d+2 = \frac{2N}{N-2} = \frac{2(2+n)}{n} = 2 + \frac{4}{n} = 2 + 4\beta, \tag{445}$$

so, measuring the parameter β inside hadronic and nuclear matters, we find corresponding (fractal) dimension.

From fundamental equation we obtain

$$(z\frac{d}{dz})^2\Psi \equiv \ddot{\Psi} = V'(\Psi)V(\Psi) = \frac{1}{2}(V^2)'$$
$$= \beta^2\Psi - \beta\gamma(n+2)\Psi^{n+1} + \gamma^2(n+1)\Psi^{2n+1} \quad (446)$$

Corresponding action Lagrangian is

$$L = \frac{1}{2}\dot{\Psi}^2 + U(\Psi),\ U = \frac{1}{2}V^2 = \frac{1}{2}\Psi^2(\beta - \gamma\Psi^n)^2$$
$$= \frac{\beta^2}{2}\Psi^2 - \beta\gamma\Psi^{2+n} + \frac{\gamma^2}{2}\Psi^{2+2n} \quad (447)$$

This potential, $-U$, has two maximums, when $V = 0$, and minimum, when $V' = 0$, at $\Psi = 0$ and $\Psi = (\beta/\gamma)^{1/n}$, and $\Psi = (\beta/(n+1)\gamma)^{1/n}$, correspondingly.

We define time-space-scale field $\Psi(t,x,\eta)$, where $\eta = \ln z-$ is scale coordinate variable, with corresponding action functional

$$A = \int dtd^d x d\eta (\frac{1}{2} g^{ab}\partial_a \Psi \partial_b \Psi + U(\Psi)) \qquad (448)$$

The renormalization constraint for this action is

$$N = 2 + 2n = \frac{2(2+d)}{2+d-2} = 2 + \frac{4}{d},\ dn = 2,\ d = 2/n = 2\beta. \qquad (449)$$

So we have two models for spase-time dimension, (445) and (449),

$$d_1 = 4\beta;\ d_2 = 2\beta \qquad (450)$$

The coordinate η characterise (multiparticle production) physical process at the (external) space-time point (x,t). The dimension of the space-time inside hadrons and nuclei, where multiparticle production takes place is

$$d + 1 = 1 + 2\beta \qquad (451)$$

Note that this formula reminds the dimension of the spin s state, $d_s = 2s + 1$. If we take $\beta(= s) = 0; 1/2; 1; 3/2; 2; ...$ We will have $d + 1 = 1; 2; 3; 4; 5; ...$

Note that as we invent Ψ as a real field, we ought to take another normalization

$$\int d^d x |\Psi|^2 = 1 \tag{452}$$

for the solutions of the motion equation. This case extra values of the parameter β is possible, $\beta > d/2$.

We can take a renormdynamic scheme were $\Psi(g)$ is running coupling constant. The variable z is a formation length and has dimension -1, RD equation for Ψ in φ^3_D model is

$$z\frac{d}{dz}\Psi = \frac{6-D}{2}\Psi + \gamma\Psi^2 \quad (453)$$

$$\beta = \frac{D-6}{2} \quad (454)$$

For high values of z, $\beta = 9$, so $D = 24$. This value of D corresponds to the physical (transverse) degrees of freedom of the relativistic string, to the dimension of the external space in which this relativistic string lives. This is also the number of the quark - lepton matter degrees of freedom, $3 \cdot 6 + 6$. So, in these high energy reactions we measured the dimension of the space-time and matter and find the values predicted by relativistic string and SM. For lower energies, in this model, D monotonically decrees until $D = 6$, than the model (may) change form on the $\varphi^4_D, \beta = D - 4$. So we have two scenarios of behavior. In one of them the dimension of the space-time inside hadrons has value 6 and higher. In another the dimension is 4 and higher.

Perturbative QCD indicates that we have a fixed point of RD in dimension slightly higher than 4, and ordinary to hadron phase transition corresponds to the dimensional phase transition from slightly lower than 4, in QED, to slightly higher than 4 dimension in QCD. In general scalar field model φ_D^n,

$$\beta = -d_g = \frac{nD}{2} - n - D. \tag{455}$$

For φ^3 model, $\beta = 9$ corresponds to $D = 24$. In tha case of the $O(N)-$sigma model

$$\beta = D - 2, \tag{456}$$

For the experimental value of $\beta = 9$, we have the dimension of the $M-$theory, $D = 11$!

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