Hidden symmetries, Killing tensors and quantum gravitational anomalies

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Outline

1. Symmetries and conserved quantities
2. Gauge covariant approach
3. Killing-Yano tensors
4. Killing-Maxwell system
5. Examples
6. Gravitational anomalies
7. Outlook

Symmetries and conserved quantities (1)

Let $(\mathcal{M}, \mathbf{g})$ be a n-dimensional manifold equipped with a (pseudo-)Riemmanian metric $\mathbf{g}$ and denote by

$$H = \frac{1}{2} g^{ij} p_i p_j \,,$$

the Hamilton function describing the motion in a curved space. In terms of the phase-space variables(x^i, p_i) the Poisson bracket of two observables P, Q is

$$\{P, Q\} = \frac{\partial P}{\partial x^i}\frac{\partial Q}{\partial p_i} - \frac{\partial P}{\partial p_i}\frac{\partial Q}{\partial x^i} \,.$$

Symmetries and conserved quantities (2)

A conserved quantity of motions expanded as a power series in momenta:

$$K = K_0 + \sum_{k=1}^{p} \frac{1}{k!} K^{i_1 \cdots i_k}(x) p_{i_1} \cdots p_{i_k} \,.$$

Vanishing Poisson bracket with the Hamiltonian, $\{K, H\} = 0$, implies

$$K^{(i_1 \cdots i_k ; i)} = 0 \,,$$

Such symmetric tensor $K_k^{i_1 \cdots i_k}$ is called a **Stäckel-Killing** (SK) tensor of rank k

Gauge covariant approach (1)

In the presence of an gauge field F_{ij} expressed (locally) in terms of the potential 1 -form A_μ

$$F = dA\,,$$

the Hamiltonian is

$$H = \frac{1}{2} g^{ij}(p_i - A_i)(p_j - A_j) + V(x)\,,$$

including a scalar potential V(x).

Gauge covariant approach (2)

Gauge covariant formulation [van Holten 2007]

Introduce the gauge invariant momenta

$$\Pi_i = p_i - A_i = \dot{x}_i \,.$$

Hamiltonian becomes

$$H = \frac{1}{2} g^{ij} \Pi_i \Pi_j + V(x) \,,$$

Covariant Poisson brackets

$$\{P, Q\} = \frac{\partial P}{\partial x^i} \frac{\partial Q}{\partial \Pi_i} - \frac{\partial P}{\partial \Pi_i} \frac{\partial Q}{\partial x^i} + F_{ij} \frac{\partial P}{\partial \Pi_i} \frac{\partial Q}{\partial \Pi_j} \,.$$

where $F_{ij} = A_{j;i} - A_{j;i}$ is the field strength.

Gauge covariant approach (3)

Fundamental Poisson brackets

$$\{x^i, x^j\} = 0\,,\ \{x^i, \Pi_j\} = \delta^i_j\,,\ \{\Pi_i, \Pi_j\} = F_{ij}\,,$$

Momenta Π_i are not canonical.
Hamilton's equations:

$$\dot{x}^i = \{x^i, H\} = g^{ij}\Pi_j\,,$$

$$\dot{\Pi}_i = \{\Pi_i, H\} = F_{ij}\dot{x}^j - V_{,i}\,.$$

Gauge covariant approach (4)

Conserved quantities of motion in terms of phase-space variables (x^i, Π_i)

$$K = K_0 + \sum_{n=1}^{p} \frac{1}{n!} K^{i_1 \cdots i_n}(x) \cdots \Pi_{i_1} \Pi_{i_n} \,,$$

Bracket

$$\{K, H\} = 0 \,.$$

vanishes for conserved quantities..

Gauge covariant approach (5)

Series of constraints:

$$K^i V_{,i} = 0\,,$$

$$K_0{}^{,i} + F_j{}^i K^j = K^{ij} V_{,j}\,.$$

$$K^{(i_1 \cdots i_l ; i_{l+1})} + F_j{}^{(i_{l+1}} K^{i_1 \cdots i_l) j} = \frac{1}{(l+1)} K^{i_1 \cdots i_{l+1} j} V_{,j}\,,$$

$$\textit{for} \;\; l = 1\,, \cdots (p-2)\,,$$

$$K^{(i_1 \cdots i_{p-1} ; i_p)} + F_j{}^{(i_p} K^{i_1 \cdots i_{p-1}) j} = 0\,,$$

$$K^{(i_1 \cdots i_p ; i_{p+1})} = 0\,.$$

Killing-Yano tensors (1)

A Killing-Yano (KY) tensor is a p -form $Y (p \leq n)$ which satisfies

$$\nabla_X Y = \frac{1}{p+1} X \lrcorner\, dY \,,$$

for any vector field X, where 'hook' operator $\lrcorner$ is dual to the wedge product.
In components,

$$Y_{i_1 \cdots i_{p-1}(i_p ; j)} = 0 \,.$$

SK and KY tensors could be related. Let $Y_{i_1 \cdots i_p}$ be a KY tensor, then the symmetric tensor field

$$K_{ij} = Y_{i i_2 \cdots i_p} Y_j^{\ \ i_2 \cdots ip} \,,$$

is a SK tensor

Killing-Yano tensors (2)

A conformal Killing-Yano (CKY) tensor is a p -form which satisfies

$$\nabla_X Y = \frac{1}{p+1} X \lrcorner \, dY - \frac{1}{n-p+1} X^\flat \wedge d^* Y \,,$$

where $X^\flat$ denotes the 1 -form dual with respect to the metric to the vector field X and d^* is the exterior co-derivative. Hodge dual maps the space of p-forms into the space of $(n-p)$-forms. Conventions:

$$* * Y = \epsilon_p Y \quad , \quad *^{-1} Y = \epsilon_p * Y \,,$$

with the number ϵ_p

$$\epsilon_p = (-1)^p *^{-1} \frac{detg}{|detg|} \,.$$

$$d^* Y = (-1)^p *^{-1} d * Y \,.$$

Killing-Yano tensors (3)

Conformal generalization of the SK tensors, namely a symmetric tensor $K_{i1\cdots ip} = K_{(i1\cdots ip)}$ is called a conformal Stackel-Killing (CSK) tensor if it obeys the equation

$$K_{(i_1\cdots i_p;j)} = g_{j(i_1}\tilde{K}_{i_2\cdots i_p)}\,,$$

where the tensor $\tilde{K}$ is determined by tracing the both sides of equation .

Similar relation between CKY and CSK tensors: If Y_{ij} is a CKY tensor

$$K_{ij} = Y_j{}^k Y_{kj}\,,$$

is a CSK tensor.

Killing-Yano tensors (4)

Remarks:

- CKY equation is invariant under Hodge duality.
- A CKY tensor is a KY tensor iff it is co-closed.
- Dual of a CKY tensor is a KY tensor iff it is closed.

Killing-Yano tensors (5)

An interesting construction involving CKY tensors Y_{ij} of rank 2 in 4 dimensions.
In this particular case:

$$Y_{i(j;k)} = -\frac{1}{3}\left(g_{jk}Y^{l}{}_{i;l} + g_{i(k}Y_{j)}{}^{l}{}_{;l}\right) ,$$

and let us denote

$$Y_k := Y^{l}{}_{k;l} .$$

This vector satisfies equation:

$$Y_{(i;j)} = \frac{3}{2}R_{l(i}Y_{j)}{}^{l} .$$

It is obvious that in a Ricci flat space ($R_{ij} = 0$) or in an Einstein space ($R_{ij} \sim g_{ij}$), Y_k is a Killing vector and we shall refer to it as the **primary Killing vector.**

Killing-Maxwell system (1)

Returning to the system of equations for the conserved quantities e should like to find the conditions of the electromagnetic tensor field F_{ij} to maintain the hidden symmetry of the system. To make things more specific, let us assume that the system admits a hidden symmetry encapsulated in a SK tensor of rank 2, K_{ij} associated with a KY tensor Y_{ij}. The sufficient condition of the electromagnetic field to preserve the hidden symmetry is

$$F_{k[i}{Y_{j]}}^k = 0\,.$$

where the indices in square bracket are to be antisymmetrized.

Killing-Maxwell system (2)

A concrete realization of this condition is presented by the Killing-Maxwell system [Carter 1997]. In Carter's construction a primary Killing vector is identified, modulo a rationalization factor, with the source current j^i of the electromagnetic field

$$F^{ij}{}_{;j} = 4\pi j^i \, .$$

Therefore the Killing-Maxwell system is defined assuming that the **electromagnetic field $\mathbf{F}_{ij}$ is a CKY tensor**. In addition, it is a closed 2-form and its Hodge dual

$$Y_{ij} = *F_{ij} \, ,$$

is a KY tensor.

Now let us consider that the hidden symmetry of the system to consider is associated with the KY tensor of the Killing-Maxwell system. It is quite simple to observe that $F_{ij}Y_k{}^j \sim F_{ij} * F_k{}^j$ is a symmetric matrix (in fact proportional with the unit matrix) and therefore above condition is fulfilled.

Examples (1)

Example I. Kerr space (1)

To exemplify the results for Killing-Maxwell system, let us consider the Kerr solution to the vacuum Einstein equations [Boyer-Lindquist coordinates (t, r, θ, ϕ)]

$$g = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2$$
$$+\frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - a\,dt]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2\,,$$

where

$$\Delta = r^2 + a^2 - 2\,m\,r\,,$$
$$\rho^2 = r^2 + a^2\cos^2\theta\,.$$

This metric describes a rotating black hole of mass m and angular momentum $J = am$.

Examples (2)

Example I. Kerr space (2)

Kerr space admits the SK tensor

$$K_{ij}dx^i dx^j = -\frac{\rho^2 a^2 \cos^2\theta}{\Delta} dr^2 + \frac{\Delta a^2 \cos^2\theta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2$$
$$+\frac{r^2\sin^2\theta}{\rho^2}[-a\,dt + (r^2 + a^2)\,d\phi]^2 + \rho^2 r^2\, d\theta^2,$$

in addition to the metric tensor g_{ij}. This tensor is associated with the KY tensor

$$Y = r\sin\theta\, d\theta \wedge [-a\,dt + (r^2 + a^2)\,d\phi]$$
$$+a\cos\theta\, dr \wedge (dt - a\sin^2\theta d\phi).$$

Examples (3)

Example I. Kerr space (3)

The dual tensor

$$*Y = a \sin\theta \cos\theta \, d\theta \wedge [-a\,dt + (r^2 + a^2)\,d\phi]$$
$$+r\,dr \wedge (-dt + a \sin^2\theta \, d\phi)$$

is a CKY tensor (electromagnetic field F_{ij} of KM system). Four-potential one-form is

$$A = \frac{1}{2}(a^2 \cos^2\theta - r^2)dt + \frac{1}{2}a\,(r^2 + a^2)\sin^2\theta \, d\phi.$$

Finally, the current is to be identified with the primary Killing vector

$$Y_k := Y^{kl}{}_{;k}\partial_l = 3\partial_t.$$

Examples (4)

Example II. Generalized Taub-NUT space (1)

The generalized Taub-NUT metric is

$$ds_4^2 = f(r)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) + g(r)(d\psi + \cos\theta d\phi)^2$$

where the curvilinear coordinates (r, θ, ϕ, ψ) are

$$x_1 = \sqrt{r}\cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}\,, \quad x_2 = \sqrt{r}\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}\,,$$
$$x_3 = \sqrt{r}\sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}\,, \quad x_4 = \sqrt{r}\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}\,.$$

Examples (5)

Example II. Generalized Taub-NUT space (2)

In Cartesian coordinates the metric is

$$ds_4^2 = 4r\,f(r)ds_0^2 + 4\left(\frac{g(r)}{r^2} - f(r)\right)(-x_2dx_1 + x_1dx_2 - x_4dx_3 + x_3dx_4)^2$$

where

$$ds_0^2 = \sum_1^4 (dx_j)^2$$

is the standard flat metric.

Examples (6)

Example II. Generalized Taub-NUT space (3)

The associated Hamiltonian in the Cartesian coordinates (x, y) of the cotangent bundle $T^\star(\mathbb{R}^4 - \{0\})$ is

$$H = \frac{1}{2}\left[\frac{1}{4rf(r)}\sum_1^4 y_j^2 + \frac{1}{4}\left(\frac{1}{g(r)} - \frac{1}{r^2 f(r)}\right)(-x_2y_1 + x_1y_2 - x_4y_3 + x_3y_4)^2\right] + V(r)$$

where a potential $V(r)$ was added for latter convenience.
The phase-space $T^\star(\mathbb{R}^4 - \{0\})$ is equipped with the standard symplectic form

$$d\Theta = \sum_1^4 dy_j \wedge dx_j\,, \quad \Theta = \sum_1^4 y_j \wedge dx_j\,.$$

Examples (7)

Example II. Generalized Taub-NUT space (4)

Let us consider the principal fiber bundle $\pi : \mathbb{R}^4 - \{0\} \rightarrow \mathbb{R}^3 - \{0\}$ with structure group $SO(2)$ lifted to a symplectic action on $T^\star(\mathbb{R}^4 - \{0\})$. The action $SO(2)$ is given by

$$(x, y) \rightarrow (T(t)x, T(t)y), \quad (x, y) \in (\mathbb{R}^4 - \{0\}) .$$

where

$$T(t) = \begin{pmatrix} R(t) & 0 \\ 0 & R(t) \end{pmatrix}, \quad R(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix} .$$

Let $\Psi : T^\star(\mathbb{R}^4 - \{0\}) \rightarrow \mathbb{R}$ be the moment map associated with the $SO(2)$ action

$$\Psi(x, y) = \frac{1}{2}(-x_2 y_1 + x_1 y_2 - x_4 y_3 + x_3 y_4) .$$

Examples (8)

Example II. Generalized Taub-NUT space (5)

Since ψ is a cyclic variable, the momentum

$$\mu = g(r)(\dot{\psi} + \cos\theta\dot{\phi}),$$

is a conserved quantity. The reduced phase-space P_μ is defined through

$$\pi_\mu : \Psi^{-1}(\mu) \to P_\mu := \Psi^{-1}(\mu)/SO(2),$$

which is diffeomorphic with $T^\star(\mathbb{R}^3 - \{0\}) \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$.

Examples (9)

Example II. Generalized Taub-NUT space (6)

The coordinates $(q_k, p_k) \in (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ are given by the Kunstaanheimo-Stiefel transformation

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ 0 \end{pmatrix} = \begin{pmatrix} x_3 & x_4 & x_1 & x_2 \\ -x_4 & x_3 & x_2 & -x_1 \\ x_1 & x_2 & -x_3 & -x_4 \\ -x_2 & x_1 & -x_4 & x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \psi/r \end{pmatrix} = \frac{1}{2r} \begin{pmatrix} x_3 & x_4 & x_1 & x_2 \\ -x_4 & x_3 & x_2 & -x_1 \\ x_1 & x_2 & -x_3 & -x_4 \\ -x_2 & x_1 & -x_4 & x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

Note that $\sum_1^3 q_k^2 = r^2$.

Examples (10)

Example II. Generalized Taub-NUT space (7)

The reduced symplectic form ω_μ is

$$\omega_\mu = \sum_{k=1}^{3} dp_k \wedge dq_k - \frac{\mu}{r^3}(q_1\, dq_2 \wedge dq_3 + q_2\, dq_3 \wedge dq_1 + q_3\, dq_1 \wedge dq_2)\,.$$

The ω_μ is the standard symplectic form on $T^\star(\mathbb{R}^3 - \{0\})$ plus the Dirac's monopole field.
The reduced Hamiltonian is determined by

$$H_\mu = \frac{1}{2f(r)} \sum_{k=1}^{3} p_k^2 + \frac{\mu^2}{2g(r)} + V(r)\,.$$

Examples (11)

Example II. Generalized Taub-NUT space (8)

Conserved quantities

- Momentum $p_\psi = \mu$ associated with the cycle variable ψ
- Angular momentum vector

$$\vec{J} = \vec{q} \times \vec{p} + \frac{\mu}{r}\vec{q}\,,$$

- In some cases the system admits additional constants of motion polynomial in momenta.

Examples (12)

Example II. Generalized Taub-NUT space (9)

Extended Taub-NUT space

$$f(r) = \frac{a + br}{r}\,, \quad g(r) = \frac{ar + br^2}{1 + cr + dr^2}\,, \quad V(r) = 0$$

with a, b, c, d real constants admits a Runge-Lenz type vector

$$\vec{A} = \vec{p} \times \vec{J} - (aE - \frac{1}{2}c\mu^2)\frac{\vec{q}}{r}\,,$$

where E is the conserved energy. If the constants a, b, c, d are subject to the constraints

$$c = \frac{2b}{a}\,, \quad d = \frac{b^2}{a^2}\,,$$

the extended metric coincides, up to a constant factor, with the original Taub-NUT metric.

Examples (13)

Example II. Generalized Taub-NUT space (10)

For

$$f(r) = 1\,, \quad g(r) = r^2\,, \quad V(r) = -\frac{\kappa}{r}$$

we recognize the MIC-Kepler problem with the Runge-Lenz type conserved vector

$$\vec{A} = \vec{p} \times \vec{J} - \kappa \frac{\vec{q}}{r}\,.$$

Moreover, for $\mu = 0$ recover the standard Coulomb - Kepler problem.

Examples (14)

Example II. Generalized Taub-NUT space (11)

It is interesting to analyze the reverse of the reduction procedure .

Using a sort of *unfolding* of the 3-dimensional dynamics imbedding it in a higher dimensional space the conserved quantities are related to the geometrical features of this manifold, namely higher rank Killing tensors.

To exemplify let us start with the reduced Hamiltonian written in curvilinear coordinates (μ a constant)

$$H_\mu = \frac{1}{2f(r)}\left[p_r^2 + \frac{1}{r^2}\left(p_\theta^2 + \frac{(p_\phi - \mu\cos\theta)^2}{\sin^2\theta}\right)\right] + \frac{\mu^2}{2g(r)} + V(r)\,,$$

on the 3-dimensional space with the metric

$$ds_3^2 = f(r)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2))$$

and the canonical symplectic form

$$d\Theta_\mu = dp_r \wedge dr + dp_\theta \wedge d\theta + dp_\phi \wedge d\phi\,.$$

Examples (15)

Example II. Generalized Taub-NUT space (12)

At each point of $T^\star(\mathbb{R}^3 - \{0\})$ we define the fiber S^1 (the circle) and on the fiber we consider the motion whose equation is

$$\frac{d\psi}{dt} = \frac{\mu}{g(r)} - \frac{\cos\theta}{f(r) r^2 \sin^2\theta}(p_\phi - \mu\cos\theta)\,.$$

The metric on $\mathbb{R}^4$ defines horizontal spaces orthoghonal to the orbits of the circle, annihilated by the connection

$$d\psi + \cos\theta d\phi\,.$$

Examples (16)

Example II. Generalized Taub-NUT space (13)

The metric on $\mathbb{R}^4$ can be written in the form

$$ds_4^2 = f(r)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) + h(r)(d\psi + \cos\theta d\phi)^2 .$$

The natural symplectic form on $T^\star(\mathbb{R}^4 - \{0\})$ is

$$d\Theta = d\Theta_\mu + dp_\psi \wedge d\psi .$$

Considering the geodesic flow of ds_4^2 and taking into account that ψ is a cycle variable

$$p_\psi = h(r)(\dot\psi + \cos\theta\dot\phi) ,$$

is a conserved quantity. To make contact with the Hamiltonian dynamics on $T^\star(\mathbb{R}^3 - \{0\})$ we must identify

$$h(r) = g(r) .$$

Quantum anomalies (1)

In the quantum case, the momentum operator is given by ∇^μ and the Hamiltonian operator for a free scalar particle is the covariant Laplacian acting on scalars

$$\mathcal{H} = \Box = \nabla_i g^{ij} \nabla_j = \nabla_i \nabla^i$$

For a CK vector we define the conserved operator in the quantized system as

$$\mathcal{Q}_V = K^i \nabla_i \,.$$

In order to identify a quantum gravitational anomaly we shall evaluate the commutator $[\Box, \mathcal{Q}_V]\Phi$, for $\Phi \in \mathcal{C}^\infty(M)$ solutions of the Klein-Gordon equation.

Quantum anomalies (2)

Explicit evaluation of the commutator:

$$[\mathcal{H}, \mathcal{Q}_V] = \frac{2-n}{n} K_k{}^{;ki} \nabla_i + \frac{2}{n} K^k{}_{;k} \Box \, .$$

In the case of ordinary K vectors, the r. h. s. of this commutator vanishes and there are no quantum gravitational anomalies. However for CK vectors, the situation is quite different. Even if we evaluate the r. h. s. of the commutator on solutions of the massless Klein-Gordon equation, $\Box \Phi(x) = 0$,the term $K_j{}^{;ji} \nabla_i$ survives. Only in a very special case, when by chance this term vanishes, the anomalies do not appear.

Quantum anomalies (3)

Quantum analog of conserved quantities for Killing tensors K^{ij}

$$\mathcal{Q}_T = \nabla_i K^{ij} \nabla_j \,,$$

Similar form for $\mathcal{Q}_{CSK}$ constructed from a conformal Stackel-Killing tensor.
Evaluation of the commutator

$$\begin{aligned}
[\Box, \mathcal{Q}_T] &= 2 \left(\nabla^{(k} K^{ij)} \right) \nabla_k \nabla_i \nabla_j \\
&+ 3 \nabla_m \left(\nabla^{(k} K^{mj)} \right) \nabla_j \nabla_k \\
&+ \left\{ -\frac{4}{3} \nabla_k \left(R_m^{[k} K^{j]m} \right) \right. \\
&+ \left. \nabla_k \left(\frac{1}{2} g_{ml} (\nabla^k \nabla^{(m} K^{lj)} - \nabla^j \nabla^{(m} K^{kl)}) + \nabla_i \nabla^{(k} K^{ij)} \right) \right\} \nabla_j \,.
\end{aligned}$$

Quantum anomalies (4)

In case of Stackel-Killing tensors the commutator simplifies:

$$[\Box, \mathcal{Q}_T] = -\frac{4}{3}\nabla_k (R_m{}^{[k} K^{j]m})\nabla_j\,.$$

There are a few notable conditions for which the commutator vanishes, i.e. No anomalies:

- Space is Ricci flat, i. e. $R_{ij} = 0$
- Space is Einstein, i. e. $R_{ij} \propto g_{ij}$
- Stackel-Killing tensors associated to Killing-Yano tensors of rank 2:

$$K_{ij} = Y_{ik} Y^k_{\ j}.$$

Quantum anomalies (5)

In case of conformal Stackel-Killing tensors there are quantum gravitational anomalies. Even if we evaluate the commutator for a conformal Stackel-Killing tensor associated to a conformal Killing-Yano tensor

$$K_{ij} = Y_{ik} Y^{k}_{j}.$$

the commutator does not vanish.

Outlook

- Non-Abelian dynamics
- Spaces with skew-symmetric torsion
- Higher order Killing tensors (rank ≥ 3)
- .....