Worldsheet supersymmetry of Pohlmeyer-reduced superstrings on $AdS_n \times S^n$

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Superstring sigma models

It is possible to describe Green-Schwarz superstring via sigma model action with target space being $\hat{F}/G$ supercoset, such that $\hat{F}/G$ is the superextension of bosonic coset, being bosonic target space, to the corresponding superspace. We are interested in the space-time configurations being $AdS_n \times S^n\,,\ n = 2, 3, 5$, considered as bosonic target space of superstring sigma models.

Supercosets

Minimal superextensions of bosonic target spaces

$$AdS_2 \times S^2 = \frac{SU(1,1) \times SU(2)}{U(1) \times U(1)} \,,$$

$$AdS_3 \times S^3 = \frac{SU(1,1) \times SU(1,1) \times SU(2) \times SU(2)}{SU(1,1) \times SU(2)} \,,$$

$$AdS_5 \times S^5 = \frac{SU(2,2) \times SU(4)}{SO(1,4) \times SO(5)}$$

are the following superspaces

$$AdS_2 \times S^2 : \qquad \frac{\hat{F}}{G} = \frac{PSU(1,1|2)}{U(1) \times U(1)} \,,$$

$$AdS_3 \times S^3 : \qquad \frac{\hat{F}}{G} = \frac{PSU(1,1|2) \times PSU(1,1|2)}{SU(1,1) \times SU(2)} \,,$$

$$AdS_5 \times S^5 : \qquad \frac{\hat{F}}{G} = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)} \,.$$

Pohlmeyer reduction

After Pohlmeyer reduction and fixation of local fermionic κ-symmetry (Grigoriev, Tseytlin, 2007) we obtain the following fermionic (g)WZW model on G/H coset, deformed by potential terms. Here the subgroup H of the group G is defined by the condition

$$[T,\, h] = 0\,, \quad h \in \mathtt{h}\,, \quad T \in \hat{f}/g$$

In case of $n = 2$

$$G = U(1) \times U(1),\ H = \emptyset,$$

in case of $n = 3$

$$G = SU(1,1) \times SU(2),\ H = U(1) \times U(1)\,,$$

in case of $n = 5$

$$G = SO(1,4) \times SO(5),\ H = [SU(2)]^4$$

Equations of motion of PR superstring

PR procedure and fixation of local fermionic κ-symmetry results in the following equations of motion for PR superstring:

$$D_-(g^{-1}D_+g) - F_{+-} = \mu^2[T, g^{-1}Tg] + \mu[\Psi_R, g^{-1}\Psi_L g],$$

$$D_-\Psi_R = \mu[T, g^{-1}\Psi_L g], \qquad D_+\Psi_L = \mu[T, g\Psi_R g^{-1}],$$

where covariant derivatives are defined as $D_\pm = \partial_\pm + [A_\pm, \cdot\,]$, and gauge field strength is given by

$$F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-],$$

where $2d$ gauge fields $A_\pm$ take values in algebra $\mathfrak{h}$ of the group H.

PR superstring action

Those equations of motion are derivable from the following action

$$S_{tot} = S_{gWZW} + \mu^2 \int d^2\sigma \mathrm{STr}(g^{-1} T g T)$$

$$+ \int d^2\sigma \mathrm{STr}(\Psi_L T D_+ \Psi_L + \Psi_R T D_- \Psi_R) + \mu \mathrm{STr}(g \Psi_R g^{-1} \Psi_L)\,,$$

where S_{gWZW} is the action of G/H gWZW model:

$$S_{gWZW} = S_{WZW} + S_{gauge}\,,$$

$$S_{WZW} = \frac{1}{2} \int d^2\sigma \mathrm{STr}\left(g^{-1}\partial_+ g g^{-1}\partial_- g\right) -$$

$$\frac{1}{6} \int d^3\sigma \varepsilon^{abc} \mathrm{STr}\left(g^{-1}\partial_a g g^{-1}\partial_b g g^{-1}\partial_c g\right)$$

$$S_{gauge} = \int d^2\sigma \mathrm{STr}(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g - g^{-1} A_+ g A_- + A_+ A_-)\,.$$

Equations of motion of PR superstring are covariant with respect to the following $H \times H$ gauge transformations:

$$g \to hg\bar{h}^{-1}\,, \quad \Psi_L \to h\Psi_L h^{-1}\,, \quad \Psi_R \to \bar{h}\Psi_R\bar{h}^{-1}\,,$$

$$A_+ \to h(A_+ + \partial_+)h^{-1}\,, \quad A_- \to \bar{h}(A_- + \partial_-)\bar{h}^{-1}\,.$$

The action is invariant with respect to the diagonal gauge transformations with H-valued parameter:

$$g \to hgh^{-1}\,, \quad \Psi_{L,R} \to h\Psi_{L,R}h^{-1}\,, \quad A_\pm \to h(A_\pm + \partial_\pm)h^{-1}\,.$$

The following algebraic constraints are the equations of motion on gauge fields $A_{\pm}$, following from PR superstring action:

$$(g^{-1}D_{+}g)_{\mathtt{h}} = 2(T\Psi_R^2)_{\mathtt{h}}\,,$$
$$(gD_{-}g^{-1})_{\mathtt{h}} = 2(T\Psi_L^2)_{\mathtt{h}}\,.$$

Worldsheet supersymmetry?

In the case of $n = 2$ the action possesses $\mathcal{N} = (2, 2)$ supersymmetry and is equivalent to the action of $\mathcal{N} = (2, 2)$ superextension of sine-Gordon model. Corresponding $\mathcal{N} = (2, 0)$ supersymmetry transformation laws may be extended (M. Grigoriev, A. Tseytlin, 2007) to the cases of other n as

$$\delta_{\epsilon_L} g = g[T, [\Psi_R, \epsilon_L]], \quad \delta_{\epsilon_L} \Psi_R = [(g^{-1} D_+ g)^{||}, \epsilon_L],$$

$$\delta_{\epsilon_L} \Psi_L = \mu[T, g\epsilon_L g^{-1}], \quad \delta_{\epsilon_L} A_+ = 0, \quad \delta_{\epsilon_L} A_- = \mu[(g^{-1}\Psi_L g)^{\perp}, \epsilon_L],$$

where matrix SUSY parameter is $\epsilon_L \in \hat{f}_1^{\perp}$ (it is similar for right-chiral supersymmetry transformation laws with parameter $\epsilon_R \in \hat{f}_3^{\perp}$). However, the action is invariant only when condition $[\epsilon_L, h] = 0$ is satisfied,which takes place only in the case of $n = 2$. The way to circumvent this problem was proposed in the work (M.G., E. Ivanov, 2011). The problem was also solved in works by Schmidt and Hollowood, Miramontes.

Modification of PR superstring action

Let us use Polyakov-Wiegmann representation for gauge fields $A_{\pm}$:

$$A_{+} = -\partial_{+} u u^{-1}\,, \qquad A_{-} = -\partial_{-} \bar{u} \bar{u}^{-1}\,,$$

where u, $\bar{u} \in H$. General $H \times H$ gauge transformations of the fields $A_{\pm}$ are reproduced by the following transformations of "prepotentials" u and $\bar{u}$:

$$u \to h u\,, \qquad \bar{u} \to \bar{h} \bar{u}\,.$$

Notice that the values of gauge fields in prepotential representation are left unchanged under right transformations of fields $u, \bar{u}$ with holomorphich and anti-holomorphic parameters (KM symmetry)

$$u \to u\, \omega(\sigma^{-})\,, \qquad \bar{u} \to \bar{u}\, \bar{\omega}(\sigma^{+})\,.$$

Consider the following modified PR superstring action:

$$S_{tot} \to S'_{tot} = S_{tot} + S_a \,,$$

where S_{tot} is the original PR superstring action (with gauge fields represented via prepotentials), and

$$S_a = S^{(H)}_{WZW}(B)$$

is WZW action for field $B = u^{-1}\bar{u} \in H$. Due to PW identity

$$S_{gWZW}(g, u, \bar{u}) = S_{WZW}(u^{-1} g \bar{u}) - S^{(H)}_{WZW}(u^{-1}\bar{u}) \,,$$

and therefore modified gWZW action is simply WZW action for "rotated" bosonic field:

$$S'_{gWZW}(g, u, \bar{u}) = S_{WZW}(u^{-1} g \bar{u}) \,.$$

Notice that the action S'_{tot} is invariant under the full $H \times H$ gauge group.

Supersymmetry of modified action

Problems with infeasible restriction $[\epsilon_L,\, h] = 0$ may be evaded if one uses prepotential representation of gauge fields, introduces the object

$$\tilde{\epsilon}_L = \bar{u}\epsilon_L\bar{u}^{-1}\,,$$

and starts with modified action S'_{tot}. Consider odd $(2(n-1),0)$ chiral transformations

$$\delta_{\epsilon_L} g = g[T,[\Psi_R,\,\tilde{\epsilon}_L]]\,, \quad \delta_{\epsilon_L}\Psi_R = [(g^{-1}D_+g)^{||},\,\tilde{\epsilon}_L]\,,$$

$$\delta_{\epsilon_L}\Psi_L = \mu[T,\,g\tilde{\epsilon}_L g^{-1}]\,, \quad \delta_{\epsilon_L} A_+ = 0\,, \quad \delta_{\epsilon_L} A_- = \mu[(g^{-1}\Psi_L g)^{\perp},\,\tilde{\epsilon}_L]$$

To show the invariance of the action S'_{tot} under those transformations it is convenient to start with "massless" $\mu = 0$ case. Then gauge fields are not transformed, and therefore original action S_{tot} (which differs from modified action S'_{tot} by term, dependent only on gauge fields), also appears to be invariant. The key relation in the invariance of the massless action is

$$D_- \tilde{\epsilon}_L = 0$$

Then, in massive case SUSY transformations of potential A_- lead to non-local transformations of prepotential $\bar{u}$. The later are obtained as solution of equation

$$D_-(\delta\bar{u}\bar{u}^{-1}) = \mu[\tilde{\epsilon}_L, (g^{-1}\Psi_L g)^{\perp}] \quad \Rightarrow$$

$$\bar{u}^{-1}\delta\bar{u} = \mu\,(\partial_-)^{-1}\,\left(\bar{u}^{-1}[\tilde{\epsilon}_L, (g^{-1}\Psi_L g)^{\perp}]\bar{u}\right)\,.$$

Holomorphic function $f(\sigma^+) \in \mathtt{h}$, arising as integration constant, may be absorbed in KM transformations of the field $\bar{u}$. This consideration is applicable to both $n = 3$ and $n = 5$ cases. In difference from the case $n = 2$, now we have gauge fields and prepotential, non-trivial transformations of which enable the invariance of total action.

$$(\partial_- + A_-)\bar{u} = 0 \quad \Rightarrow \quad \bar{u} = P\exp\{-\int^{\sigma^-} d\sigma^{-\prime} A_-(\sigma^{-\prime}, \sigma^+)\}\,\bar{\omega}(\sigma^+)\,.$$

On-shell supersymmetry

Variations of PR superstring equations are given by expressions

$$\delta\left(D_+\Psi_L - \mu[T,\, g\Psi_R g^{-1}]\right) = 2\mu T\left(g[\mathcal{O}_+,\, \tilde{\epsilon}]g^{-1}\right)^{||}\,,$$

$$\delta\left(D_-\Psi_R - \mu[T,\, g^{-1}\Psi_L g]\right) = 0\,,$$

$$\delta\left(D_-(g^{-1}D_+g) - F_{+-} - \mu^2[T,\, g^{-1}Tg] - \mu[\Psi_R,\, g^{-1}\Psi_L g]\right) =$$
$$\mu[g^{-1}\Psi_L g,\, [\tilde{\epsilon},\, \mathcal{O}_+]]\,,$$

where

$$\mathcal{O}_+ = (g^{-1}\partial_+ g + g^{-1}A_+ g - 2T\Psi_R^2 - \tilde{A}_+)_{\mathtt{h}}\,.$$

Define current $\tilde{\mathcal{O}}_+ = \bar{u}^{-1}\mathcal{O}_+\bar{u}$. On shell $\partial_-\tilde{\mathcal{O}}_+ = 0$. Then we represent it as

$$\tilde{\mathcal{O}}_+ = \hat{\tilde{\omega}}\,\partial_+\,\hat{\tilde{\omega}}^{-1}$$

and replace SUSY parameter by

$$\epsilon_L \;\Rightarrow\; \hat{\tilde{\omega}}\,\epsilon_L\,\hat{\tilde{\omega}}^{-1}\,.$$

With such modified parameter equations of motion are invariant.

Current $\tilde{\mathcal{O}}_+$ is invariant under gauge $H \times H$ transformations, but behaves as gauge connection under action of holomorphic KM transformations:

$$\tilde{\mathcal{O}}_+ \quad \rightarrow \quad \bar{\omega}^{-1}\,(\tilde{\mathcal{O}}_+ + \partial_+)\bar{\omega}\,,$$

or, in terms of prepotential $\hat{\bar{\omega}}$,

$$\hat{\bar{\omega}} \quad \rightarrow \quad \bar{\omega}^{-1}\,\hat{\bar{\omega}}\,.$$

therefore it is possible to choose the following on-shell gauge

$$\hat{\bar{\omega}} = I \quad \Leftrightarrow \quad \tilde{\mathcal{O}}_+ = \mathcal{O}_+ = 0\,,$$

which is preserved, if every supersymmetry transformation is accompanied with compensating KM transformation. In this gauge equations of motion are covariant.

Conserved supercurrent

Supersymmetry transformations with localized parameter $\epsilon_L \to \epsilon_L(\sigma^+, \sigma^-)$ give supercurrent components, entering variation of action via relation

$$\delta S_{tot} = \int d^2\sigma (\partial_+ \epsilon_L J_- + \partial_- \epsilon_L J_+) \, .$$

Explicitly

$$J_+ = \bar{u}^{-1}[(g^{-1}D_+g)^{||}, [T, \Psi_R]]\bar{u} + \mu[\partial_-^{-1}(\bar{u}^{-1}(g^{-1}\Psi_L g)^{\perp}\bar{u}), \tilde{\mathcal{O}}_+] \, ,$$

$$J_- = -\mu \bar{u}^{-1}(g^{-1}\Psi_L g)^{\perp}\bar{u} \, .$$

On shell it takes place conservation law:

$$\partial_+ J_- + \partial_- J_+ = 0 \, .$$

Closure

Odd $(4,0)$ ($(8,0)$) transformations in $n=3$ ($n=5$) model are split into $(2,0)$ transformations with complex Grassmann parameter η. Each of $(2,0)$ brackets are closed on translations with parameter $a^+ = \eta_{(2)}\eta^\dagger_{(1)} - \eta_{(1)}\eta^\dagger_{(2)}$ and half of gauge transformations

$$
\begin{aligned}
(\delta_1\delta_2 - \delta_2\delta_1)\, g &= -2a^+\partial_+ g - 2a^+ A_+\, g + g(2a^+ A_+ + \tilde{Q})\,,\\
(\delta_1\delta_2 - \delta_2\delta_1)\, \Psi_R &= -2a^+\partial_+\Psi_R + [\Psi_R,\, 2a^+ A_+ + \tilde{Q}]\,,\\
(\delta_1\delta_2 - \delta_2\delta_1)\, \Psi_L &= -2a^+\partial_+\Psi_L - [2a^+ A_+,\, \Psi_L]\,,\\
(\delta_1\delta_2 - \delta_2\delta_1)\, A_- &= -2a^+\partial_+ A_- + D_-(2a^+ A_+ + \tilde{Q})\,,\\
(\delta_1\delta_2 - \delta_2\delta_1)\, A_+ &= -2a^+\partial_+ A_+ + D_+(2a^+ A_+) = 0\,.
\end{aligned}
$$

The similar formulas are valid for right-chiral $(0,2)$ supersymmetry transformations.

Supersymmetry of the original action

Notice that the choice of gauge $u = \bar{u} = I$ in the action S'_{tot} may be equivalently interpreted as change of variables:

$$g = u\tilde{g}\bar{u}^{-1}, \quad \Psi_L = u\tilde{\Psi}_L u^{-1}, \quad \Psi_R = \bar{u}\tilde{\Psi}_R\bar{u}^{-1},$$

where u and $\bar{u}$ are not assumed to be equal to I. Then, SUSY transformations in the gauge $u = \bar{u} = I$

$$\delta_{\epsilon_L} g = g([T, [\Psi_R, \epsilon_L]] + \hat{\delta}\bar{h}), \quad \delta_{\epsilon_L} \Psi_R = [(g^{-1}\partial_+ g)^{||}, \epsilon_L] + [\Psi_R, \hat{\delta}\bar{h}],$$

$$\delta_{\epsilon_L} \Psi_L = \mu[T, g\epsilon_L g^{-1}], \quad \delta A_{\pm} = 0,$$

with all fields, replaced by those with tildas, obviously leave invariant the actions S_{tot} and S_a in S'_{tot} independently, because u and $\bar{u}$, and, consequently, additional term $S_a(u^{-1}\bar{u})$ are not transformed at all.

Then, returning back to the original variables in S_{tot}, we can easily check, that non-local transformations, leaving action S_{tot} invariant, look as:

$$\delta_{\epsilon_L} g = g([T,[\Psi_R,\, \tilde{\epsilon}_L]] + \hat{\delta}\overline{h})\,, \quad \delta_{\epsilon_L} \Psi_R = [(g^{-1}D_+ g)^{||},\, \tilde{\epsilon}_L] + [\Psi_R,\, \hat{\delta}\overline{h}]\,,$$

$$\delta_{\epsilon_L} \Psi_L = \mu[T,\, g\tilde{\epsilon}_L g^{-1}]\,, \quad \delta A_\pm = 0\,,$$

where now $\hat{\delta}\overline{h} = \mu(D_-)^{-1}\,[\tilde{\epsilon}_L, (g^{-1}\Psi_L g)^{\perp}]\,.$