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Instantons and Chern-Simons flows in d=6,7 and 8

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Self-duality in higher dimensions

generalized Yang-Mills anti-self-duality on a Riemannian manifold M^d:

$$*F \;=\; -\Sigma\wedge F \qquad \text{for} \qquad F=\mathrm{d}A+A\wedge A \qquad \text{and} \qquad \Sigma\in\Lambda^{d-4}(M)$$

apply gauge-covariant derivative $D=\mathrm{d}+[A,\cdot]$:

$$D{*F} \;+\; \mathrm{d}\Sigma\wedge F \;=\; 0 \qquad \Longleftrightarrow \qquad \text{Yang-Mills with torsion} \quad \mathcal{H}=*\mathrm{d}\Sigma\in\Lambda^3(M)$$

follows from the action

$$\begin{aligned} S_{\mathrm{YM}}+S_{\mathrm{CS}} \;&=\; \int_M \mathrm{tr}\Big\{F\wedge *F \;+\; (-)^{d-3}\Sigma\wedge F\wedge F\Big\} \\ &=\; \int_M \mathrm{tr}\Big\{F\wedge *F \;+\; \tfrac12 \mathrm{d}\Sigma\wedge\Big(A\,\mathrm{d}A+\tfrac23 A^3\Big)\Big\} \end{aligned}$$

can also consider the gradient Chern-Simons flow on M

$$*\frac{\mathrm{d}A}{\mathrm{d}\tau} \;=\; *\frac{\delta}{\delta A}S_{\mathrm{CS}} \;=\; \mathrm{d}\Sigma\wedge F$$

this follows from ASD on $\widetilde{M}=\mathbb{R}_\tau\times M$ (in $A_\tau{=}0$ gauge)

Q: which manifolds admit a global $(d-4)$ form? A: (weak) G-structure manifolds

key examples:

d=6: SU(3)-structure, e.g. nearly-Kähler 6-manifolds, like $S^6 = \frac{G_2}{\mathrm{SU}(3)}$

structure: 2-form $\Sigma =: \omega$ with $\mathrm{d}\omega \sim \mathrm{Im}\Omega$ and $\mathrm{dRe}\Omega \sim \omega^2$ for 3-form Ω

d=7: G_2-structure, e.g. nearly-parallel G_2-manifolds, like $X_{k,\ell} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)_{k,\ell}}$

structure: 3-form $\Sigma =: \psi$ with $\mathrm{d}\psi \sim *\psi$

d=8: Spin(7)-structure, e.g. cones over G_2-manifolds

special cases: Calabi-Yau 4-folds (SU(4)), hyper-Kähler (Sp(2))

for our coset-space examples $M = \frac{G}{H}$ we take the gauge group to be G

Six dimensions: nearly-Kähler coset spaces

all known compact examples are nonsymmetric coset spaces:

$$S^6 = \frac{G_2}{\mathrm{SU}(3)}\,, \quad \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\times\mathrm{U}(1)}\,, \quad \frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}\,, \quad S^3\times S^3 = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{SU}(2)}$$

coset structure: $H \lhd G \quad \longrightarrow \quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$

3-symmetry: $S : G \to G$ with $S^3 = \mathrm{id}$ automorphism

$\longrightarrow \quad s : \mathfrak{g} \to \mathfrak{g}$ with $s|_{\mathfrak{h}} = \mathbb{1}\,, \quad s|_{\mathfrak{m}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} J \qquad \frac{2\pi}{3}$ rotation

Lie-algebra basis: $\{I_{a=1,\ldots,6}\,,\, I_{i=7,\ldots,\dim G}\}$ with $[I_a, I_b] = f^i_{ab} I_i + f^c_{ab} I_c$

Cartan-Killing form: $\langle\cdot,\cdot\rangle_{\mathfrak{g}} = -\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(\cdot)\circ\mathrm{ad}(\cdot)) = 3\,\langle\cdot,\cdot\rangle_{\mathfrak{h}} = 3\,\langle\cdot,\cdot\rangle_{\mathfrak{m}} = \mathbb{1}$

canonical 1-forms framing $T^*(G/H)$: e^a , furthermore: $e^i = e^i_a\, e^a \qquad \longrightarrow$

$$g = \delta_{ab}\, e^a e^b\,, \qquad \omega = \tfrac{1}{2} J_{ab}\, e^a \wedge e^b\,, \qquad \Omega = -\tfrac{1}{\sqrt{3}}(f + \mathrm{i}Jf)_{abc}\, e^a \wedge e^b \wedge e^c$$

nearly-Kähler accident:

$*F \;=\; -\omega\wedge F \qquad \Longleftrightarrow \qquad 0 \;=\; \mathrm{d}\omega\wedge F \;\sim\; \mathrm{Im}\Omega\wedge F \qquad \Longleftrightarrow \qquad$ DUY equations

in components: $\quad \frac{1}{2}\epsilon_{abcdef}F_{ef} \;=\; -J_{[ab}F_{cd]} \qquad \Longleftrightarrow \qquad 0 \;=\; f_{abc}F_{bc}$

for this ASD equation we have an action: $\quad S_{\mathrm{CS}} = -\int_M \mathrm{tr}\{\omega\wedge F\wedge F\}$

hence, each CS flow $\quad \dot{A}_a \sim f_{abc}F_{bc} \quad$ on M^6 ends in an instanton

consequences of ASD: $\quad \omega\lrcorner F = 0\;,\quad \mathrm{Re}\Omega\wedge F = 0\;,\quad D{*}F = 0$

in components: $\quad \omega_{ab}F_{ab} = 0\;,\quad (Jf)_{abc}F_{bc} = 0\;,\quad D_aF_{ab} = 0$

H-instantons: unique G-inv. connection is canonical, $A^{\text{can}} = e^i\, I_i \quad \longrightarrow$

$$F^{\text{can}} = -\tfrac{1}{2} f^i_{ab}\, e^a \wedge e^b\, I_i \quad \checkmark$$

G-instantons: G-invariant connection on G/H reads $A = e^i\, I_i + e^a\, \Phi_{ab}\, I_b$

ansatz (general on S^6): $\Phi_{ab} = \phi_1\, \delta_{ab} + \phi_2\, J_{ab} \quad \Leftrightarrow \quad \Phi = \phi_1 \mathbb{1} + \phi_2 J$

curvature: $F_{ab} = F^{1,1}_{ab} + F^{2,0\oplus 0,2}_{ab} = (|\Phi|^2 - 1)\, f^i_{ab}\, I_i + [(\bar{\Phi}^2 - \Phi)\, f]_{abc}\, I_c$

3-symmetric! DUY $\Leftrightarrow$ $\bar{\Phi}^2 = \Phi \quad \Rightarrow \quad \Phi = 0$ or $\Phi = \exp\{\frac{2\pi k}{3} J\}$

$\rightarrow$ three (flat) G-instanton connections $A^{(k)} = e^i\, I_i + e^a\, (s^k I)_a \quad k = 0, 1, 2$

besides canonical (curved) connection $A^{\text{can}} = e^i\, I_i$

Seven dimensions: cylinder over nearly-Kähler cosets

nearly-parallel G_2 accident:

$$*F \;=\; -\psi\wedge F \qquad\Longleftrightarrow\qquad 0 \;=\; \mathrm{d}\psi\wedge F \;\sim\; *\psi\wedge F \qquad\Longleftrightarrow\qquad 0 \;=\; \psi\lrcorner F$$

in components: $\frac{1}{2}\epsilon_{abcdefg}F_{fg} \;=\; -\psi_{[abc}F_{de]} \qquad\Longleftrightarrow\qquad 0 \;=\; \psi_{abc}F_{bc}$

again, there exists an action: $S_{\mathrm{CS}} = \int_M \mathrm{tr}\{\psi\wedge F\wedge F\}$

hence, each CS flow $\dot{A}_a \sim \psi_{abc}F_{bc}$ on M^7 ends in an instanton

consequence of ASD: $D{*}F = 0$ (in components: $D_aF_{ab} = 0$)

consider $\widetilde{M} = \mathbb{R}_\tau \times \frac{G}{H}$ with $\frac{G}{H}$ nearly-Kähler and metric $g = (\mathrm{d}\tau)^2 + \delta_{ab}\, e^a e^b$

natural G_2-structure 3-form:

$$\psi \;=\; \mathrm{d}\tau\wedge\omega + \mathrm{Im}\Omega \qquad \overset{\mathrm{Im}\Omega\sim\mathrm{d}\omega}{\Longrightarrow} \qquad S_{\mathrm{CS}} \;=\; \int_{\widetilde{M}} \psi\wedge\widetilde{F}\wedge\widetilde{F} \;=\; \int_{\widetilde{M}} \mathrm{d}\tau\wedge\omega\wedge\widetilde{F}\wedge\widetilde{F}$$

reduce ASD from $\widetilde{M} = \mathbb{R}_\tau \times M$ to M:

$$\tilde{*}\widetilde{F} \;=\; -\psi\wedge\widetilde{F} \qquad \Longleftrightarrow \qquad *\dot{A} \;=\; \mathrm{d}\omega\wedge F \qquad \& \qquad \omega\lrcorner F \;=\; 0$$

operator $\tilde{*}(\psi\wedge\cdot)$ has (eigenvalue, dimension) = (-1,14) and (2,7) $\Rightarrow$ 7 equations

flow endpoints $\mathrm{d}\omega\wedge F = 0$ are instantons on $\frac{G}{H}$

Yang-Mills with torsion:

$$D{*}\widetilde{F} + \mathrm{d}\tau\wedge\mathrm{d}\omega\wedge\widetilde{F} \;=\; 0$$

torsion $\mathcal{H} = -\frac{1}{3}\kappa \,*\,(\mathrm{d}\tau\wedge\mathrm{d}\omega) \quad \Longleftrightarrow \quad T_{abc} = \kappa\, f_{abc}$ with $\kappa = 3$

we will allow κ to deviate from the ASD value 3 ansatz for connection: $A \;=\; \mathrm{d}\tau\, A_0 + e^i\, I_i + e^a\, [\Phi(\tau)\, I]_a$ gauge $A_0 = 0$

curvature: $F_{0a} = [\dot{\Phi}\, I]_a$ and $F_{ab} = (|\Phi|^2{-}1)\, f^i_{ab}\, I_i \;+\; [(\bar{\Phi}^2{-}\Phi)\, f]_{abc}\, I_c$

YM + torsion $\Rightarrow$ $\ddot{\Phi} \;=\; (\kappa{-}1)\Phi \;-\; (\kappa{+}3)\bar{\Phi}^2 \;+\; 4\bar{\Phi}\Phi^2$ ϕ^4-type!

follows from the action $S[A(\Phi)] \;\sim\; \int_{\frac{G}{H}}\mathrm{vol}\; \int_{\mathbb{R}}\mathrm{d}\tau \left\{3|\dot{\Phi}|^2 + V(\Phi)\right\}$

with $V(\Phi) \;=\; (3{-}\kappa) \;+\; 3(\kappa{-}1)|\Phi|^2 \;-\; (3{+}\kappa)(\Phi^3{+}\bar{\Phi}^3) \;+\; 6|\Phi|^4$

this is a complex Φ^4 potential without rotational symmetry but with 3-symmetry!

equation of motion $3\ddot{\Phi} = \frac{\partial V}{\partial \bar{\Phi}}$ is Newtonian mechanics on $\mathbb{C}$ with potential $-V$

this gives all solutions for the case $\frac{G}{H} = S^6 = \frac{G_2}{\mathrm{SU}(3)}$

for $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ the general G-invariant connection contains 3 parameters $\Phi_i \in \mathbb{C}$

the action reads $\quad S[A(\{\Phi\})] \sim \int_{\frac{G}{H}} \mathrm{vol} \int_{\mathbb{R}} \mathrm{d}\tau \left\{ |\dot\Phi_1|^2 + |\dot\Phi_2|^2 + |\dot\Phi_3|^2 + V(\{\Phi\}) \right\}$

$$
\begin{aligned}
V(\{\Phi\}) \;=\; & (3-\kappa) + (\kappa-1)\Big(|\Phi_1|^2+|\Phi_2|^2+|\Phi_3|^2\Big) - (\kappa+3)2\mathrm{Re}\Big(\Phi_1\Phi_2\Phi_3\Big) \\
& + |\Phi_1\Phi_2|^2 + |\Phi_2\Phi_3|^2 + |\Phi_3\Phi_1|^2 + |\Phi_1|^4 + |\Phi_2|^4 + |\Phi_3|^4
\end{aligned}
$$

the equations of motion $\ddot\Phi_i = \frac{\partial V}{\partial \bar\Phi_i}$ (3 particles on $\mathbb{C}$) must be supplemented with

the U(1)×U(1) Noether charge conservation $\quad \Phi_i \dot{\bar\Phi}_i - \dot\Phi_i \bar\Phi_i = \Phi_j \dot{\bar\Phi}_j - \dot\Phi_j \bar\Phi_j$

due to symmetry $(\Phi_1, \Phi_2, \Phi_3) \to (\mathrm{e}^{\mathrm{i}\delta_1}\Phi_1, \mathrm{e}^{\mathrm{i}\delta_2}\Phi_2, \mathrm{e}^{\mathrm{i}\delta_3}\Phi_3)$ with $\delta_1+\delta_2+\delta_3 = 0$

specialization: $\quad \Phi_1 \equiv \Phi_2 \quad \to \quad \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\times\mathrm{U}(1)} \quad , \quad \Phi_1 \equiv \Phi_2 \equiv \Phi_3 \quad \to \quad \frac{G_2}{\mathrm{SU}(3)}$

Seven dimensions: solutions

finite action $\Leftrightarrow$ trajectories between points $\hat{\Phi}$ with $\mathrm{d}V(\hat{\Phi}) = 0 = V(\hat{\Phi})$

these are precisely (with two exotic exceptions) the DUY solutions on $\frac{G}{H}$:

$\hat{\Phi}_i = \mathrm{e}^{2\pi\mathrm{i}k_i/3}$ with $\sum_i k_i{=}0$ (for any κ)

$\hat{\Phi}_i = 0$ (for $\kappa{=}3$: $V(0) = V(1) = 0$)

two types of trajectories:

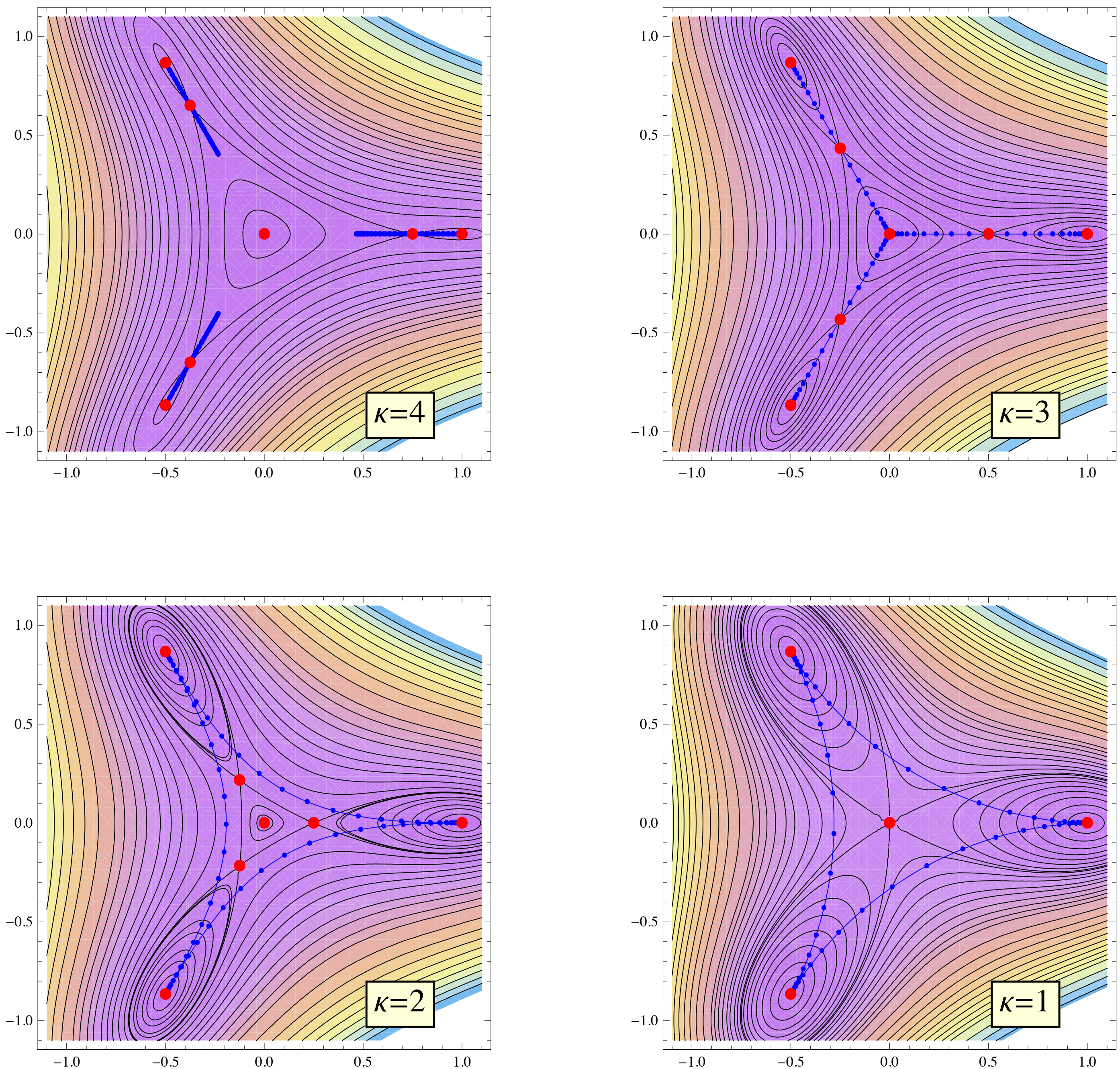
radial type:	$(0,0,0) \;\rightarrow\; (1,1,1)$	carry over to S^6
[only for $\kappa{=}3$]	$(0,0,0) \;\rightarrow\; (1,\mathrm{e}^{2\pi\mathrm{i}/3},\mathrm{e}^{-2\pi\mathrm{i}/3})$	
transversal type:	$(1,1,1) \;\rightarrow\; \mathrm{e}^{2\pi\mathrm{i}/3}(1,1,1)$	carry over to S^6
[for any κ-value]	$(1,\mathrm{e}^{2\pi\mathrm{i}/3},\mathrm{e}^{-2\pi\mathrm{i}/3}) \;\rightarrow\; (\mathrm{e}^{2\pi\mathrm{i}/3},\mathrm{e}^{-2\pi\mathrm{i}/3},1)$	

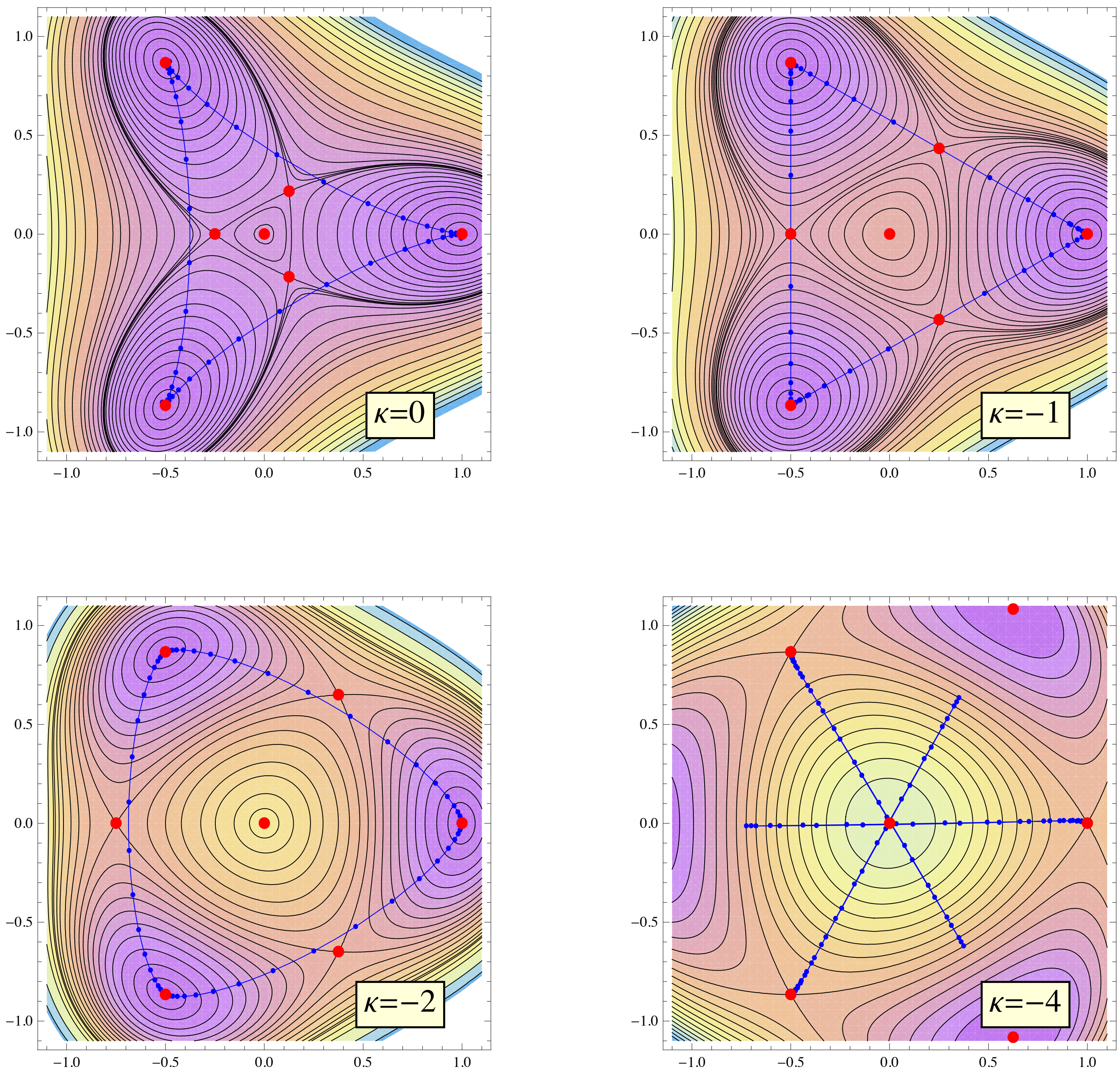
numerical analysis:

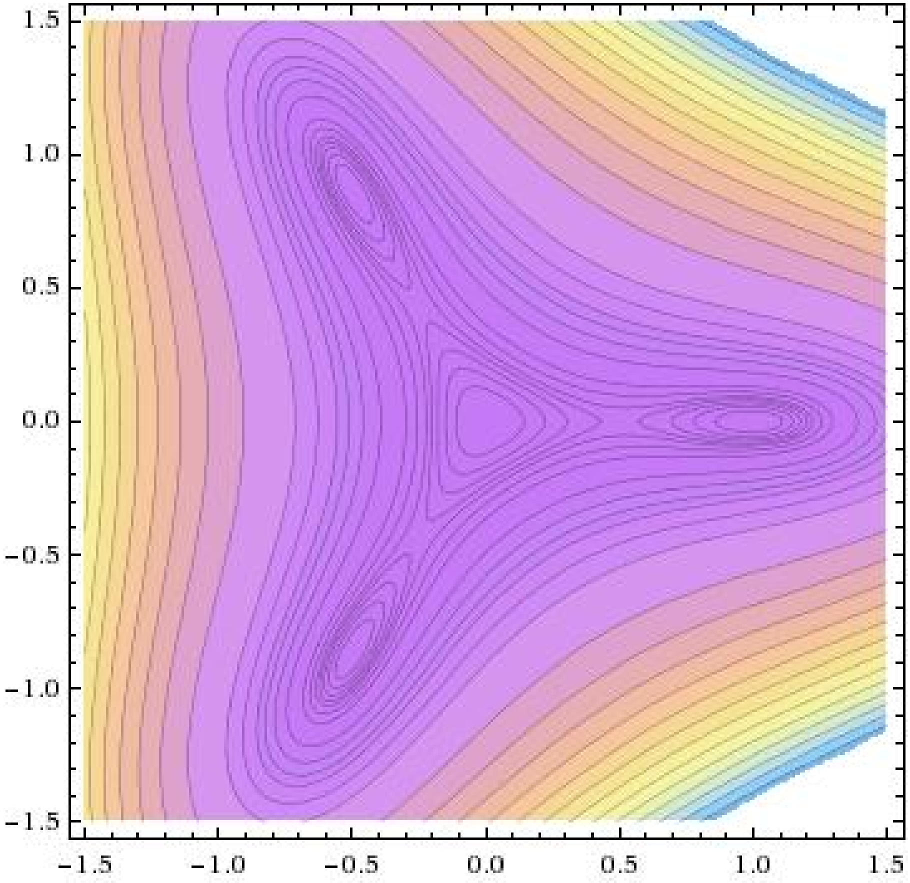
$|\kappa| > 3$: radial bounces

$\kappa = +3$: radial instantons — $\kappa = -3$: Mexican hat

$|\kappa| < 3$: transversal instantons


κ=4
κ=3
κ=2
κ=1


κ=0
κ=−1
κ=−2
κ=−4
1.0
0.5
0.0
−0.5
−1.0



contour plot of $V(\Phi)$ for $\kappa = +3$

$\kappa = +3$: gradient flow

let us specialize to

$$\Phi_1 = \Phi_2 = \Phi_3 =: \Phi \quad \Leftrightarrow \quad S^6$$

$$3\ddot{\Phi} = \frac{\partial V}{\partial \bar{\Phi}} \qquad \Leftarrow \qquad \sqrt{2}\dot{\Phi} = \pm\frac{\partial W}{\partial \bar{\Phi}}$$

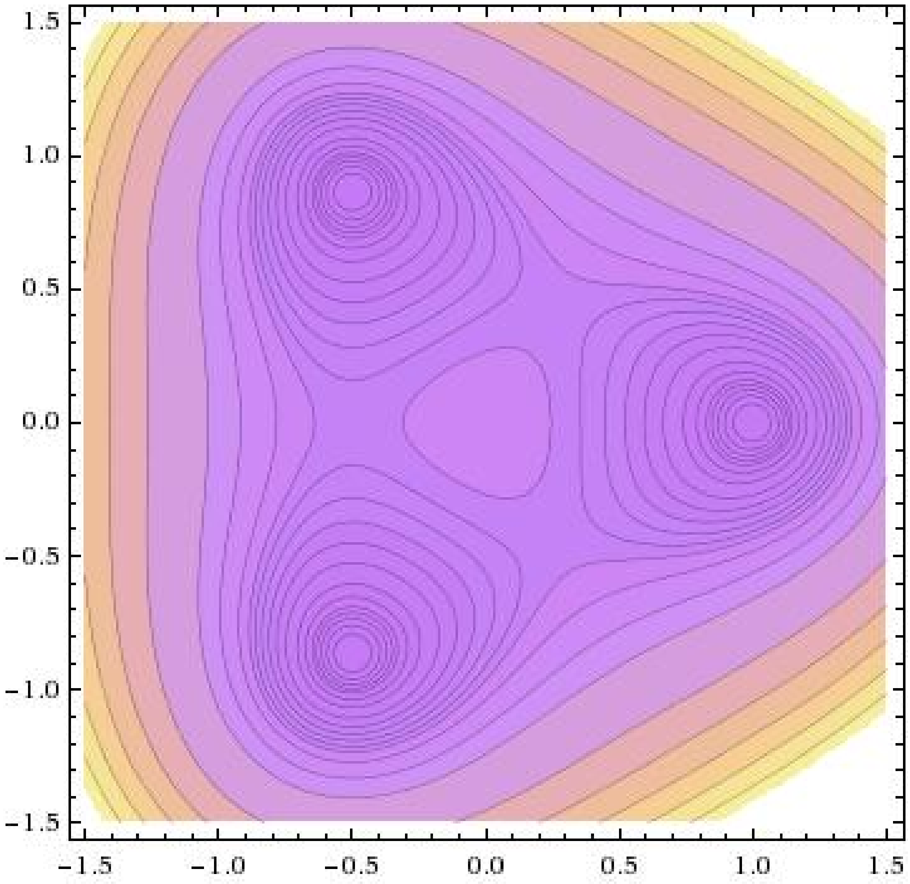
with $\quad W = \frac{1}{3}(\Phi^3 + \bar{\Phi}^3) - |\Phi|^2$

admits analytic solution:

$$\Phi = \mathrm{e}^{\frac{2\pi \mathrm{i} k}{3}}\left(\frac{1}{2} \pm \frac{1}{2}\tanh\frac{\tau}{2\sqrt{3}}\right)$$

real function W is a superpotential:

$$V = 6\left|\frac{\partial W}{\partial \bar{\Phi}}\right|^2 \qquad \text{for} \quad \kappa = +3$$



contour plot of $V(\Phi)$ for $\kappa = -1$

$\kappa = -1$: hamiltonian flow

remain specialized to S^6

$3\ddot{\Phi} = \frac{\partial V}{\partial \bar{\Phi}} \qquad \Leftarrow \qquad \sqrt{2}\dot{\Phi} = \pm \mathrm{i}\, \frac{\partial H}{\partial \bar{\Phi}}$

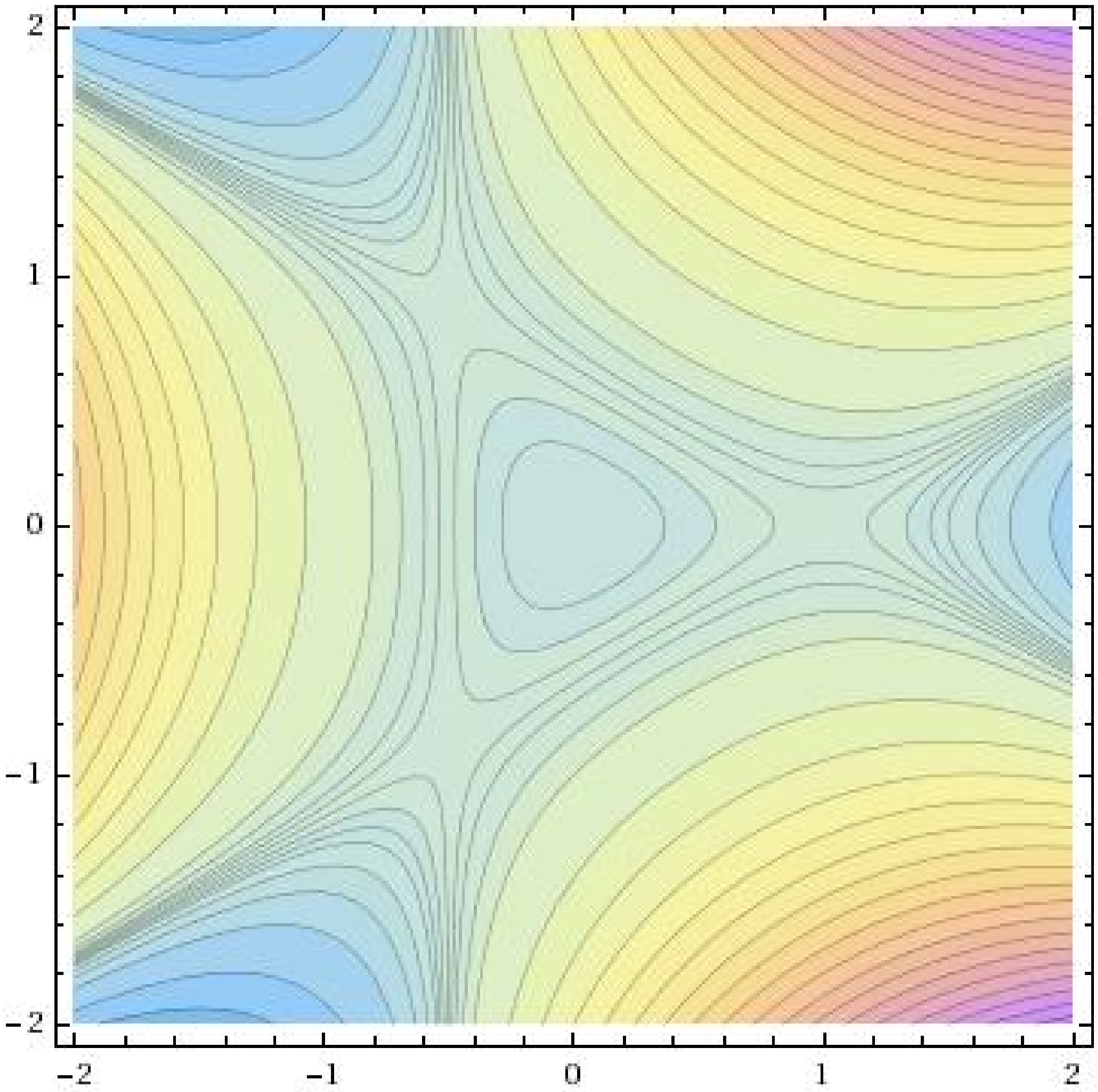
with $\quad H = \frac{1}{3}(\Phi^3 + \bar{\Phi}^3) - |\Phi|^2$

admits straight transverse trajectories:

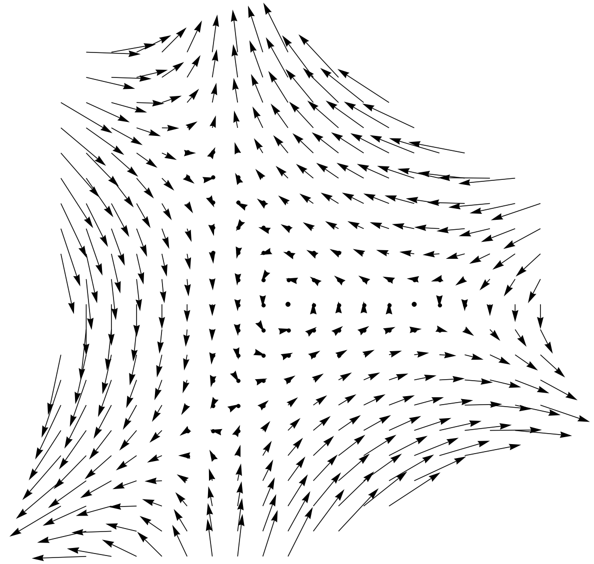
$\Phi = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, \mathrm{i}\, (\tanh \frac{\tau}{2})$

and images under 3-symmetry action

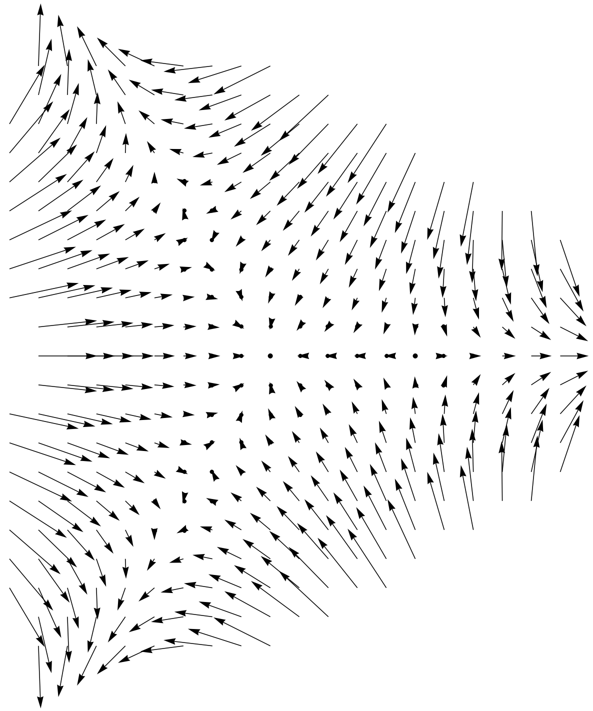
why does this analytic solution exist?



contour plot of superpotential $W(\Phi)$
(applies to $\kappa = +3$ and $\kappa = -1$)



hamiltonian vector field



gradient vector field

What kind of gauge bundles $\mathcal{E}$ emerge from our solutions $\Phi(\tau)$?

$$F(\tau) \;=\; \mathrm{d}\tau\wedge e^a\,[\dot\Phi\, I]_a \;+\; \tfrac12 e^a\wedge e^b\Big\{(|\Phi|^2-1)\,f^i_{ab}\,I_i \;+\; [(\bar\Phi^2-\Phi)\,f]_{abc}\,I_c\Big\}$$

interpolates between critical points of V: $F(-\infty) \;\to\; F(0) \;\to\; F(+\infty)$

hamiltonian flow: $\mathcal{E}_{\text{flat}} \to \mathcal{E}_{\text{saddle}} \to \mathcal{E}_{\text{flat}}$, for $\kappa=-1$:

$$F(\tau) \;=\; \tfrac{\sqrt3}{4}(\cosh^{-2}\tfrac{\tau}{2})\Big\{\pm\mathrm{d}\tau\wedge e^a\,[J\,I]_a \;-\; \tfrac{\sqrt3}{2}\,e^a\wedge e^b(f^i_{ab}\,I_i - f_{abc}\,I_c)\Big\}$$

gradient flow: $\mathcal{E}_{\text{can}} \to \mathcal{E}'_{\text{saddle}} \to \mathcal{E}_{\text{flat}}$, for $\kappa=+3$:

$$F(\tau) \;=\; \tfrac{1}{4\sqrt3}(\cosh^{-2}\tfrac{\tau}{2\sqrt3})\Big\{\pm\mathrm{d}\tau\wedge e^a\,I_a \;-\; \tfrac{\sqrt3}{2}\,e^a\wedge e^b(f^i_{ab}\,I_i + f_{abc}\,I_c)\Big\}$$
$$-\tfrac14(1\mp\tanh\tfrac{\tau}{2\sqrt3})\,e^a\wedge e^b\,f^i_{ab}\,I_i$$

plus 3-sym. action $I_a \;\mapsto\; [s^k I]_a = [\mathrm{e}^{\frac{2\pi k}{3}J} I]_a$

The story in eight dimensions

consider $\mathcal{M}_8 = \mathbb{R}^2 \times \frac{G}{H}$ with metric $g_8 = (\mathrm{d}\tau)^2 + (\mathrm{d}\sigma)^2 + \delta_{ab}\, e^a e^b$

Spin(7) structure defined by $\Sigma = \frac{1}{2}\omega\wedge\omega + \omega\wedge\mathrm{d}\tau\wedge\mathrm{d}\sigma - \mathrm{Re}\Omega\wedge\mathrm{d}\tau + \mathrm{Im}\Omega\wedge\mathrm{d}\sigma$

Σ-anti-self-duality: $\Sigma \wedge F = -\ast_8 F \quad \Leftrightarrow \quad \begin{cases} (\omega + \mathrm{d}\tau\wedge\mathrm{d}\sigma) \lrcorner F = 0 \\ F_{\bar\alpha\bar\beta} = -\frac{1}{2}\epsilon_{\bar\alpha\bar\beta\bar\gamma\bar\delta}F^{\bar\gamma\bar\delta} \end{cases}$

reduce over σ to $\mathbb{R}_\tau \times \frac{G}{H}$: $\Sigma \longrightarrow \psi = \omega \wedge \mathrm{d}\tau + \mathrm{Im}\Omega$ G_2 structure

get ψ-anti-self-duality: $\psi \wedge F = -\ast_7 F \quad \Leftrightarrow \quad \begin{cases} J_{ab}F_{ab} = 0 \\ \partial_\tau A_a \sim f_{abc}F_{bc} \end{cases}$

reduce over τ to $\mathbb{R}_\sigma \times \frac{G}{H}$: $\Sigma \longrightarrow \tilde\psi = \omega \wedge \mathrm{d}\sigma + \mathrm{Re}\Omega$ $\tilde G_2$ structure

get $\tilde\psi$-anti-self-duality: $\tilde\psi \wedge F = -\ast_7 F \quad \Leftrightarrow \quad \begin{cases} J_{ab}F_{ab} = 0 \\ \partial_\sigma A_a \sim [Jf]_{abc}F_{bc} \end{cases}$

differentiate new self-duality equations:

$$\psi\wedge F = -*_7 F \quad\longrightarrow\quad D*F - \mathrm{d}\tau\wedge\mathrm{d}\omega\wedge F = 0 \quad\Leftrightarrow\quad \kappa=+3$$

$$\tilde\psi\wedge F = -*_7 F \quad\longrightarrow\quad D*F + \tfrac{1}{3}\mathrm{d}\tau\wedge\mathrm{d}\omega\wedge F = 0 \quad\Leftrightarrow\quad \kappa=-1 \quad !$$

ansatz $A = e^i\,I_i + e^a\,[\Phi\,I]_a$ obeys $J_{ab}F_{ab}=0$ and yields

$$\partial_\tau A_a \sim f_{abc}\,F_{bc} \quad\longrightarrow\quad \sqrt{2}\dot\Phi = \pm\frac{\partial W}{\partial\bar\Phi}$$ gradient flow

$$\partial_\sigma A_a \sim [Jf]_{abc}\,F_{bc} \quad\longrightarrow\quad \sqrt{2}\dot\Phi = \pm\mathrm{i}\,\frac{\partial H}{\partial\bar\Phi}$$ hamiltonian flow

remark: d=7 flow is actually equivalent to Spin(7)-anti-self-duality in d=8:

$$*_8F_8 \;=\; -\Sigma\wedge F_8 \quad\Longleftrightarrow\quad *_7\dot A_7 \;=\; \mathrm{d}\psi\wedge F_7$$

operator $*_8(\Sigma\wedge\cdot)$ has (eigenvalue, dim.) = $(-1,21)$ and $(?,7)$ $\Rightarrow$ 7 equations

A new instanton on $\mathbb{R}^7$

cone $\mathcal{C}\left(\frac{G_2}{\mathrm{SU}(3)}\right) = \mathcal{C}(S^6)$ is topologically equivalent to $\mathbb{R}^7 \backslash \{0\}$

metric $\quad \mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \delta_{ab} e^a e^b = \mathrm{e}^{2\tau}(\mathrm{d}\tau^2 + \delta_{ab} e^a e^b)$

fundamental forms

$$\omega = e^1{\wedge}e^2 + e^3{\wedge}e^4 + e^5{\wedge}e^6$$

$$\Omega = (e^1{+}\mathrm{i}e^2){\wedge}(e^3{+}\mathrm{i}e^4){\wedge}(e^5{+}\mathrm{i}e^6)$$

G_2-structure $\quad \psi = r^2 \omega{\wedge}\mathrm{d}r + r^3 \mathrm{Im}\Omega = \mathrm{e}^{3\tau}\left(\tfrac{1}{2}\psi_{0ab}\mathrm{d}\tau{\wedge}e^a{\wedge}e^b + \tfrac{1}{6}\psi_{abc}e^a{\wedge}e^b{\wedge}e^c\right)$

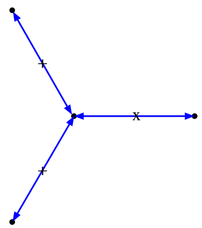
with $\mathrm{d}\psi = 0 \ \Leftrightarrow \ \mathcal{H} = 0$ $\qquad$ $\psi \ldots$ are the octonionic structure constants

G_2-instanton eq. $\quad \psi \lrcorner F = 0 \quad \Longleftrightarrow \quad \omega \lrcorner F = 0 \quad \& \quad \dot{A}_a = -\tfrac{1}{2}\mathrm{Re}\Omega_{abc}F_{bc} = 0$

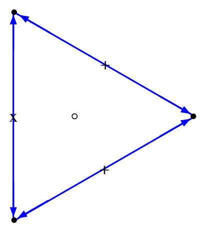
G_2-invariant solution $\quad \Phi = \tfrac{1}{2}(1 - \tanh\tau) = \frac{1}{1+r^2} \quad \Longrightarrow$

$$F_{0a} = -\frac{4\sqrt{3}}{(1+r^2)^2} I_a \qquad \& \qquad F_{ab} = -\frac{12(2+r^2)}{(1+r^2)^2} f^i_{ab} I_i - \frac{12}{(1+r^2)^2} f^c_{ab} I_c$$

this is smooth at $r{=}0$ $\qquad$ and agrees with the Günaydin/Nicolai G_2 instanton



Partial summary



$$\Sigma \wedge F = - *_8 F$$

$\swarrow$ $\searrow$

$\psi \wedge F = - *_7 F$ on $\mathbb{R} \times \frac{G}{H}$ $\tilde{\psi} \wedge F = - *_7 F$

$\partial_\tau A_a \sim f_{abc} F_{bc}$ $\partial_\sigma A_a \sim [J f]_{abc} F_{bc}$

$\downarrow$ ansatz $A = e^i I_i + e^a [\Phi\, I]_a$ $\downarrow$

$\sqrt{2}\dot{\Phi} = \pm \frac{\partial W}{\partial \bar{\Phi}}$ $\sqrt{2}\dot{\Phi} = \pm \mathrm{i}\, \frac{\partial H}{\partial \bar{\Phi}}$

$\downarrow$ $W = \frac{1}{3}(\Phi^3 + \bar{\Phi}^3) - |\Phi|^2 = H$ $\downarrow$

$$F(\tau) = \mathrm{d}\tau \wedge e^a [\dot{\Phi}\, I]_a + \tfrac{1}{2} e^a \wedge e^b \Big\{ (|\Phi|^2 - 1)\, f^i_{ab}\, I_i + [(\bar{\Phi}^2 - \Phi)\, f]_{abc}\, I_c \Big\}$$

are G_2-instantons for Yang-Mills with torsion $D * F + (*\mathcal{H}) \wedge F = 0$

from $S[A] = \int_{\mathbb{R} \times \frac{G}{H}} \mathrm{tr}\Big\{ F \wedge *F + \tfrac{1}{3}\kappa\, \mathrm{d}\tau \wedge \omega \wedge F \wedge F \Big\}$ with $\kappa = +3, -1$

and obey gradient/hamiltonian flow equations for $\int_{\frac{G}{H}} \mathrm{tr}\, \{\omega \wedge F \wedge F\} \propto W(\Phi) + \frac{1}{3}$

THANK YOU !

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