BRST renormalization

Peter Lavrov

Tomsk State Pedagogical University

Dubna, SQS'11, 23 July 2011

Based on PL, I. Shapiro, Phys. Rev. D81, 2010

Plan

- Introduction
- BV-quantization: main features
- General gauge theories in curved space
- Renormalization in curved space
- Non-covariant gauges
- Conclusion

Renormalization of Yang-Mills theories:

- Non-abelian gauge fields (Yang, Mills, 1951)
- Non-unitarity of S-matrix (Feynman, 1963)
- Faddeev-Popov quantization (Faddeev, Popov, 1967)
- Gauge invariant renormalization ('t Hooft, Veltman, 1971)
- Slavnov-Taylor identity (Taylor,1971; Slavnov, 1972)
- BRST symmetry (Becchi, Rouet, Stora, 1975, Tyutin, 1975)
- Zinn-Justin equation (Zinn-Justin, 1977)

Renormalization of general gauge theories

- Supergravity theories (Freedman, van Nieuwenhuizen, Ferrara, 1976; Deser, Zumino, 1976) $\rightarrow$ Gauge theories with open algebras
- Reducible gauge theories (Townsend, 1979)
- BV-quantization (Batalin, Vilkovisky,1981,1983)
- Gauge invariant renormalizability of general gauge theories (Voronov, Tyutin, 1982; Voronov, PL, Tyutin, 1982)

Renormalization in curved space-time

Utiyama, DeWitt, 1962
Renormalization of a classical gravitational field interacting with quantized matter fields

Panangaden, 1981
One Loop Renormalization of Quantum Electrodynamics in Curved Space-Time

Toms, 1982,1983
Renormalization of Interacting Scalar Field Theories in Curved Space-Time;
The Background - Field Method and the Renormalization of Nonabelian Gauge Theories in Curved Space-Time

Buchbinder, 1984
Renormalization Group Equations In Curved Space-Time

BV quantization: main features

[Batalin, Vilkovisky,1981,1983; Voronov, Tyutin, 1982; Voronov,PL,Tyutin, 1982]

Initial action

The starting point of the BV method is a theory of fields $A^i\,(i = 1, 2, ..., n)$ with Grassmann parities $\varepsilon(A^i) = \varepsilon_i$, for which the initial classical action $S_0(A)$ is assumed to have at least one stationary point A_0^i

$$S_{0,i}(A)|_{A_0} = 0$$

and to be regular in the neighborhood of A_0.

Gauge invariance

$$\delta A^i = R^i_\alpha(A)\xi^\alpha$$

$$S_{0,i}(A)\,R^i_\alpha(A) = 0\,, \quad \alpha = 1, 2, ..., m\,, \quad 0 < m < n\,, \quad \varepsilon(\xi^\alpha) = \varepsilon_\alpha\,.$$

Here ξ^α are arbitrary functions of space-time coordinates , and $R^i_\alpha(A)$ are generators of gauge transformations.

Gauge algebra

$$R^i_{\alpha,j}R^j_\beta - (-1)^{\varepsilon_\alpha\varepsilon_\beta}R^i_{\beta,j}R^j_\alpha = -R^i_\gamma F^\gamma_{\alpha\beta} - S_{0,j}M^{ij}_{\alpha\beta},$$

where $F^\gamma_{\alpha\beta} = F^\gamma_{\alpha\beta}(A)$ are structure functions with the following symmetry properties:

$$F^\gamma_{\alpha\beta}(A) \;=\; -\,(-1)^{\varepsilon_\alpha\varepsilon_\beta}F^\gamma_{\beta\alpha}(A)$$

and $M^{ij}_{\alpha\beta} = M^{ij}_{\alpha\beta}(A)$ are satisfying the conditions

$$M^{ij}_{\alpha\beta}(A) = -(-1)^{\varepsilon_i\varepsilon_j}M^{ji}_{\alpha\beta}(A) = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}M^{ij}_{\beta\alpha}(A)\,.$$

Configuration space

$$\phi^A = (A^i, ...), \quad \varepsilon(\phi^A) = \varepsilon_A, \quad gh(A^i) = 0$$

Antifields

$$\phi^*_A = \Big(A^*_i, ...\Big), \quad \varepsilon(\phi^*_A) = \varepsilon_A + 1, \quad gh(\phi^*_A) = -1 - gh(\phi^A)$$

Antibracket

$$(F, G) \;\equiv\; \frac{\delta F}{\delta \phi^A} \frac{\delta G}{\delta \phi^*_A} - (F \leftrightarrow G)\, (-1)^{(\varepsilon(F)+1)\cdot(\varepsilon(G)+1)} \; .$$

$$\varepsilon((F, G)) = \varepsilon(F) + \varepsilon(G) + 1;$$

Properties of antibracket

1) Generalized antisymmetry

$$(F,G) = -(G,F)(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)};$$

2) Leibniz rule

$$(F,GH) = (F,G)H + (F,H)G(-1)^{\varepsilon(G)\varepsilon(H)};$$

3) Generalized Jacobi identity

$$((F,G),H)(-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \mathsf{cycle}(F,G,H) \equiv 0\,.$$

4) Invariance under the *anticanonical* transformation generated by an odd functional $X = X(\phi,\phi^*)$, $\varepsilon(X) = 1$,

$$\phi'^{A} = \frac{\delta X(\phi,\phi^{*'})}{\delta \phi^{*'}_A}, \quad \phi^*_A = \frac{\delta X(\phi,\phi^{*'})}{\delta \phi^A}.$$

BV quantization: main features

Δ-operator

$$\Delta = (-1)^{\varepsilon_A} \frac{\delta_l}{\delta \phi^A} \, \frac{\delta}{\delta \phi^*_A} \,, \qquad \Delta^2 = 0 \,, \quad \varepsilon(\Delta) = 1 \,.$$

$$\Delta[F \cdot G] = (\Delta F) \cdot G + F \cdot (\Delta G)(-1)^{\varepsilon(F)} + (F, G)(-1)^{\varepsilon(F)} \,.$$

Quantum master equation

$$\frac{1}{2}(S, S) = i\hbar \Delta S \quad \Leftrightarrow \quad \Delta \exp\left\{\frac{i}{\hbar} S\right\} = 0, \quad \varepsilon(S) = gh(S) = 0$$

$$S|_{\phi^* = \hbar = 0} = S_0(A).$$

BV quantization: main features

Generating functional of Green's functions

$$Z(J) = \int d\phi \exp\Big\{\frac{i}{\hbar}\big[S_{eff}(\phi) + J_A\phi^A\big]\Big\},$$

$$S_{eff}(\phi) = S\Big(\phi,\, \phi^* \,=\, \frac{\delta\Psi}{\delta\phi}\Big)$$

$\Psi = \Psi(\phi)$ is a fermionic gauge functional.
The gauge-fixing procedure can be described in terms of anticanonical transformation of the variables ϕ, ϕ^* in $S(\phi, \phi^*)$ with the generating functional X

$$X(\phi, \phi^*) = \phi^*_A\phi^A + \Psi(\phi).$$

BV quantization: main features

BRST symmetry

$$Z(J) = \int d\phi\, d\phi^*\, d\lambda \exp\left\{\frac{i}{\hbar}\left[S(\phi,\phi^*) + \left(\phi^*_A - \frac{\delta\Psi}{\delta\phi^A}\right)\lambda^A + J_A\phi^A\right]\right\}$$

λ^A ($\varepsilon(\lambda^A) = \varepsilon_A + 1$) are the auxiliary (Nakanishi-Lautrup) fields.

$$\delta\phi^A = \lambda^A\mu\,, \quad \delta\phi^*_A = \mu\frac{\delta S}{\delta\phi^A}\,, \quad \delta\lambda^A = 0\,.$$

Gauge invariance of S-matrix

$$Z_{\Psi+\delta\Psi} = Z_\Psi \quad (Z_\Psi = Z(0))$$

Ward identity

Extended generating functional of the Green functions

$$\mathcal{Z}(J,\phi^*) = \int d\phi \exp\Big\{\frac{i}{\hbar}\big[S_{ext}(\phi,\phi^*) + J_A\phi^A\big]\Big\},$$

$$S_{ext}(\phi,\phi^*) = S\Big(\phi,\ \phi^* + \frac{\delta\Psi}{\delta\phi}\Big), \qquad \frac{1}{2}(S_{ext},S_{ext}) = i\hbar\Delta S_{ext}.$$

$$\mathcal{Z}(J,\phi^*)|_{\phi^*=0} = Z(J),$$

$$J_A\frac{\delta\mathcal{Z}}{\delta\phi^*_A} = 0$$

$$(\Gamma,\Gamma) = 0, \quad \Gamma = \Gamma(\phi,\phi^*)$$

$$\Gamma(\phi,\ \phi^*) = \frac{\hbar}{i} ln\mathcal{Z}(J,\phi^*) - J_A\phi^A\,, \quad \phi^A = \frac{\hbar}{i\mathcal{Z}}\frac{\delta\mathcal{Z}}{\delta J_A}\,, \quad \frac{\delta\Gamma}{\delta\phi^A} = -J_A\,.$$

BV quantization: main features

Gauge dependence

$$\Psi \rightarrow \Psi + \delta\Psi$$

$$\delta\Gamma = (<\delta\Psi>, \Gamma) = \frac{\delta\Gamma}{\delta\phi^A}\delta Y^A$$

Gauge invariant renormalizability

$$\frac{1}{2}(S,S) = i\hbar\Delta S, \quad (\Gamma,\Gamma) = 0$$

$$S \rightarrow S_R, \quad \Gamma \rightarrow \Gamma_R$$

$$\frac{1}{2}(S_R,S_R) = i\hbar\Delta S_R, \quad (\Gamma_R,\Gamma_R) = 0$$

Action

Consider a theory of gauge fields A^i in an external gravitational field $g_{\mu\nu}$.

$$S_0 = S_0(A, g)$$

gauge invariance

$$S_{0,i}R^i_a = 0, \quad \delta A^i = R^i_a(A, g)\lambda^a\,, \quad \lambda^a = \lambda^a(x) \quad (a = 1, 2, ..., n)\,,$$

general covariance

$$\delta_g S_0 = \frac{\delta S_0}{\delta A^i}\delta_g A^i + \frac{\delta S_0}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} = 0$$

$$\begin{aligned}\delta_g g_{\mu\nu} &= -g_{\mu\alpha}\partial_\nu\xi^\alpha - g_{\nu\alpha}\partial_\mu\xi^\alpha - \partial_\alpha g_{\mu\nu}\xi^\alpha = \\ &= -\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu\end{aligned}$$

ξ^α are the parameters of the coordinates transformation,
$\xi^\alpha = \xi^\alpha(x) \quad (\alpha = 1, 2, ..., d)\,.$

Generating functional of the Green functions

$$\mathcal{Z}(J,\phi^*,g)=\int d\phi\,\exp\Big\{\frac{i}{\hbar}\Big[S_{ext}(\phi,\phi^*,g)+J_A\phi^A\Big]\Big\}$$

$$S_{ext}(\phi,\phi^*,g)=S\Big(\phi,\phi^*+\frac{\delta\Psi(\phi,g)}{\delta\phi},g\Big)$$

$$(S,S)=0\,,\quad S(\phi,\phi^*,g)|_{\phi^*=0}=S_0(A,g)$$

$$(S_{ext},S_{ext})=0.$$

General covariance

We assume the general covariance of $S = S(\phi, \phi^*, g)$, under arbitrary local transformations of coordinates $x^\mu \to x^\mu + \xi^\mu(x)$.

$$\delta_g S(\phi, \phi^*, g) = \frac{\delta S}{\delta \phi^A}\delta_g \phi^A + \delta_g \phi^*_A \frac{\delta S}{\delta \phi^*_A} + \frac{\delta S}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} = 0$$

Let us choose the gauge fixing functional $\Psi = \Psi(\phi, g)$ in a covariant form

$$\delta_g \Psi = 0\,,$$

then the quantum action $S_{ext} = S_{ext}(\phi, \phi^*, g)$ obeys the general covariance too

$$\delta_g S_{ext} = 0\,.$$

General covariance of generating functionals

$$\delta_g(J_A\phi^A) = (\delta_g J_A)\phi^A + J_A(\delta_g\phi^A) = 0\,,$$

$$\delta_g \mathcal{Z}(J,\phi^*,g) = 0$$

$$\delta_g \Gamma(\phi,\phi^*,g) == 0.$$

General covariance

The next step is to prove the general covariance for renormalized generating functionals.

$$\Gamma = S_{ext} + \bar{\Gamma}^{(1)} = S_{ext} + \hbar\big[\bar{\Gamma}^{(1)}_{div} + \bar{\Gamma}^{(1)}_{fin}\big] + O(\hbar^2)\,,$$

$\bar{\Gamma}^{(1)}_{div}$ and $\bar{\Gamma}^{(1)}_{fin}$ denote the divergent and finite parts of the one-loop approximation for Γ.

$$S_{ext} \to S_{ext1} = S_{ext} - \hbar\bar{\Gamma}^{(1)}_{div}.$$

$$\delta_g\big[\bar{\Gamma}^{(1)}_{div} + \bar{\Gamma}^{(1)}_{fin}\big] = 0 \quad \to \quad \delta_g\bar{\Gamma}^{(1)}_{div} = 0\,, \quad \delta_g\bar{\Gamma}^{(1)}_{fin} = 0$$

The one-loop renormalized action S_{ext1} (i.e., classical action, renormalized at the one-loop level) is covariant

$$\delta_g S_{ext1} = 0\,.$$

Constructing the generating functional of one-loop renormalized Green functions $\mathcal{Z}_1(J,\phi^*,g)$, with the action $S_{ext1}=S_{ext1}(\phi,\phi^*,g)$, we arrive at the relation

$$\delta_g\mathcal{Z}_1=0\,,\,,\quad \delta_g\Gamma_1=0\,.$$

The generating functional of vertex functions $\Gamma_1=\Gamma_1(\phi,\phi^*,g)$ which is finite in the one-loop approximation, can be presented in the form

$$\Gamma_1=S_{ext}+\hbar\bar{\Gamma}^{(1)}_{fin}+\hbar^2\big[\bar{\Gamma}^{(2)}_{1,div}+\bar{\Gamma}^{(2)}_{1,fin}\big]+O(\hbar^3)\,.$$

This functional contains a divergent part $\bar{\Gamma}^{(2)}_{1,div}$ and defines renormalization of the action S_{ext} in the two-loop approximation

$$S_{ext}\rightarrow S_{ext2}=S_{ext1}-\hbar^2\bar{\Gamma}^{(2)}_{1,div}\,.$$

$$\delta_g \bar{\Gamma}^{(2)}_{1,div} \,=\, 0\,, \quad \delta_g \bar{\Gamma}^{(2)}_{1,fin} = 0 \quad \delta_g S_{ext2} = 0.$$

Applying the induction method we arrive at the following results:

a) The full renormalized action,

$$S_{extR} \,=\, S_{ext} - \sum_{n=1}^{\infty} \hbar^n \bar{\Gamma}^{(n)}_{n-1,div}\,,$$

which is local in each finite order in $\hbar$, obeys the general covariance

$$\delta_g S_{extR} \,=\, 0\,;$$

b) The renormalized generating functional of vertex functions

$$\Gamma_R \,=\, S_{ext} + \sum_{n=1}^{\infty} \hbar^n \bar{\Gamma}^{(n)}_{n-1,fin}\,,$$

which is finite in each finite order in $\hbar$, is covariant

$$\delta_g \Gamma_R \,=\, 0\,.$$

Let us investigate the problem of general covariant renormalizability for general gauge theories in the presence of an external gravitational field, when one uses non-covariant gauge fixing functional $\Psi = \Psi(\phi, g)$,

$$\delta_g \Psi \neq 0\,.$$

Non-covariance of S_{ext} can be described in the form of anticanonical infinitesimal transformation with the odd generating functional

$$X(\phi, \phi^*, g) = \phi^*_A \phi^A + \delta_g \Psi(\phi, g)\,,$$

$$\Phi^A = \frac{\delta X(\phi', \phi^*, g)}{\delta \phi^*_A} = \Phi^{A'}, \quad \phi^{*'}_A = \frac{\delta X(\phi', \phi^*, g)}{\delta \phi^{'}_A} = \phi^*_A + \frac{\delta \delta_g \Psi}{\delta \phi^A}\,,$$

$$\delta_g S_{ext} = \frac{\delta \delta_g \Psi}{\delta \phi^A} \frac{\delta S_{ext}}{\delta \phi^*_A} = (\delta_g \Psi, S_{ext})\,.$$

Non-covariant gauges

$$\delta_g \mathcal{Z} = \frac{i}{\hbar} J_A \frac{\delta}{\delta \phi^*_A} \delta_g \Psi\Big(\frac{\hbar}{i}\frac{\delta}{\delta J}, g\Big) \mathcal{Z}\,,$$

$$\delta_g \Gamma = (\langle\langle \delta_g \Psi \rangle\rangle, \Gamma)\,,$$

$$\langle\langle \delta_g \Psi \rangle\rangle = \delta_g \Psi\Big(\phi + i\hbar (\Gamma^{''})^{-1} \frac{\delta_l}{\delta \phi}, g\Big)\,,$$

$$\Gamma^{''}_{AB} = \frac{\delta_l}{\delta \phi^A}\frac{\delta}{\delta \phi^B}\Gamma.$$

These results can be immediately reproduced in the renormalized theory. The corresponding variation of renormalized action $\delta_g S_{extR}$ can be presented in the form

$$\delta_g S_{extR} = (\delta_g \Psi_R, S_{extR})$$

of the anticanonical transformation with local generating functional $X = \phi^*_A \phi^A + \delta_g \Psi_R$,

$$\delta_g \Psi_R(\phi, \phi^*, g) = \delta_g \Psi(\phi, g) - \sum_{n=1}^{\infty} \hbar^n \delta_g \Psi^{(n)}_{n-1,div}(\phi, \phi^*, g)\,,$$

while the variation of renormalized vertex generating functional $\delta_g \Gamma_R$ has the form

$$\delta_g \Gamma_R = (\langle\langle \delta_g \Psi_R \rangle\rangle_R, \Gamma_R),$$

which corresponds to finite anticanonical transformation with generating function

$$X = \phi^*_A \phi^A + \langle\langle \delta_g \Psi_R \rangle\rangle_R, \quad \langle\langle \delta_g \Psi_R \rangle\rangle_R = \delta_g \Psi(\phi, g) + \sum^{\infty} \hbar^n \delta_g \Psi^{(n)}_{n-1,fin}.$$

The interpretation of these relations is that the theory with external gravitational field may have non-covariance in the renormalized effective action, but it comes only from the possible non-covariance of the arguments. Therefore, the violation of the general coordinate symmetry which can occur because of the non-covariant gauge-fixing can be always included into the arguments.

Conclusion

It is proved that for general gauge theories in covariant gauges

- the full renormalized action, S_{extR} being local in each finite order in $\hbar$, obeys the general covariance and
- the renormalized generating functional of vertex functions Γ_R which is finite in each finite order in $\hbar$, is covariant.
- It is proved that in non-covariant gauges the gauge dependence of the full renormalized action, S_{extR}, and the renormalized generating functional of vertex functions Γ_R is described in terms of anticanonical transformations of variables of configuration space.

Thank you!