# Surprises in AdS

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SQS'2011, JINR, Dubna, 18 July, 2011

Based on: Daniel Butter and SMK
*$\mathcal{N}=2$ AdS supergravity and supercurrents*, arXiv:1104.2153
*$\mathcal{N}=2$ supersymmetric $\sigma$-models in AdS*, arXiv:1105.3111
*A dual formulation of supergravity-matter theories*, arXiv:1106.3038

# Outline

# 4D $\mathcal{N} = 1$ AdS superspace

- $\mathcal{N} = 1$ AdS superspace as a coset space (nonlinear realizations etc.)

$$\mathrm{AdS}^{4|4} = \mathrm{OSp}(1|4)/\mathrm{SO}(3,1)$$

Keck (75)
Zumino (77)
Ivanov & Sorin (80)

- $\mathcal{N} = 1$ AdS superspace as a maximally symmetric solution of old minimal supergravity with a cosmological term.

Gates, Grisaru, Roček & Siegel, *Superspace* (83)
Buchbinder & SK, *Ideas & Methods of SUSY & SUGRA* (95)

# 4D $\mathcal{N}=1$ AdS superspace

AdS supergravity action

$$S=-3\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, E+\mu\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}+\bar\mu\int \mathrm{d}^4x\mathrm{d}^2\bar\theta\,\bar{\mathcal{E}}$$

Covariant derivatives

$$\mathcal{D}_{\rm A}=(\mathcal{D}_{\rm a},\mathcal{D}_\alpha,\bar{\mathcal{D}}^{\dot\alpha})=E_{\rm A}{}^M\partial_M+\frac{1}{2}\phi_{\rm A}{}^{\rm bc}M_{\rm bc}$$

$\mathcal{N}=1$ AdS superspace is characterized by the following algebra of covariant derivatives ($\mu=\mathrm{const}$):

$$\{\mathcal{D}_\alpha,\mathcal{D}_\beta\}=-4\bar\mu M_{\alpha\beta}\ ,\qquad \{\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\beta}\}=-2\mathrm{i}(\sigma^{\rm c})_{\alpha\dot\beta}\mathcal{D}_{\rm c}$$

$$[\mathcal{D}_{\rm a},\mathcal{D}_\beta]=-\frac{\mathrm{i}}{2}\bar\mu(\sigma_{\rm a})_{\beta\dot\gamma}\bar{\mathcal{D}}^{\dot\gamma}\ ,\qquad [\mathcal{D}_{\rm a},\mathcal{D}_{\rm b}]=-|\mu|^2M_{\rm ab}$$

# Massless higher spin off-shell supermultiplets in AdS

Two dually equivalent off-shell formulations
for arbitrary superspin $s \geq 1$

SMK & Sibiryakov (94)

Dynamical variables include transverse and longitudinal linear superfields
*Transverse linear* superfield $\Gamma_{\alpha(k)\dot\alpha(l)}$

$$\bar{\mathcal{D}}^{\dot\beta}\,\Gamma_{\alpha(k)\dot\beta\dot\alpha(l-1)} = 0 \ , \qquad l > 0$$

*Longitudinal linear* superfield $G_{\alpha(k)\dot\alpha(l)}$

$$\bar{\mathcal{D}}_{(\dot\alpha_1}\,G_{\alpha(k)\dot\alpha_2\dots\dot\alpha_{l+1})} = 0$$

Ivanov & Sorin (80)

# Massless higher spin off-shell supermultiplets in AdS

Massless multiplet of half-integer superspin $s+1/2$, $s=1,2\dots$

SMK & Sibiryakov (94)

Transverse formulation

$$\mathcal{V}^{\perp}_{s+1/2} = \Big\{ H_{\alpha(s)\dot\alpha(s)} \ , \ \Gamma_{\alpha(s-1)\dot\alpha(s-1)} \ , \ \bar\Gamma_{\alpha(s-1)\dot\alpha(s-1)} \Big\}$$

Longitudinal formulation

$$\mathcal{V}^{\parallel}_{s+1/2} = \Big\{ H_{\alpha(s)\dot\alpha(s)} \ , \ G_{\alpha(s-1)\dot\alpha(s-1)} \ , \ \bar G_{\alpha(s-1)\dot\alpha(s-1)} \Big\}$$

Both series terminate at $s=1$ (supergravity multiplet)

$$(\bar{\mathcal{D}}^2-4\mu)\Gamma = 0 \qquad \text{complex linear}$$
$$G=\phi\ , \qquad \bar{\mathcal{D}}_{\dot\alpha}\phi = 0 \qquad \text{chiral}$$

# Massless higher spin off-shell supermultiplets in AdS

Longitudinal formulation: massless superspin-3/2 multiplet

$$S^{||}_{(3/2)} = -\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\,\Big\{\frac{1}{16}H^{\dot\alpha\alpha}\mathcal{D}^\beta(\bar{\mathcal{D}}^2-4\mu)\mathcal{D}_\beta H_{\alpha\dot\alpha} - \frac{1}{48}([\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot\alpha}]H^{\dot\alpha\alpha})^2 + \frac{1}{4}(\mathcal{D}_{\alpha\dot\alpha}H^{\dot\alpha\alpha})^2 + \frac{1}{4}\mu\bar\mu H^{\dot\alpha\alpha}H_{\alpha\dot\alpha} + \mathrm{i}H^{\dot\alpha\alpha}\mathcal{D}_{\alpha\dot\alpha}(\phi-\bar\phi) + 3(\phi\bar\phi-\phi^2-\bar\phi^2)\Big\}$$

gauge invariance

$$\delta H_{\alpha\dot\alpha} = \mathcal{D}_\alpha\bar{L}_{\dot\alpha} - \bar{\mathcal{D}}_{\dot\alpha}L_\alpha\ , \qquad \delta\phi = -\frac{1}{12}(\bar{\mathcal{D}}^2-4R)\mathcal{D}^\alpha L_\alpha$$

This is exactly the linearized action of old minimal supergravity with a cosmological term.

# Massless higher spin off-shell supermultiplets in AdS

Transverse formulation: massless superspin-3/2 multiplet

$$S^{\perp}_{(3/2)} = -\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\,\Big\{\frac{1}{16}H^{\dot\alpha\alpha}\mathcal{D}^{\beta}(\bar{\mathcal{D}}^2 - 4\mu)\mathcal{D}_{\beta}H_{\alpha\dot\alpha} + \frac{1}{4}\mu\bar\mu H^{\dot\alpha\alpha}H_{\alpha\dot\alpha} + \frac{1}{2}H^{\alpha\dot\alpha}(\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\Gamma - \bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha\bar\Gamma) + \bar\Gamma\Gamma + \Gamma^2 + \bar\Gamma^2\Big\}$$

Gauge invariance

$$\delta H_{\alpha\dot\alpha} = \mathcal{D}_\alpha\bar{L}_{\dot\alpha} - \bar{\mathcal{D}}_{\dot\alpha}L_\alpha\ , \qquad \delta\Gamma = -\frac{1}{4}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}^2\bar{L}^{\dot\alpha}$$

In the Minkowski limit ($\mu \to 0$), $S^{\perp}_{(3/2)}$ arises as a linearization of non-minimal ($n = -1$) supergravity.

- General incorrect belief (*Superspace*, 1983 – June 2011): Non-minimal supergravity does not allow a supersymmetric AdS solution $\Longrightarrow$
  $S^{\perp}_{(3/2)}$ cannot be obtained by linearizing a non-minimal supergravity action around a supersymmetric AdS solution.

# $\mathcal{N} = 1$ supercurrents in AdS

- Supercurrent is a supermultiplet containing the energy-momentum tensor and the supersymmetry current(s).

  Ferrara & Zumino (75)

- Supercurrent is the source of supergravity.

  Ogievetsky & Sokatchev (77)

- Consistent $\mathcal{N} = 1, 2$ supercurrents can be associated with the linearized off-shell supergravity actions.

  SMK (2010); Butter & SMK (2011)

- AdS supercurrents are more restrictive than those corresponding to the Poincaré supersymmetry. There are only two irreducible AdS supercurrents. They are related to each other by a well-defined improvement transformation.

  Butter & SMK (2011)

# $\mathcal{N} = 1$ supercurrents

- Revival of interest to supercurrent multiplets was inspired by
  Komargodski & Seiberg (2009,10)
- One of their main results was the observation that the Ferrara-Zumino supercurrent multiplet

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = D_\alpha X \ , \qquad \bar{D}_{\dot{\alpha}} X = 0$$

  is not well defined in some supersymmetric field theories.
  Ferrara & Zumino (75)
- In particular, they showed that FZ- multiplet does not exist in the case of $\sigma$-models with a Kähler form that is not exact, or for theories with non-zero Fayet-Iliopoulos terms.
- Related studies
  Dienes &Thomas (2009)
  SMK (2009,10)

# $\mathcal{N} = 1$ supercurrents in AdS

Minimal $(12 + 12)$ supercurrent
(AdS extension of the Ferrara-Zumino multiplet)

$$\bar{\mathcal{D}}^{\dot\alpha} J_{\alpha\dot\alpha} = \mathcal{D}_\alpha X \ , \qquad \bar{\mathcal{D}}_{\dot\alpha} X = 0$$

Supercurrent $J_{\alpha\dot\alpha}$ and trace multiplet $X$ are defined as

$$J_{\alpha\dot\alpha} = \frac{\delta S}{\delta H^{\dot\alpha\alpha}} \ , \qquad X = -\frac{1}{3}\frac{\delta S}{\delta \phi}$$

Non-minimal $(20 + 20)$ supercurrent

$$\bar{\mathcal{D}}^{\dot\alpha} \mathbb{J}_{\alpha\dot\alpha} = -\frac{1}{4}\bar{\mathcal{D}}^2 \zeta_\alpha \ , \qquad \mathcal{D}_{(\beta}\zeta_{\alpha)} = 0$$

where $\zeta_\alpha$ is calculated by $(\bar\Gamma = \mathcal{D}^\alpha \psi_\alpha + \cdots)$

$$\zeta_\alpha = \frac{\delta S}{\delta \psi^\alpha} = -\mathcal{D}_\alpha \frac{\delta S}{\delta \bar\Gamma} = \mathcal{D}_\alpha (V + \mathrm{i}\, U)$$

Improvement transformation

$$J_{\alpha\dot\alpha} = \mathbb{J}_{\alpha\dot\alpha} + \frac{1}{6}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}]V - \mathcal{D}_{\alpha\dot\alpha} U \ , \qquad X = \frac{1}{12}(\bar{\mathcal{D}}^2 - 4\mu)(V - 3\mathrm{i}\, U)$$

# $\mathcal{N}=1$ supercurrents in Minkowski space

Most general $(20+20)$ supercurrent multiplet in field theory:

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} = \chi_\alpha + \mathrm{i}\,\eta_\alpha + D_\alpha X$$

$$\bar{D}_{\dot\alpha}\chi_\alpha = \bar{D}_{\dot\alpha}\eta_\alpha = \bar{D}_{\dot\alpha} X = 0 \ , \qquad D^\alpha\chi_\alpha - \bar{D}_{\dot\alpha}\bar\chi^{\dot\alpha} = D^\alpha\eta_\alpha - \bar{D}_{\dot\alpha}\bar\eta^{\dot\alpha} = 0$$

$J_{\alpha\dot\alpha}$ supercurrent; $X$, $\chi_\alpha$ and $\eta_\alpha$ trace multiplets

SMK (2010)

Somewhat different form of the conservation equation, based on the Noether procedure, was proposed earlier:

Magro, Sachs & Wolf (2002)

The three terms in r.h.s. emphasize the fact that there exist exactly three linearized actions for minimal $(12+12)$ supergravity

Gates, SMK & Phillips (2003)

- Only one minimal $(12+12)$ action exists in AdS

# $\mathcal{N} = 1$ supercurrents in Minkowski space

$(20 + 20)$ Supercurrent

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} = \chi_\alpha + \mathrm{i}\,\eta_\alpha + D_\alpha X$$
$$\bar{D}_{\dot\alpha}\chi_\alpha = \bar{D}_{\dot\alpha}\eta_\alpha = \bar{D}_{\dot\alpha} X = 0 \,, \qquad D^\alpha\chi_\alpha - \bar{D}_{\dot\alpha}\bar\chi^{\dot\alpha} = D^\alpha\eta_\alpha - \bar{D}_{\dot\alpha}\bar\eta^{\dot\alpha} = 0$$

$(12 + 12)$ Supercurrents

- $\chi_\alpha = 0$ & $\eta_\alpha = 0$ Ferrara-Zumino multiplet
  Old minimal SUGRA [Wess & Zumino (78); Stelle & West (78); Ferrara & van Nieuwenhuizen (78)]
- $X = 0$ & $\eta_\alpha = 0$ $\mathcal{R}$-multiplet
  New minimal SUGRA [Akulov, Volkov & Soroka (77); Sohnius & West (81)]
- $X = 0$ & $\chi_\alpha = 0$
  'Very new' min SUGRA [Buchbinder, Gates, Linch & Phillips (2002)]

$(16 + 16)$ Supercurrent (s)

- $\eta_\alpha = 0$ $\mathcal{S}$-multiplet
  Komargodski & Seiberg (2010)

# $\mathcal{N} = 1$ supercurrents in Minkowski space

Non-minimal $(20 + 20)$ supercurrent

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} = -\frac{1}{4}\bar{D}^2\zeta_\alpha - \frac{1}{4}\frac{n+1}{3n+1} D_\alpha \bar{D}_{\dot\beta}\bar{\zeta}^{\dot\beta} , \qquad D_{(\alpha}\zeta_{\beta)} = 0$$

where $n \neq -1/3, 0$ pararmetrizes the different versions of non-minimal supergravity

Breitenlohner (77); Gates, Grisaru, Roček & Siegel, *Superspace* (83)

$$\zeta_\alpha = D_\alpha Z \quad \Longrightarrow \quad -\frac{1}{4}\frac{n+1}{3n+1} D_\alpha \bar{D}_{\dot\beta}\bar{\zeta}^{\dot\beta} = \chi_\alpha + \mathrm{i}\eta_\alpha$$

Only the case $n = -1$ is allowed in AdS

- The Ferrara-Zumino multiplet exist for most theories.
- It has been argued that the $\mathcal{S}$-multiplet (i) exists always; and (ii) is most general.
  Komargodski & Seiberg (2010); Dumitrescu & Seiberg (2011)
- The $\mathcal{S}$-multiplet does not have any natural extension to AdS.
  Butter & SMK (2011)

# $\mathcal{N}=1$ supersymmetric $\sigma$-models in AdS

General $\mathcal{N}=1$ supersymmetric nonlinear $\sigma$-model in Minkowski space

$$S=\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, K(\phi,\bar\phi)+\int \mathrm{d}^4x\mathrm{d}^2\theta\, W(\phi)+\int \mathrm{d}^4x\mathrm{d}^2\bar\theta\, \bar W(\bar\phi)$$

$K(\phi^a,\bar\phi^{\bar b})$ Kähler potential defined modulo Kähler transformations

$$K(\phi,\bar\phi)\to K(\phi,\bar\phi)+F(\phi)+\bar F(\bar\phi)$$

$W(\phi^a)$ superpotential, invariant under the Kähler transfromations.
The $\sigma$-model target space, $\mathcal{M}$, is an arbitrary Kähler manifold.

Zumino (79)

- Crucial difference of AdS superspace from flat superspace:

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\, F(\phi)=\mu\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathcal{E}\, F(\phi)\neq 0$$
$$\Longrightarrow \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathcal{E}\, W(\phi)=\frac{1}{\mu}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\, W(\phi)$$

# $\mathcal{N}=1$ supersymmetric $\sigma$-models in AdS

General $\mathcal{N}=1$ nonlinear $\sigma$-model in AdS

$$S=\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,E\,\mathcal{K}(\phi^a,\bar{\phi}^{\bar{b}})\ ,\qquad \bar{\mathcal{D}}_{\dot{\alpha}}\phi^a=0$$

Unlike in the Minkowski case, the action does not possess Kähler invariance. Nevertheless, Kähler invariance naturally emerges if we represent the Lagrangian as

$$\mathcal{K}(\phi,\bar{\phi})=K(\phi,\bar{\phi})+\frac{1}{\mu}W(\phi)+\frac{1}{\bar{\mu}}\bar{W}(\bar{\phi})\ ,$$

for some Kähler potential $K$ and superpotential $W$. Under a Kähler transformation, these transform as

$$K(\phi,\bar{\phi})\rightarrow K(\phi,\bar{\phi})+F(\phi)+\bar{F}(\bar{\phi}),\qquad W(\phi)\rightarrow W(\phi)-\mu F(\phi)$$

The Kähler metric

$$g_{a\bar{b}}:=\partial_a\partial_{\bar{b}}\mathcal{K}=\partial_a\partial_{\bar{b}}K$$

is invariant under the Kähler transformations.

# $\mathcal{N} = 1$ supersymmetric $\sigma$-models in AdS

## Implications

The $\sigma$-model couplings in AdS are more restrictive than in flat space.

- Because of

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, E\, F(\phi) = \mu \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathcal{E}\, F(\phi) \neq 0$$

  the $\sigma$-model Lagrangian $\mathcal{K}(\phi, \bar{\phi})$ should be a globally defined function on the target space $\mathcal{M}$.

- The Kähler two-form

$$\Omega = \mathrm{i}\, g_{a\bar{b}}\, \mathrm{d}\phi^a \wedge \mathrm{d}\bar{\phi}^{\bar{b}}\ , \qquad g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K}$$

  is exact, and hence $\mathcal{M}$ is necessarily non-compact.

Adams, Jockers, Kumar & Lapan, arXiv:1104.3155
Festuccia & Seiberg, arXiv:1105.0689
Butter & SMK, arXiv:1104.2153, arXiv:1105.3111

# $\mathcal{N} = 2$ supersymmetric $\sigma$-models in AdS

- Poincaré supersymmetry: The most general 4D $\mathcal{N} = 2$ nonlinear $\sigma$-models were constructed in terms of $\mathcal{N} = 1$ chiral superfields 25 years ago
  Hull, Karlhede, Lindström & Roček (86)
- Extension to 5 dimensions (using 4D $\mathcal{N} = 1$ chiral superfields)
  Bagger & Xiong (2006)
- Extension to 6 dimensions (using 4D $\mathcal{N} = 1$ chiral superfields)
  Gates, Penati & Tartaglino-Mazzucchelli (2006)
- AdS supersymmetry: the most general 4D $\mathcal{N} = 2$ nonlinear $\sigma$-models in terms of $\mathcal{N} = 1$ chiral superfields
  Butter & SMK arXiv:1105.3111
- Independent analysis in five dimensions
  Bagger & Xiong, arXiv:1105.4852

# $\mathcal{N}=2$ supersymmetric $\sigma$-models in AdS

- Start from general $\mathcal{N}=1$ nonlinear $\sigma$-model in AdS

$$S=\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\,\mathcal{K}(\phi^a,\bar{\phi}^{\bar{b}})\;, \qquad \bar{\mathcal{D}}_{\dot\alpha}\phi^a=0$$

and look for those restrictions on the target space geometry which guarantee that the theory is $\mathcal{N}=2$ supersymmetric

- Ansatz for second supersymmetry

$$\delta_\varepsilon\phi^a=\frac{1}{2}(\bar{\mathcal{D}}^2-4\mu)(\varepsilon\bar{\Omega}^a)\;, \qquad \bar{\Omega}^a=\bar{\Omega}^a(\phi,\bar{\phi})$$

- Supersymmetry parameter

$$\bar{\varepsilon}=\varepsilon\;, \quad (\bar{\mathcal{D}}^2-4\mu)\varepsilon=\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha\varepsilon=0 \quad \Longrightarrow \quad \mathcal{D}_{\alpha\dot\alpha}\varepsilon=0$$

Gates, SMK & Sibiryakov (97)

# $\mathcal{N} = 2$ supersymmetric $\sigma$-models in AdS

The $\theta$-dependent parameter $\varepsilon$ contains two components:

- a bosonic parameter $\xi$ which is defined by $\varepsilon|_{\theta=0} = \xi|\mu|^{-1}$ and describes the O(2) rotations;
- a fermionic parameter $\epsilon_\alpha := \mathcal{D}_\alpha \varepsilon|_{\theta=0}$ along with its conjugate (Killing spinor), which generate the second supersymmetry.

$$\mathrm{OSp}(2|4) \to \mathrm{OSp}(1|4)$$

Schematically, $\varepsilon$ looks like

$$\varepsilon \sim \frac{\xi}{|\mu|} + \epsilon^\alpha \theta_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} - \xi \Big( \frac{\bar{\mu}}{|\mu|} \theta^2 + \frac{\mu}{|\mu|} \bar{\theta}^2 \Big)$$

$|\mu|^2$ AdS curvature

# $\mathcal{N}=2$ supersymmetric $\sigma$-models in AdS

Conditions for $\mathcal{N}=2$ supersymmetry:

- The existence of a covariantly constant two-form

$$\omega_{ab} := g_{a\bar{b}}\,\Omega^{\bar{b}}{}_{,b} = -\omega_{ba}$$
$$\nabla_c\,\omega_{ab} = \nabla_{\bar{c}}\,\omega_{ab} = 0 \quad \Longrightarrow \quad \omega_{ab} = \omega_{ab}(\phi)\ ;$$

$\omega_{ab}$ holomorphic two-form

- The existence of a certain Killing vector field

$$V^a := \mu\,\omega^{ab}\mathcal{K}_b\ , \qquad V^{\bar{a}} := \bar{\mu}\,\bar{\omega}^{\bar{a}\bar{b}}\mathcal{K}_{\bar{b}}$$
$$\nabla_a V_{\bar{b}} + \nabla_{\bar{b}} V_a = 0\ , \qquad \nabla_a V_b = -\bar{\mu}\,\omega_{ab} = -\nabla_b V_a$$

The first condition occurs both in the Minkowski and AdS cases. The second condition is characteristic of AdS supersymmetry only.

# $\mathcal{N}=2$ supersymmetric $\sigma$-models in AdS

$\mathrm{OSp}(1|4)$ isometries of $\mathrm{AdS}^{4|4}$ are generated by Killing vector fields defined as

$$\Lambda = \lambda^{\rm a}\mathcal{D}_{\rm a} + \lambda^\alpha \mathcal{D}_\alpha + \bar\lambda_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} ~, \qquad [\Lambda + \frac{1}{2}\omega^{\rm bc}M_{\rm bc}, \mathcal{D}_{\rm A}] = 0$$

for some Lorentz transformation $\omega^{\rm bc}$, $\omega_{\alpha\beta} = \mathcal{D}_{(\alpha}\lambda_{\beta)}$.

Closure of the supersymmetry algebra

- $\mathrm{OSp}(1|4)$ transformation of the chiral fields

$$\delta_\Lambda \phi^a = \Lambda\phi^a$$

- commutator of two second SUSY transformations

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}]\phi^a = -\omega^{ac}\omega_{cb}\Lambda_{[\varepsilon_2,\varepsilon_1]}\phi^b \quad \Longrightarrow \quad \boxed{\omega^{ac}\omega_{cb} = -\delta^a{}_b}$$

  where $\tilde\lambda^{\alpha\dot\alpha} := 4{\rm i}\big(\varepsilon_2^\alpha\bar\varepsilon_1^{\dot\alpha} - \varepsilon_1^\alpha\bar\varepsilon_2^{\dot\alpha}\big)$ and $\tilde\lambda^\alpha := 2\mu\big(\varepsilon_1^\alpha\varepsilon_2 - \varepsilon_2^\alpha\varepsilon_1\big)$

- commutator of $\mathrm{OSp}(1|4)$ transformation and second SUSY

$$[\delta_\Lambda, \delta_\varepsilon]\phi^a = -\delta_{\Lambda\varepsilon}\phi^a$$

The algebra of $\mathrm{OSp}(2|4)$ transformations closes off the mass shell

# $\mathcal{N} = 2$ supersymmetric $\sigma$-models in AdS

The implications of

$$\omega^{ac}\omega_{cb} = -\delta^a{}_b$$

are the same as in the super-Poincaré case

Hull, Karlhede, Lindström & Roček (86)

In addition to the canonical complex structure

$$J_3 = \begin{pmatrix} \mathrm{i}\,\delta^a{}_b & 0 \\ 0 & -\mathrm{i}\,\delta^{\bar a}{}_{\bar b} \end{pmatrix},$$

we may construct two more integrable complex structures

$$J_1 = \begin{pmatrix} 0 & \omega^a{}_{\bar b} \\ \omega^{\bar a}{}_b & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & \mathrm{i}\,\omega^a{}_{\bar b} \\ -\mathrm{i}\,\omega^{\bar a}{}_b & 0 \end{pmatrix}$$

such that $\mathcal{M}$ is Kähler with respect to each of them. The operators $J_A = (J_1, J_2, J_3)$ obey the quaternionic algebra

$$J_A J_B = -\delta_{AB}\mathbb{I} + \varepsilon_{ABC} J_C$$

Target space is hyperkähler manifold

# $\mathcal{N}=2$ supersymmetric $\sigma$-models in AdS

## Geometry of $\mathcal{N}=2$ AdS $\sigma$-models

- AdS supersymmetry demands the existence of the vector field

$$V^{\nu}=(V^{a},V^{\bar a})~,\qquad V^{a}=\mu\omega^{ab}\mathcal{K}_{b}~,\qquad V^{\bar a}=\bar\mu\bar\omega^{\bar a\bar b}\mathcal{K}_{\bar b}$$

which obeys the Killing equations

$$\nabla_a V_b+\nabla_b V_a=\nabla_a V_{\bar b}+\nabla_{\bar b}V_a=0$$

- $V$ is not holomorphic with respect to $J_3$.
- However, $V$ *is* holomorphic with respect to a certain linear combination of $J_1$ and $J_2$, namely

$$J:=\frac{\mathrm{Re}~\mu}{|\mu|}J_1+\frac{\mathrm{Im}~\mu}{|\mu|}J_2=\frac{1}{|\mu|}\begin{pmatrix}0 & \mu\,\omega^{a}{}_{\bar b}\\ \bar\mu\,\omega^{\bar a}{}_{b} & 0\end{pmatrix}~,\quad J^2=-\mathbb{1}$$

# Non-minimal AdS supergravity

Traditional approach to non-minimal supergravity

Breitenlohner (77); Siegel & Gates (79)

- Any $\mathcal{N}=1$ supergravity-matter system, including the new minimal ($n=0$) and non-minimal ($n \neq -1/3, 0$) supergravity theories, can be realized as a super-Weyl invariant coupling of the old minimal supergravity to matter.

  Ferrara, Girardello, Kugo & Van Proeyen (83)

  Buchbinder, SMK & Soloviev (89)

- To describe the non-minimal supergravity, the old minimal supergravity is coupled to a complex linear superfield $\Sigma$

$$(\bar{\mathcal{D}}^2 - 4R)\Sigma = 0$$

  with the super-Weyl transformation law

$$\delta_\sigma \Sigma = \Big[\frac{3n-1}{3n+1}\sigma - \bar{\sigma}\Big]\Sigma \ , \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\sigma = 0$$

- The super-Weyl invariant action for pure non-minimal supergravity is

$$S_{\text{non-minimal}} = \frac{1}{n}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\,(\Sigma\bar{\Sigma})^{(3n+1)/2}$$

# Non-minimal AdS supergravity

The traditional approach cannot be used to realize AdS supergravity
Gates, Grisaru, Roček & Siegel, *Superspace* (83)

New approach

Butter & SMK (2011)

- Only in the case $n = -1$, we can deform the linear constraint

$$-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Sigma = 0 \quad \longrightarrow \quad -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Gamma = W(\varphi)$$

with $W(\varphi)$ super-Weyl invariant matter superpotential

- General SUGRA-matter system

$$S = -3\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,E\,\phi\bar{\phi}\,\mathrm{e}^{-K/3} + \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathcal{E}\,\phi^3 W + \int \mathrm{d}^4x\,\mathrm{d}^2\bar{\theta}\,\bar{\mathcal{E}}\,\bar{\phi}^3\bar{W}$$

proves to be dual to

$$S_{\rm dual} = -\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,E\,\mathrm{e}^{-K}(\Gamma\bar{\Gamma})^{-1}$$

# Non-minimal AdS supergravity

- Improved linear constraint corresponding to AdS supergravity

$$-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Gamma = \mu = \mathrm{const}$$

- Minimal AdS supergravity action

$$S_{\mathrm{AdS,min}} = -3\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,E\,\phi\bar{\phi} + \mu\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathcal{E}\,\phi^3 + \bar{\mu}\int \mathrm{d}^4x\,\mathrm{d}^2\bar{\theta}\,\bar{\mathcal{E}}\,\bar{\phi}^3$$

is dual to

$$S_{\text{AdS,non-min}} = -\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,E\,(\Gamma\bar{\Gamma})^{-1}$$

- Linearizing $S_{\text{AdS,non-min}}$ around the AdS background gives transverse formulation for massless superspin-3/2 multiplet

$$S^{\perp}_{(3/2)} = -\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,E\,\Big\{\frac{1}{16}H^{\dot{\alpha}\alpha}\mathcal{D}^{\beta}(\bar{\mathcal{D}}^2 - 4\mu)\mathcal{D}_{\beta}H_{\alpha\dot{\alpha}} + \frac{1}{4}\mu\bar{\mu}H^{\dot{\alpha}\alpha}H_{\alpha\dot{\alpha}}$$
$$+\frac{1}{2}H^{\alpha\dot{\alpha}}(\mathcal{D}_{\alpha}\bar{\mathcal{D}}_{\dot{\alpha}}\Sigma - \bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_{\alpha}\bar{\Sigma}) + \bar{\Sigma}\Sigma + \Sigma^2 + \bar{\Sigma}^2\Big\}$$

# Summary (List of AdS surprises)

- The FZ supercurrent multiplet is the most general in AdS.
- The target spaces corresponding to arbitrary $\mathcal{N} = 1$ supersymmetric $\sigma$-models in AdS are characterized by exact Kähler forms.
- For arbitrary $\mathcal{N} = 2$ supersymmetric $\sigma$-models in AdS formulated in terms of $\mathcal{N} = 1$ chiral superfields, the algebra of $\mathrm{OSp}(2|4)$ transformations closes off the mass shell.
- In spite of the 28-year old belief, there exists non-minimal $\mathcal{N} = 1$ supergravity. (The way around the "no-go" analysis of *Superspace* is deformation of the linear constraint.)