SQM with non-Abelian self-dual fields

Maxim Konyushikhin

(with Evgeny Ivanov and Andrei Smilga)

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$\mathcal{N} = 4$ supersymmetric quantum mechanics

A particle in external self-dual gauge field:

$$H = \frac{1}{2}\left(\hat{p}_\mu - \mathcal{A}_\mu\right)^2 + \frac{i}{4}\,\mathcal{F}_{\mu\nu}\,\psi\sigma^\dagger_\mu\sigma_\nu\bar{\psi}$$

Supercharges:

$$Q_\alpha = \left(\sigma_\mu\bar{\psi}\right)_\alpha\left(\hat{p}_\mu - \mathcal{A}_\mu\right)$$

$$\bar{Q}^\alpha = \left(\psi\sigma^\dagger_\mu\right)^\alpha\left(\hat{p}_\mu - \mathcal{A}_\mu\right)$$

$$\{Q_\alpha, Q_\beta\} = 0, \qquad \left\{Q_\alpha, \bar{Q}^\beta\right\} = 2\delta^\beta_\alpha H$$

Notations:

$$\sigma_\mu = \{i, \vec{\sigma}\}, \qquad \sigma^\dagger_\mu = \{-i, \vec{\sigma}\}$$

$$\mu, \nu = 1, \ldots, 4 \qquad \alpha, \beta = 1, 2$$

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In spinorial notations

$$\mathrm{SO}(4) \simeq \mathrm{SU}(2)\times\mathrm{SU}(2), \qquad \mathcal{F}_{\mu\nu} \to \left\{\mathcal{F}_{\alpha\beta}, \mathcal{F}_{\dot\alpha\dot\beta}\right\}, \quad \mathcal{F}_{\dot\alpha\dot\beta} = 0$$

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E. Ivanov and O. Lechtenfeld, 2003

A. Kirchberg, J. D. Lange and A. Wipf, 2005

M. Konyushikhin and A. Smilga, 2009

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$\mathcal{F}_{\mu\nu}$ can have color indices (be a matrix)

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Add two complex auxiliary fields φ_α ($\mathrm{SU}(2)$ gauge group example):

$$\mathcal{A}_\mu \dot{x}_\mu \qquad \longrightarrow \qquad i\bar{\varphi}^\alpha \dot{\varphi}_\alpha + (\mathcal{A}_\mu)^\alpha_\beta \, \varphi_\alpha \bar{\varphi}^\beta \, \dot{x}_\mu$$

Quantization:

$$\left[\varphi_\alpha, \bar{\varphi}^\beta\right] = \delta^\beta_\alpha, \qquad T_a = \frac{1}{2}\bar{\varphi}\sigma_a\varphi, \qquad [T_a, T_b] = i\varepsilon_{abc} T_c$$

Representation fixing:

$$\bar{\varphi}^\alpha \varphi_\alpha = k, \qquad T_a T_a = \frac{k}{2}\left(\frac{k}{2}+1\right), \qquad k - \text{integer}$$

A. P. Balachandran, Per Salomonson, Bo-Sture Skagerstam, Jan-Olof Winnberg, 1977

S. Fedoruk, E. Ivanov, O. Lechtenfeld, 2009

How to write the Lagrangian in non-Abelian case

Basic idea:

$$\mathcal{A}_\mu \dot{x}_\mu \qquad \longrightarrow \qquad i\bar{\varphi}^\alpha \dot{\varphi}_\alpha + (\mathcal{A}_\mu)^\alpha_\beta \, \varphi_\alpha \bar{\varphi}^\beta \, \dot{x}_\mu$$

A. P. Balachandran, Per Salomonson, Bo-Sture Skagerstam, Jan-Olof Winnberg, 1977

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$\mathcal{N} = 4$ supersymmetric quantum mechanics (flat metric case):

$$L = \frac{1}{2}\, \dot{x}_\mu \dot{x}_\mu + i\bar{\psi}^{\dot{\alpha}} \dot{\psi}_{\dot{\alpha}} + i\bar{\varphi}^\alpha \left(\dot{\varphi}_\alpha + iB\varphi_\alpha\right) + kB$$

$$+ (\mathcal{A}_\mu)^\alpha_\beta \, \varphi_\alpha \bar{\varphi}^\beta \, \dot{x}_\mu - \frac{i}{4} \, (\mathcal{F}_{\mu\nu})^\alpha_\beta \, \varphi_\alpha \bar{\varphi}^\beta \, \psi \sigma^\dagger_\mu \sigma_\nu \bar{\psi}$$

E. Ivanov, M. Konyushikhin, A. Smilga, 2009

Harmonic superspace

Idea: extend space coordinates with R-symmetry group:

$$\mathbb{R}^4 \to \mathbb{R}^4 \times \mathrm{SU}(2)_\mathrm{R}$$

Harmonics – coordinates on 3-sphere:

$$u^{+\alpha} u^-_\alpha = 1, \qquad u^-_\alpha \equiv \left(u^{+\alpha}\right)^*, \qquad \alpha = 1,2$$

Superspace:

$$\left\{t, \theta_\alpha, \bar\theta^\beta, u^\pm_\gamma\right\} \qquad \longleftrightarrow \qquad \left\{t_\mathrm{A}, \theta^\pm, \bar\theta^\pm, u^\pm_\alpha\right\}$$

(standard basis) *(analytical basis)*

$$t_\mathrm{A} = t + i\left(\theta^+\bar\theta^- + \theta^-\bar\theta^+\right), \qquad \theta^\pm = u^\pm_\alpha \theta^\alpha, \qquad \bar\theta^\pm = u^\pm_\alpha \bar\theta^\alpha$$

A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, 1984

A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, *Harmonic Superspace*, 2001

E.Ivanov and O.Lechtenfeld, 2003

Superfield description involves two prepotentials:

- First term produces kinetic term with conformally flat metric:

$$S_{\rm kin} = \int dt\, d^4\theta du\, R(x,u) = \int dt\, \{ f^{-2}(x)\dot{x}_\mu \dot{x}_\mu + \ldots \}$$

- The second term produces non-Abelian self-dual field:

$$S_{\rm int} = -\frac{1}{2}\int dt\, d^2\theta du\, K\,(x,u,v)$$

$$= \int dt \left\{ i\bar{\varphi}^\alpha\,(\dot{\varphi}_\alpha + iB\varphi_\alpha) + (\mathcal{A}_\mu)_\beta^\alpha \varphi_\alpha \bar{\varphi}^\beta\, \dot{x}_\mu + \ldots \right\}$$

(v – auxiliary superfield for φ_α)

- Third term – Fayet–Iliopoulos term:

$$S_{\rm FI} = -\frac{ik}{2}\int dt\, d^2\theta du\, V^{++} = k\int dt\, B(t)$$

There is a relation between self-dual SU(2) gauge field $\mathcal{A}_\mu(x)$ and function $K(x, u, v)$

For example,

$$\int du\, d^2\theta K^{++}\left(x^{+\dot\alpha}, u^\pm_\alpha, v^+\widetilde{v^+}\right) = \int du\, d\bar\theta^+ d\theta^+\, K\left(x^{+\dot\alpha}, u^\pm_\alpha\right) v^+\widetilde{v^+}$$

gives t'Hooft *ansatz* form:

$$\mathcal{A}^a_\mu(x) = -\bar\eta^a_{\mu\nu}\partial_\nu \ln h(x), \qquad h(x) = \int du\, K\left(x^{+\dot\alpha}, u^\pm_\alpha\right)$$

$h(x)$ – arbitrary harmonic function, $\partial^2_\mu h(x) = 0$.

E. Ivanov, M. Konyushikhin, A. Smilga, '09

Why is it interesting?

More on harmonics: solution of self-duality equation

$$\mathcal{F}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\mathcal{F}^a_{\rho\lambda}$$

The solutions $(\mathcal{A}_{\alpha\dot{\alpha}})^i_j(x)$ are expressed through arbitrary function $(K^{++})^i_j\left(x^{+\dot{\alpha}}, u\right)$.

$$K^{++} = -ie^{-i\lambda}\partial^{++}e^{i\lambda} \qquad \longleftrightarrow \qquad \mathcal{A}^{\alpha}_{\dot{\alpha}}\, u^+_\alpha = -ie^{-i\lambda}\partial_{-\dot{\alpha}}e^{i\lambda}$$

A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, 1988

Bridge $\lambda\left(x^{\pm\dot{\alpha}}, u\right)$ is a matrix. The self-duality equation is equivalent to

$$\partial_{-\dot{\alpha}}K^{++} = 0$$

Notations:

$$\partial^{++} = u^+_\alpha\frac{\partial}{\partial u^-_\alpha}, \qquad \partial_{-\dot{\alpha}} = \frac{\partial}{\partial x^{-\dot{\alpha}}}$$

General n-instanton solution, SU(2) gauge group

$$\left(K^{++}\right)^i_j = i\sum_{A=1}^{n} \frac{\rho_A^2}{\left(x_A^{+-}\right)^2}\left(M_A\right)^i_k u^{+k} u^+_\ell \left(M_A^\dagger\right)^\ell_j$$

S. Kalitzin, E. Sokatchev, 1987

Instanton moduli are explicit.

Connection with ADHM construction is unknown

Connection with our construction is unknown

Summary and perspectives

- A new $\mathcal{N} = 4$ SQM model is constructed.
- Superfield description is given for the particular case of t'Hooft ansatz. Generalization to general self-dual fields? Generalization to $\mathrm{SU}(N)$ gauge group?
- Non-Abelian self-dual gauge fields can be described in terms of functions on extended space.