How should we revise BRST ?

Arthemy V. Kiselev

(Groningen, ISPU)

A. K. + J. van de Leur (Utrecht):

- TMPh'2011: `math.DG/1006.4227`,
- Protaras'2010: `math-ph/0904.1555`,
- TMPh'2010: `nlin.SI/0902.3624`.

Relations between equations.

$$\boldsymbol{x} = \begin{cases} x^0 = t \\ x^1, \dots, x^{n-1} \end{cases} \qquad \boldsymbol{q} = \begin{cases} \Sigma^n \longrightarrow M^{3,1} \\ \text{fields } A_\alpha \end{cases} \qquad \text{Action } S\big[\boldsymbol{q}(\boldsymbol{x})\big]$$

Equations: $\mathcal{E} = \left\{ F = \dfrac{\delta S}{\delta \boldsymbol{q}(x)} = 0 \right\}, \qquad F \in P_0$

(Bianchi) identities: $\Phi[F] \equiv 0 \ \forall \ \boldsymbol{q}(x), \qquad \Phi \in P_1$

Relations: $\Phi_{i+1}[\Phi_i] \equiv 0 \ \forall \ \Phi_{i-1}, \qquad \Phi_{i+1} \in P_{i+1}.$

Relations $\rightleftarrows$ operators.

Noether sym $\quad \dot{\boldsymbol{q}} = \varphi\big(\boldsymbol{x}, [\boldsymbol{q}]\big)$: $\quad \dot{S} = 0$.

Evolution $\quad \partial_\varphi^{(\boldsymbol{q})} = \varphi \dfrac{\partial}{\partial \boldsymbol{q}} + \tfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}(\varphi) \dfrac{\partial}{\partial \boldsymbol{q}_{\boldsymbol{x}}} + \cdots$.

Relation $\quad \Phi[F] \equiv 0$.

Chain rule: $\quad \dot{\Phi} = \partial^{(F)}_{\partial_\varphi^{(\boldsymbol{q})}(F)}(\Phi) \equiv 0 \quad \Longleftrightarrow \quad \underline{\ell_\Phi^{(F)} \circ \ell_F^{(\boldsymbol{q})}}(\varphi) \equiv 0.$

Helmholz: $\quad F = \dfrac{\delta S}{\delta \boldsymbol{q}(\boldsymbol{x})} \quad \Longleftrightarrow \quad \ell_F^{(\boldsymbol{q})} = \Big(\ell_F^{(\boldsymbol{q})}\Big)^\dagger$

<u>By parts</u>: $\quad \ell_F^{(\boldsymbol{q})} \circ \Big(\ell_\Phi^{(F)}\Big)^\dagger(\boldsymbol{p}) \equiv 0, \qquad \boldsymbol{p} \in \widehat{P}_1$

Operator $\quad G_\alpha := \Big(\ell_{\Phi_\alpha}^{(F)}\Big)^\dagger \qquad \operatorname{im} G_\alpha \subseteq \ker \ell_{\mathcal{E}} \simeq \operatorname{sym} \mathcal{E}$.

Gauge symmetries $\varphi = G_i(\boldsymbol{p}_i)$.

Commutation closure

$$\big[G_i(\boldsymbol{p}_i), G_i(\boldsymbol{p}_j)\big] = \sum_{k=1}^{\dim \mathfrak{g}} G_k\big(\mathbf{c}_{ij}^k(\boldsymbol{p}_i, \boldsymbol{p}_j)\big)$$

Structure "const" $\quad \mathbf{c}_{ij}^k =$ bi-diff. operators:

$$\mathbf{c}_{ij}^k(\boldsymbol{p}_i, \boldsymbol{p}_j) = \delta_j^k \cdot \partial^{(\boldsymbol{q})}_{G_i(\boldsymbol{p}_i)}(\boldsymbol{p}_j) - \delta_i^k \cdot \partial^{(\boldsymbol{q})}_{G_j(\boldsymbol{p}_j)}(\boldsymbol{p}_i) + \Gamma_{ij}^k(\boldsymbol{p}_i, \boldsymbol{p}_j)$$

Bi-diff Christoffel symbols:

$$\Gamma \longmapsto g\Gamma g^{-1} + \mathrm{d}g\, g^{-1}$$

under $\boldsymbol{p} \mapsto g\,\boldsymbol{p}$.

BRST differential $\boldsymbol{Q}$.

Parity $\Pi\colon \boldsymbol{p}_i \rightleftarrows \boldsymbol{b}_i$ $\longleftarrow$ odd **ghost**.

$$\boldsymbol{Q} = \partial^{(\boldsymbol{q})}_{\sum_{i=1}^{\dim\mathfrak{g}}} G_i(\boldsymbol{b}_i) - \frac{1}{2}\sum_{k=1}^{\dim\mathfrak{g}} \partial^{(\boldsymbol{b}_k)}_{\sum_{i,j=1}^{\dim\mathfrak{g}}} \Gamma^k_{ij}(\boldsymbol{b}_i, \boldsymbol{b}_j)$$

$$\boldsymbol{Q}^2 = 0.$$

Ex. Maxwell: $\Phi_1 =$ Bianchi identities, $\Gamma^k_{ij} = 0$.

Conclusion.

- BRST = Koszul $+\boldsymbol{Q}+\cdots$,
  $(F \in P_0,\ \Phi_1 \in P_1,\ \ldots,\ \Phi_k \in P_k)$.
- $\dim \mathfrak{g} = 1$: Variational Lie algebroids $\supsetneq$
  Poisson algebroids.
- Bi-Ham. (G_1, G_2) $\longleftarrow$ **no** pattern for integrability.
- Jet space $\longrightarrow$ $\boldsymbol{Q}^2 = 0$ $\rightsquigarrow$ path integral.