Wedge Dislocations in the Geometric Theory of Defects

M. O. Katanaev
Steklov Mathematical Institute, Moscow

Katanaev, Volovich Ann. Phys. 216(1992)1; ibid. 271(1999)203
Katanaev Theor.Math.Phys.135(2003)733; ibid. 138(2004)163
Phisics – Uspekhi 48(2005)675.

Notations

$\mathbb{R}^3$ - continuous elastic media = Euclidean three-dimensional space

$x^i, y^i \quad i=1,2,3$ - Cartesian coordinates

δ_{ij} - Euclidean metric

$u^i(x)$ - displacement vector field

$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ - strain tensor

σ^{ij} - stress tensor

Elasticity theory of small deformations

$\partial_i \sigma^{ij} + f^j = 0$ - Newton's law

$\sigma^{ij} = \lambda \delta^{ij} \varepsilon_k{}^k + 2\mu \varepsilon^{ij}$ - Hooke's law

$f^i(x)$ - density of nonelastic forces $(f^i = 0)$

λ, μ - Lame coefficients

Affine geometry $(\mathbb{M}, g, \Gamma)$

$\mathbb{M}\ (\sim \mathbb{R}^m),\ \dim \mathbb{M} = m$ - manifold $\qquad x^\mu,\ \mu = 1, \ldots, m$ - local coordinates

Metric

$g_{\mu\nu}(x),\ \ g_{\mu\nu} = g_{\nu\mu},\ \ \det g_{\mu\nu} \neq 0$ - metric $\quad (X, Y) = X^\mu Y^\nu g_{\mu\nu}$ -scalar product

Affine connection

$\Gamma_{\mu\nu}{}^\rho(x)$ - affine connection

$T_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho$ - torsion tensor

Covariant derivatives:

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + X^\rho \Gamma_{\mu\rho}{}^\nu$$

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\mu\nu}{}^\rho A_\rho$$

Riemann-Cartan geometry $(\mathbb{M}, g, T)$

$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}{}^\sigma g_{\sigma\rho} - \Gamma_{\mu\rho}{}^\sigma g_{\nu\sigma} = 0$ - metricity condition

$$\Gamma_{\mu\nu\rho} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) + \frac{1}{2}(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu})$$

$R_{\mu\nu\rho}{}^\sigma = \partial_\mu \Gamma_{\nu\rho}{}^\sigma - \Gamma_{\mu\rho}{}^\lambda \Gamma_{\nu\lambda}{}^\sigma - (\mu \leftrightarrow \nu)$ - curvature tensor

$R_{\mu\rho} = R_{\mu\nu\rho}{}^\nu$ - Ricci tensor $\qquad R = g^{\mu\rho} R_{\mu\rho}$ - scalar curvature

Cartan variables

$e_\mu{}^i(x)$ - vielbein $\omega_\mu{}^{ij}(x),\ \omega_\mu{}^{ij} = -\omega_\mu{}^{ji}$ - SO(m)-connection $i, j = 1,\ldots,m$

$g_{\mu\nu} = e_\mu{}^i e_\nu{}^j \delta_{ij}$ - definition of vielbein

$e_i = e^\mu{}_i \partial_\mu$ - orthonormal basis $(e_i, e_j) = e^\mu{}_i e^\nu{}_j g_{\mu\nu} = \delta_{ij}$

$X = X^\mu \partial_\mu = X^i e_i$ - vector field

$\nabla_\mu e_\nu{}^i = \partial_\mu e_\nu{}^i - \Gamma_{\mu\nu}{}^\rho e_\rho{}^i + e_\nu{}^j \omega_{\mu j}{}^i = 0$ - definition of SO(m)-connection

Covariant derivatives:

$$\nabla_\mu X^i = \partial_\mu X^i + X^j \omega_{\mu j}{}^i$$

$$\nabla_\mu X_i = \partial_\mu X_i - \omega_{\mu i}{}^j X_j$$

Cartan variables (continued)

$$T_{\mu\nu}{}^{i} = \partial_{\mu} e_{\nu}{}^{i} - e_{\mu}{}^{j}\omega_{\nu j}{}^{i} - (\mu \leftrightarrow \nu)$$ - torsion

$$R_{\mu\nu j}{}^{i} = \partial_{\mu}\omega_{\nu j}{}^{i} - \omega_{\mu j}{}^{k}\omega_{\nu k}{}^{i} - (\mu \leftrightarrow \nu)$$ - curvature

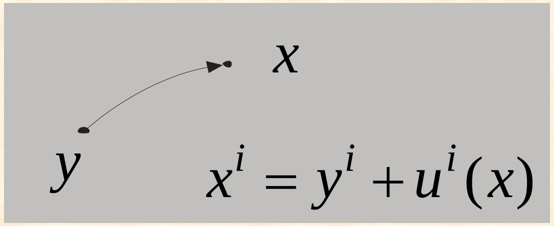
$$T_{\mu\nu}{}^{i} = T_{\mu\nu}{}^{\rho} e_{\rho}{}^{i}, \quad R_{\mu\nu j}{}^{i} = R_{\mu\nu\rho}{}^{\sigma} e^{\rho}{}_{j} e_{\sigma}{}^{i}$$

Theorem (local). If $R_{\mu\nu j}{}^{i} = 0$, then there exists the rotational angle field $\omega_{j}{}^{i}(x)$ such that $\omega_{\mu j}{}^{i} = \partial_{\mu} S^{-1}{}_{j}{}^{k} S_{k}{}^{i}$

Theorem (local). If $R_{\mu\nu j}{}^{i} = 0$ and $T_{\mu\nu}{}^{i} = 0$, then there exists the coordinate system $y^{i}(x)$ and the rotational angle field $\omega_{j}{}^{i}(x)$ such that $\omega_{\mu j}{}^{i} = \partial_{\mu} S^{-1}{}_{j}{}^{k} S_{k}{}^{i}$ and $e_{\mu}{}^{i} = \partial_{\mu} y^{j} S_{j}{}^{i}$

$S_{i}{}^{j}(\omega) \in \mathbb{SO}(m)$ - orthogonal matrix

Differential geometry of elastic deformations



$$y^i \to x^i(y)$$ - diffeomorphism: $\mathbb{R}^3 \to \mathbb{R}^3$

$$y^i \mapsto x^i$$

$$\delta_{ij} \qquad g_{ij}$$

$$g_{ij}(x) = \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\delta_{kl} \approx \delta_{ij} - \partial_i u_j - \partial_j u_i = \delta_{ij} - 2\varepsilon_{ij}$$ - induced metric (*)

$$\tilde{\Gamma}_{ijk} = \tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \neq 0$$ - Christoffel's symbols

$$\tilde{R}_{ijk}{}^{l} = \partial_i \tilde{\Gamma}_{jk}{}^{l} - \tilde{\Gamma}_{ik}{}^{m}\tilde{\Gamma}_{jm}{}^{l} - (i \leftrightarrow j) = 0$$ - curvature tensor

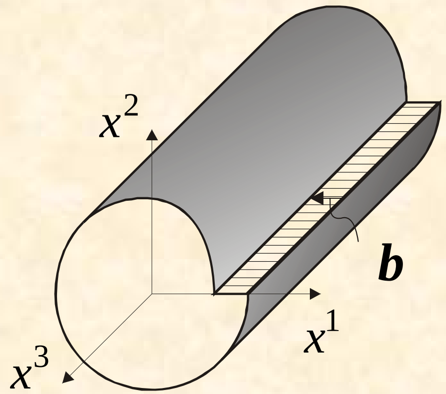
$$\ddot{x}^i = -\tilde{\Gamma}_{jk}{}^{i}\dot{x}^j\dot{x}^k$$ - extremals (geodesics)

$$R_{ijk}{}^{l} = 0$$ - Saint-Venant integrability conditions of (*)

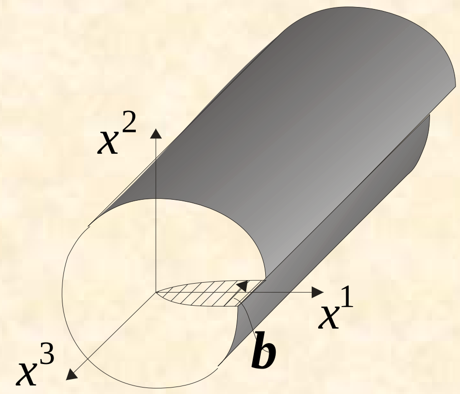
$$T_{ij}{}^{k} = \tilde{\Gamma}_{ij}{}^{k} - \tilde{\Gamma}_{ji}{}^{k} = 0$$ - torsion tensor

Dislocations

Linear defects:



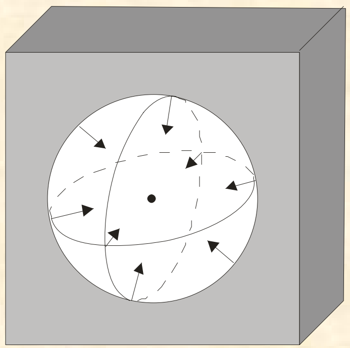
Edge dislocation



Screw dislocation

$\boldsymbol{b}$ - Burgers vector

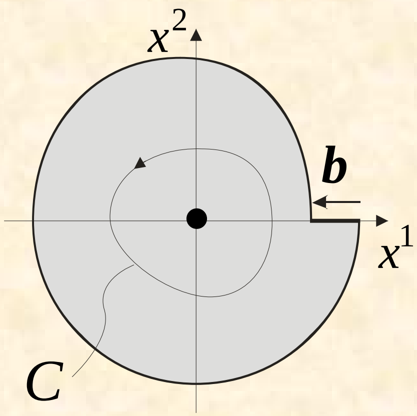
Point defects:



Vacancy

$$u^i(x) \begin{cases} \text{is continuous} & = \text{elastic deformations} \\ \text{is not continuous} & = \text{dislocations} \end{cases}$$

Edge dislocation



$$\oint_C dx^\mu \partial_\mu u^i = -\oint_C dx^\mu \partial_\mu y^i = -b^i \qquad (*)$$

$x^\mu,\ \mu = 1,2,3$ - arbitrary curvilinear coordinates

$y^i(x)$ - is not continuous !

$$e_\mu{}^i(x) = \begin{cases} \partial_\mu y^i & \text{- outside the cut} \\ \lim \partial_\mu y^i & \text{- on the cut} \end{cases}$$

- triad field (continuous on the cut)

$$(*) \Rightarrow b^i = \oint_C dx^\mu e_\mu{}^i = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i)$$ - Burgers vector in elasticity

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$ - torsion

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$ - curvature

$$\omega_\mu{}^{ij} = -\omega_\mu{}^{ji}$$

SO(3)-connection

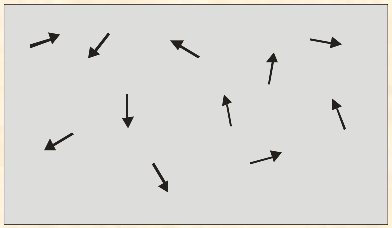
$$b^i = \iint dx^\mu \wedge dx^\nu T_{\mu\nu}{}^i$$

- definition of the Burgers vector in the geometric theory

Back to elasticity: if $R_{\mu\nu}{}^{ij} = 0$ then $\omega_\mu{}^{ij} \to 0$

Disclinations

Ferromagnets



$n^i(x)$ - unit vector field $\qquad n_0^i$ - fixed unit vector

$$n^i = n_0^j S_j{}^i(\omega)$$

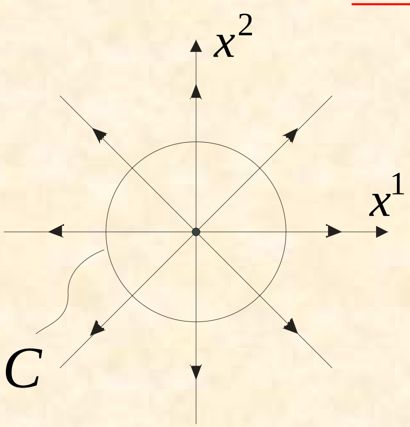
$S_i{}^j \in \mathbb{SO}(3)$ - orthogonal matrix

$\omega^{ij} = -\omega^{ji} \in \mathfrak{so}(3)$ - Lie algebra element (spin structure)

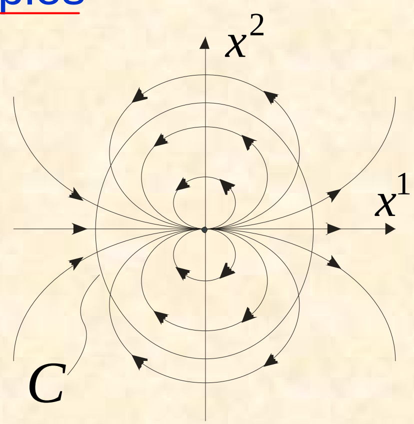
$\omega_i = \frac{1}{2}\varepsilon_{ijk}\omega^{jk}$ - rotational angle

ε_{ijk} - totally antisymmetric tensor $(\varepsilon_{123} = 1)$

Examples



$\Theta = 2\pi$



$\Theta = 4\pi$

$$\Omega^{ij} = \oint_C dx^\mu \partial_\mu \omega^{ij}$$

$\Theta_i = \varepsilon_{ijk}\Omega^{jk}$ - Frank vector (total angle of rotation)

$$\Theta = \sqrt{\Theta^i \Theta_i}$$

Summary of the geometric approach (physical interpretation)

Media with dislocations and disclinations $=$

$= \mathbb{R}^3$ with a given Riemann-Cartan geometry

Independent variables $\begin{cases} e_\mu{}^i & \text{- triad field} \\ \omega_\mu{}^{ij} & \text{- SO(3)-connection} \end{cases}$

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$

- torsion (surface density of the Burgers vector)

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$

- curvature (surface density of the Frank vector)

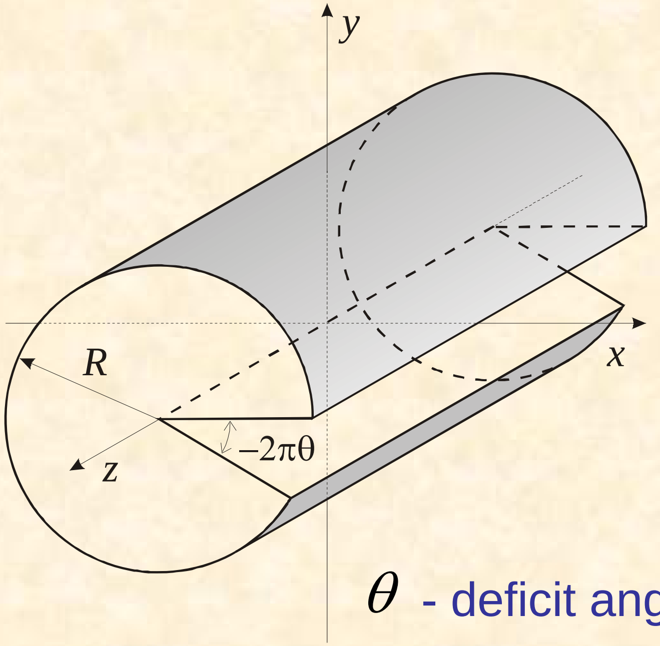
Elastic deformations: $R_{\mu\nu}{}^{ij} = 0, \quad T_{\mu\nu}{}^i = 0$

Dislocations: $R_{\mu\nu}{}^{ij} = 0, \quad T_{\mu\nu}{}^i \neq 0$

Disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \quad T_{\mu\nu}{}^i = 0$

Dislocations and disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \quad T_{\mu\nu}{}^i \neq 0$

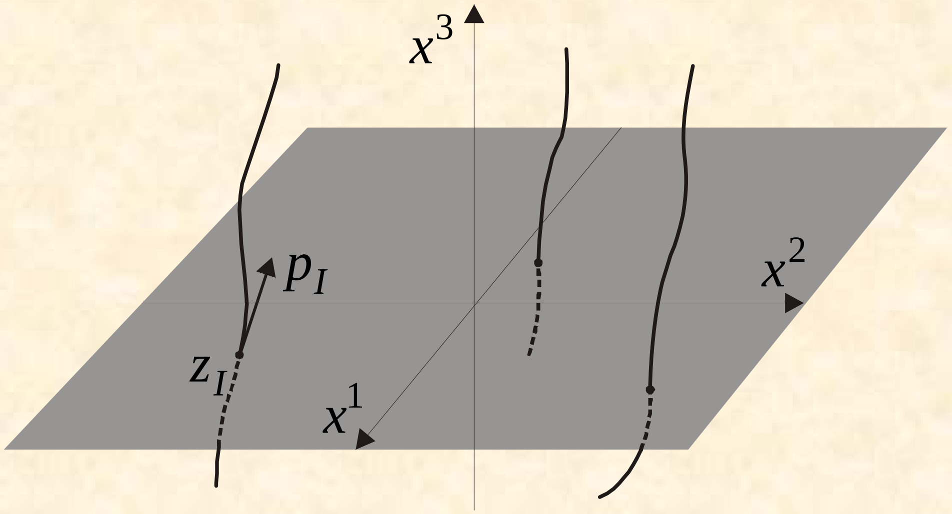
Wedge dislocation



θ - deficit angle

Staruszkiewicz (1963)
Clement (1976)
Deser, Jackiw, ‘t Hooft (1984)

Bellini, Ciafaloni, Valtancoly (1995)
Welling (1995)
Menotti, Seminara (1999)



The free energy

$$S=\int d^3x\sqrt{g}L,\quad \sqrt{g}=\det e_\mu{}^i$$

$$L=\kappa\tilde{R}-\gamma R_{[ij]}R^{[ij]}$$

$\tilde{R}(e)$ - the Hilbert-Einstein action

$R_{[ij]}(e,\omega)$ - antisymmetric part of the Ricci tensor

Katanaev, Volovich Ann. Phys. 216(1992)1

$q_I(\tau)=\left\{q_I^\alpha(\tau)\right\}$ - wedge dislocation axis $\qquad \dot{q}_I:=\frac{dq_I}{d\tau}$ - notation

$$S=\int d^3x\sqrt{g}\tilde{R}+\sum_I^N m_I\int d\tau\sqrt{\dot{q}_I^\alpha\dot{q}_I^\beta g_{\alpha\beta}}$$

Equations of equilibrium

$$\tilde{R}_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\tilde{R}=\frac{1}{2}T_{\alpha\beta},$$

where $T_{\alpha\beta}=\frac{1}{\sqrt{g}}\sum_I\frac{m_I\dot{q}_{I\alpha}\dot{q}_{I\beta}}{\dot{q}_I^3}\delta(\mathbf{x}-\mathbf{q})$

$$\ddot{q}_I^\alpha=-\Gamma_{\beta\gamma}{}^\alpha\dot{q}_I^\beta\dot{q}_I^\gamma,$$

$$\delta(\mathbf{x}-\mathbf{q}):=\delta(x^1-q^1)\delta(x^2-q^2)$$

Canonical Formulation

$(x^1, x^2, x^3) \mapsto (x^3, x^1, x^2)$ - reordering of coordinates

$\alpha, \beta, \ldots = 3,1,2$
$\mu, \nu, \ldots = 1,2$ - notations

$$g_{\alpha\beta} = \begin{pmatrix} N^2 + N^\rho N_\rho & N_\nu \\ N_\mu & g_{\mu\nu} \end{pmatrix}$$ - ADM parameterization of 3D metric

where N - lapse and N_μ - shift functions

$g_{\mu\nu}$ - 2D metric on slices $x^3 = \text{const}$

$(g_{\mu\nu}, p^{\mu\nu})$ $(q_I^\alpha, p_{I\alpha})$ - coordinates and conjugate momenta

$$S_{\text{HE}} = \int d^3x \left(p^{\mu\nu} \dot{g}_{\mu\nu} - N H_\perp^{(0)} - N^\mu H_\mu^{(0)} \right)$$ - the Hilbert-Einstein action

$$H_\perp^{(0)} = \frac{1}{\sqrt{g}} \left(p^{\mu\nu} p_{\mu\nu} - p^2 \right) - \sqrt{g} \hat{R}$$

$$H_\mu^{(0)} = -2 \hat{\nabla}_\nu p^\nu{}_\mu$$

- general relativity constraints

$$S_I = \int d\tau \left(p_{I\alpha} \dot{q}_I^\alpha - N \,|\, \dot{q}_I^3 \,|\, G_I \right)$$ - the action for wedge dislocations

$$G_I := \frac{1}{N} \,|\, p_{I3} - N^\mu p_{I\mu} \,| - \sqrt{m_I^2 - \hat{p}_I^2} = 0$$ - first class constraints $I = 1, \dots, N$

$$\hat{p}_I^2 := p_{I\mu} p_{I\nu} \hat{g}^{\mu\nu}$$ $\tau_I \mapsto \tau_I'(\tau_I)$

The gauge $\dot{q}_I^3 = 1 \implies$

$$S_I = \int d\tau \left(p_{I\mu} \dot{q}_I^\mu + N \sqrt{m_I^2 - \hat{p}_I^2} + N^\mu p_{I\mu} \right)$$

$p_N = 0, \qquad p_{N_\mu} = 0$ - primary constraints

$$H_\perp = \frac{1}{\sqrt{g}} \left(p^{\mu\nu} p_{\mu\nu} - p^2 \right) - \sqrt{g} \hat{R} - \sum_I \sqrt{m_I^2 - \hat{p}_I^2} \, \delta(\mathbf{x} - \mathbf{q}) = 0$$

- secondary constraints

$$H_\mu = -2 \hat{\nabla}_\nu p^\nu{}_\mu - \sum_I p_{I\mu} \delta(\mathbf{x} - \mathbf{q}) = 0$$

$$S_{\mathrm{T}} = \int d^3x \left(p^{\mu\nu} \dot{g}_{\mu\nu} + \sum_I p_{I\mu} \dot{q}_I^\mu \delta(\mathbf{x} - \mathbf{q}) - N H_\perp - N^\mu H_\mu \right)$$ - total Hamiltonian

Secondary constraints

$$H_{\perp} = \frac{1}{\sqrt{g}}\left(p^{\mu\nu} p_{\mu\nu} - p^2\right) - \sqrt{g}\hat{R} - \sum_I \sqrt{m_I^2 - \hat{p}_I^2}\,\delta(\mathbf{x}-\mathbf{q}) = 0$$

$$H_{\mu} = -2\hat{\nabla}_{\nu} p^{\nu}{}_{\mu} - \sum_I p_{I\mu}\,\delta(\mathbf{x}-\mathbf{q}) = 0$$

Equations of equilibrium

$$\dot{g}_{\mu\nu} = \frac{2N}{\sqrt{g}} p_{\mu\nu} - \frac{2N}{\sqrt{g}} g_{\mu\nu} p + \hat{\nabla}_{\mu} N_{\nu} + \hat{\nabla}_{\nu} N_{\mu},$$

$$\dot{p}^{\mu\nu} = \frac{N}{2\sqrt{g}} \hat{g}^{\mu\nu}\left(p^{\rho\sigma} p_{\rho\sigma} - p^2\right) - \frac{2N}{\sqrt{g}}\left(p^{\mu\rho} p^{\nu}{}_{\rho} - p^{\mu\nu} p\right) +$$

$$+\sqrt{g}\left(\Delta N \hat{g}^{\mu\nu} - \hat{\nabla}^{\mu}\hat{\nabla}^{\nu} N\right) - p^{\mu\rho}\hat{\nabla}_{\rho} N^{\nu} - p^{\nu\rho}\hat{\nabla}_{\rho} N^{\mu} - \hat{\nabla}_{\rho}\left(N^{\rho} p^{\mu\nu}\right) +$$

$$+ N \sum_I \frac{p_I^{\mu} p_I^{\nu}}{\sqrt{m_I^2 - \hat{p}_I^2}}\,\delta(\mathbf{x}-\mathbf{q}_I)$$

Complex coordinates

$$(x^1, x^2) \mapsto (z, \overline{z})$$ where $$z := x^1 + ix^2, \qquad \overline{z} := x^1 - ix^2$$

$$g_{\mu\nu} = e^{2\phi}\delta_{\mu\nu}$$ - conformally flat metric $$\delta_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$p := p^{\mu\nu} g_{\mu\nu} = 0$$ - the third gauge condition

$$p^z{}_{\overline{z}} = p^1{}_1 + ip^1{}_2, \qquad p^{\overline{z}}{}_z = p^1{}_1 - ip^1{}_2,$$

$$p^z{}_z = 0, \qquad p^{\overline{z}}{}_{\overline{z}} = 0$$

$$g_{\mu\nu},\ p^{\mu\nu} \mapsto \phi, p^z{}_{\overline{z}}$$

Solution of the kinematical constraints

$$H_\mu = 0 \Longrightarrow \partial_{\overline{z}} p^{\overline{z}}{}_z = -\frac{1}{2}\sum_I p_{Iz}\delta(z - z_I) \Longrightarrow p^{\overline{z}}{}_z = -\frac{1}{2\pi}\sum_I \frac{p_{Iz}}{z - z_I}$$

Solution of the Dynamical Constraint

$$H_\perp = 0 \quad \Longrightarrow \quad 2\Delta\phi = 2 p^{\bar{z}}{}_z p^z{}_{\bar{z}} e^{-2\phi} - \sum_I \sqrt{m_I^2 - 4 p_{Iz} p_{I\bar{z}} e^{-2\phi}}\ \delta(z - z_I)$$

$$2\tilde{\phi} := 2\phi - \ln\left(2 p^{\bar{z}}{}_z p^z{}_{\bar{z}}\right)$$ - ansatz

Central of mass coordinate system: $\sum_I p_{Iz} = 0$

$$p^{\bar{z}}{}_z = \frac{P_{N-2}(z)}{\prod_I (z - z_I)} = C \frac{\prod_A (z - z_A)}{\prod_I (z - z_I)}$$ where $C(z_I, p_{Iz}) := \frac{1}{2\pi} \sum_I p_{Iz} \sum_{J \neq I} z_J$

$$2\Delta\tilde{\phi} = e^{-2\tilde{\phi}} - 4\pi \sum_I (a_I - 1)\ \delta(z - z_I) - 4\pi \sum_A \delta(z - z_A)$$

$$4\pi a_I := \sqrt{m_I^2 - \frac{2 p_{Iz} p_{I\bar{z}}}{p^{\bar{z}}{}_z p^z{}_{\bar{z}}} e^{-2\tilde{\phi}}}$$

Solution of the Dynamical Constraint

For one wedge dislocation at the origin of the coordinate system:

$$2\Delta\tilde{\phi} = e^{-2\tilde{\phi}} - 4\pi(a_I - 1)\delta(z) \qquad |z| \to 0: \quad e^{-2\tilde{\phi}} \sim \frac{8a_I^2}{\Lambda^2}\left(\frac{z\bar{z}}{\Lambda^2}\right)^{a_I - 1}$$

and $$p^{\bar{z}}{}_z p^{z}{}_{\bar{z}} \sim \frac{1}{z\bar{z}}$$

$$2\Delta\tilde{\phi} = e^{-2\tilde{\phi}} - 4\pi\sum_I(\mu_I - 1)\,\delta(z - z_I) - 4\pi\sum_A \delta(z - z_A) \qquad 4\pi\mu_I = |m_I|$$

$$e^{2\phi} = 2|C|^2 \frac{\prod_A (z - z_A)(\bar{z} - \bar{z}_A)}{\prod_I (z - z_I)(\bar{z} - \bar{z}_I)} e^{2\tilde{\phi}}$$

Lapse function

$$\dot{p} = \dot{p}^{\mu\nu} g_{\mu\nu} + p^{\mu\nu} \dot{g}_{\mu\nu} = 0 \quad \Longrightarrow$$

$$\sqrt{g}\Delta N + \frac{N}{\sqrt{g}} p^{\mu\nu} p_{\mu\nu} + N \sum_I \frac{p_I^\mu p_{I\mu}}{|m_I|} \delta(\mathbf{x} - \mathbf{q_I}) = 0$$

$$\Downarrow$$

$$\Delta N = -2 p^{\bar{z}}{}_z p^z{}_{\bar{z}} e^{-2\phi} N - e^{-2\phi} N \sum_I \frac{4 p_{Iz} p_{I\bar{z}}}{|m_I|} \delta(z - z_I)$$

One can prove that $N(z_I)$ is finite

$$\Delta N = -\frac{1}{2\pi^2} \sum_I \frac{p_{Iz} p_{I\bar{z}}}{(z - z_I)(\bar{z} - \bar{z}_I)} e^{-2\phi} N$$

One dislocation at the origin:

$$\Delta N = -\frac{1}{2\pi^2} \frac{p_{Iz} p_{I\bar{z}}}{z\bar{z}} e^{-2\phi} N \qquad N \sim 1 - 4\left(\frac{z\bar{z}}{\Lambda^2}\right)^{\mu_I}$$

Shift functions

$$\dot{g}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(g^{\rho\sigma}\dot{g}_{\rho\sigma}) = 0$$

$$\Downarrow$$

$$\frac{2N}{\sqrt{g}} p_{\mu\nu} + \nabla_\mu N_\nu + \nabla_\nu N_\mu - g_{\mu\nu}\nabla_\rho N^\rho = 0$$

$$\Downarrow$$

$$\partial_{\bar{z}} N^z = -Ne^{-2\phi} p^z{}_{\bar{z}} \qquad \times p^z{}_{\bar{z}}$$

$$\Downarrow$$

$$p^{\bar{z}}{}_z \partial_{\bar{z}} N^z = 2\partial_{\bar{z}}\partial_z N$$

$$\Downarrow$$

$$N^z = \frac{2}{p^{\bar{z}}{}_z}\partial_z N + f(z)$$

Reduction to the Riemann—Hilbert problem

$$g_{\alpha\beta} = e_\alpha{}^a e_\beta{}^b \delta_{ab}$$

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\varepsilon}\varepsilon_{\gamma\delta\varsigma}R^{\varepsilon\varsigma}$$

$\varepsilon_{\alpha\beta\gamma}$ - totally antisymmetric tensor

$$R_{\alpha\beta\gamma\delta} = 0 \Longrightarrow e_\alpha{}^a = \partial_\alpha y^a$$

The gauge: $S = \int d^2 z \sqrt{\det h}\, h^{\alpha\beta}\partial_\alpha y^a \partial_\beta y_a \Longrightarrow \partial_z \partial_{\bar{z}} y^a = 0$

$$\Downarrow$$

$$y^a = F^a(z, x^3) + G^a(\bar{z}, x^3) + H^a(x^3)$$

$e_z{}^a := \partial_z y^a = e_z{}^a(z, x^3)$ - holomorphic

$e_{\bar{z}}{}^a := \partial_{\bar{z}} y^a = e_{\bar{z}}{}^a(\bar{z}, x^3)$ - antiholomorphic $\qquad e_{\bar{z}}{}^a = \overline{e_z{}^a}$

$$e_3{}^a := \partial_3 y^a = C^a(x^3) + \int dz\, e_z{}^a + \int d\bar{z}\, e_{\bar{z}}{}^a$$

Reduction to the Riemann—Hilbert problem

Everything is defined by $e_z{}^a(z, x^3)$

Let γ_I be a closed loop around the dislocation axis at z_I

$$y^a(z_0) \mapsto \tilde{y}(z_0) = \int_{z_0}^{z_I} dz\, e_z{}^a + \oint_{\gamma_I} dz\, e_z{}^a + \int_{z_I}^{z_0} dz\, e_z{}^a$$

$$= (1 - M_I) Y_I^a + \oint_{\gamma_I} dz\, e_z{}^a$$

$M_I \in \mathbb{SO}(3)$ - the monodromy matrix

$$\pi\left(\mathbb{C} \backslash \left\{z_1, \ldots, z_N, z_\infty\right\}; z_0\right) \to \mathbb{SO}(3) \subset \mathbb{GL}(3, \mathbb{C})$$

the Riemann—Hilbert problem

Show that for any representation $\pi\left(\mathbb{C} \backslash \left\{z_1, \ldots, z_N, z_\infty\right\}; z_0\right) \to \mathbb{GL}(p, \mathbb{C})$

there is a Fuchsian system of equations with a given monodromy

Conclusion

1 Wedge dislocations in elastic medea are described by 3-dimensional Euclidean gravity coupled to point particles.

2 Solving Einstein Equations is reduced to solving the Riemann-Hilbert problem

3 For two wedge dislocations the problem is solved analytically in terms of the hypergeometric functions

4 For arbitrary number of wedge dislocations much can be done analytically