What is the partition bundle?

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A quantum theory is characterized by its partition function Z.

In a Hamiltonian formalism

$$Z = \mathsf{Tr}_{\mathcal{H}} \left(e^{-\beta H + \gamma J + \dots} \right)$$

with

$$\begin{aligned} \mathcal{H} &= \text{Hilbert space} \\ H, J, \dots &= \text{commuting observables} \\ \beta, \gamma, \dots &= \text{formal parameters.} \end{aligned}$$

In a Lagrangian formalism with periodic time

$$Z = \int \mathcal{D}q \dots e^{-\int_0^\beta dt L}$$

with

$$q, \dots = \text{dynamical variables}$$

But what is the counterpart of Z for theories with no classical description (no Lagrangian or even equations of motion)?

The best known examples are the (2, 0) superconformal theories in six dimensions:

- Completely classified by the type

$$\Phi \in \text{ADE} \simeq \{\text{simply laced Lie algebras}\}$$

- Realized in type IIB string theory at codimension 4 singularity.

- A-series (D-series) realized on coincident M5-branes (with orientifold plane).

- Holographic representation of A-series as M-theory on $\text{AdS}_7 \times S^4$.

- $\text{OSp}(6,2|4)$ superconformal algebra in flat space with $\text{so}(6,2) \oplus \text{sp}(4)$ even subalgebra.

But $(2,0)$ theories can also be defined on an arbitrary six-manifold M endowed with some additional data.

- Data related to the geometry of M:

$$
\begin{aligned}
\sigma &\in \Sigma \\
&= \{\text{orientations on } M\} \\
&= \text{affine space over } H^0(M, \mathbb{Z}_2) \\
s &\in \mathcal{S} \\
&= \{\text{spin structures on } M\} \\
&= \text{affine space over } H^1(M, \mathbb{Z}_2) \\
[g] &\in \mathcal{G} \\
&= \{\text{conformal structures on } M\} \\
&= \text{infinite dimensional real manifold}
\end{aligned}
$$

- Data related to the $\mathrm{sp}(4) \simeq \mathrm{so}(5)$ R-symmetry (neglected in this talk).

- Data related to observables defined on two- and four-dimensional submanifolds of M (also neglected here).

- Q: What kind of object is Z, and how does it depend on the geometric data?

- A: We will describe it for the A_{N-1} model.

  The leading term in the IR-limit of its holographic dual is a Schwarz-type topological field theory with action

  $$S = N \int_{AdS_7} C \wedge dC,$$

  where C is an abelian three-form gauge field.

  Geometric quantization of this TFT leads to a holomorphic prequantum line bundle and a finite-dimensional space V of holomorphic sections.

  The 'partition vector' Z of $(2,0)$ theory is an element of the Hilbert space V of the TFT.

More precisely:

The data $(\sigma, s, [g])$ in the infinite-dimensional space $\Sigma \times \mathcal{S} \times \mathcal{G}$ determines data (ω, u, J) in a finite-dimensional space $\Omega \times \mathcal{U} \times \mathcal{J}$:

$$\begin{aligned}
\omega &\in \Omega \\
&= \{\text{symplectic structures on } H^3(M,\mathbb{R}) \\
&\quad\ \text{induced from the intersection form}\} \\
&= \text{set with 2 elements} \\
u &\in \mathcal{U} \\
&= \{\text{non}-\text{degenerate quadratic forms on} \\
&\quad H^3(M,\mathbb{Z}_2) \text{ polarized by } \omega\} \\
&= \text{set with } 2^{2n} \text{ elements} \\
J &\in \mathcal{J} \\
&= \{\text{translation invariant complex structures on} \\
&\quad H^3(M,\mathbb{R})\} \\
&= \text{complex space of dimension } \frac{1}{2}n(n+1).
\end{aligned}$$

Here $n = \frac{1}{2}b_3(M)$ (the third Betti number of M).

In more detail:

- The symplectic structure ω on $H^3(M,\mathbb{R})$ is given by the wedge product followed by integration over M.

- The non-degenerate quadratic form u on $H^3(M,\mathbb{Z}_2)$ is defined as
$$(-1)^{u(\gamma)} = \exp\left(2\pi i \frac{1}{2}\int_{S^1\times M} C \wedge dC\right).$$
Here C is an abelian three-form gauge field on $S^1 \times M$ determined by a straight line from 0 to $\gamma \in H^3(M,\mathbb{Z}) \subset H^3(M,\mathbb{R})$. Because of $\frac{1}{2}$, to make sense of this expression requires a spin structure s on M.

- The complex structure J on $H^3(M,\mathbb{R})$ is given by the Hodge duality operator $*$, which obeys $** = -1$ for a Euclidean signature on M.

The data (ω, u, J) determine a Hermitian line bundle $\mathcal{L}$ over the intermediate Jacobian torus

$$T = H^3(M, \mathbb{R})/H^3(M, \mathbb{Z}).$$

(T parametrizes abelian three-form gauge fields on M.)

- The curvature of $\mathcal{L}$ is given by ω.

- The holonomy of $\mathcal{L}$ along a closed curve on T obtained from a straight line from 0 to $\gamma \in H^3(M, \mathbb{Z})$ is given by $(-1)^{u(\gamma)}$.

For the A_{N-1} model, the TFT prequantum line bundle is $\mathcal{L}^N$ and the Hilbert space is

$$V = H^0(T, \mathcal{L}^N)$$

of dimension

$$\dim V = N^n$$

(by the index theorem).

The partition vector Z is an element of V.

$\mathcal{L}^N$ is invariant under the commuting translations

$$T_c: T \to T$$

by elements $c \in \frac{1}{N} H^3(M, \mathbb{Z})$. Clearly $T_c^N = \mathbb{1}$.

But the induced operators

$$T_c^*: V \to V$$

fulfill the Heisenberg relations

$$\begin{aligned} (T_c^*)^N &= (-1)^{u(Nc)} \\ T_c^* T_{c'}^* &= T_{c'}^* T_c^* \exp\left(2\pi i N \int_M c \wedge c'\right). \end{aligned}$$

The spin structure s determines the choice of square root signs in the Heisenberg algebra

$$T_c^* T_{c'}^* = \pm \sqrt{\exp\left(2\pi i N \int_M c \wedge c'\right)} T_{c+c'}^*.$$

The vector space V carries an irreducible representation of this Heisenberg algebra.

- Q: What happens to the vector space V as the geometric data $(\sigma, s, [g])$ are varied in the space $\Sigma \times S \times \mathcal{G}$?

- A: We have described a map

$$\phi: \Sigma \times S \times \mathcal{G} \to \Omega \times \mathcal{U} \times \mathcal{J}.$$

  $V = H^0(T, \mathcal{L}^N)$ is the fiber of a rank N^n holomorphic vector bundle over the latter finite dimensional space.

  Pullback by ϕ gives a 'partition bundle' over the former space.

- Eventually, one would like to compute the precise 'partition section' Z of this bundle, but this goal is still out of reach.

- But for the moment, we can gain a better understanding of the holomorphic vector bundle:

There is a homomorphism from the mapping class group of M to an $\mathrm{Sp}_{2n}(\mathbb{Z})$ group of transformation on $H^3(M,\mathbb{Z}) \simeq \mathbb{Z}^{2n}$. This preserves the symplectic structure ω and permutes the possible quadratic forms u in two orbits:

- The first orbit consists of u which give $H^3(M,\mathbb{Z}_2)$ the structure of a direct sum of n hyperbolic planes.

  There is then a Lagrangian decomposition
  $$H^3(M,\mathbb{Z}) = A \oplus B$$
  with
  $$u(a+b) = \int_M a \wedge b \quad \text{for} \quad a \in A, b \in B.$$

- The second orbit consists of u which give $H^3(M,\mathbb{Z}_2)$ the structure of a direct sum of $n-1$ hyperbolic planes and a two-dimensional anisotropic space. (We conjecture that no u on this orbit arise from a spin structure on M as described above.)

- We will describe a holomorphic vector bundle over the space

$$\mathcal{J} = \overline{\mathcal{J}} / \mathsf{Sp}_{2n}(\mathbb{Z})$$

  of complex structures on the intermediate Jacobian torus $T = H^3(M, \mathbb{R})/H^3(M, \mathbb{Z})$.

  We do this by an explicit construction of a holomorphic frame for a bundle over the universal covering space $\overline{\mathcal{J}}$.

- $\overline{\mathcal{J}}$ can be identified with the genus n Siegel upper half space.

  The holomorphic frame then amounts to a kind of vector-valued Siegel modular forms that do not seem to have been much considered before.

In terms of the decomposition

$$H^3(M, \mathbb{Z}) = A \oplus B,$$

the complex structure on $H^3(M, \mathbb{R})$ can be described by a map

$$\tau: A \to B \otimes \mathbb{C}$$

subject to a certain self-adjointness property and with positive definite imaginary part.

The intermediate Jacobian torus can then be identified as

$$T = \frac{B \otimes \mathbb{C}}{B \oplus \tau A}$$

The fiber $V = H^0(T, \mathcal{L}^N)$ can be identified with the space of holomorphic functions

$$\psi(\tau|.): B \otimes \mathbb{C} \to \mathbb{C}$$

subject to the double quasi-periodicity conditions

$$\psi(\tau|z+m+\tau n) = \psi(\tau|z) \exp\left(-i\pi N \int_M n \wedge \tau n + 2n \wedge z\right)$$

for $z \in B \otimes \mathbb{C}$, $n \in A$, and $m \in B$.

We define a (up to a common factor) unique holomorphic frame $\{\psi_{[a]}\}$ indexed by $[a] \in \frac{1}{N}A/A$ by requiring the following behaviour under the Heisenberg translations:

$$\psi_{[a]}(\tau|z + b' + \tau a') = \psi_{[a+a']}(\tau|z) \\ \times \exp\left(-i\pi N \int_M a' \wedge \tau a' + 2a' \wedge z - 2a \wedge b'\right)$$

for $a' \in \frac{1}{N}A$ and $b' \in \frac{1}{N}B$.

The solution is

$$\psi_{[a]}(\tau|z) = \frac{1}{\theta(\tau|0)} \sum_{n \in A} \exp\left(i\pi N \int_M \right. \\ \left. (n+a) \wedge \tau(n+a) + 2(n+a) \wedge z\right)$$

(Here

$$\theta(\tau|z) = \sum_{n \in A} \exp\left(n \wedge \tau n + n \wedge z\right).$$

is the Riemann theta function.)

With $H^3(M,\mathbb{Z}) = A \oplus B$, a symplectic map $S: H^3(M,\mathbb{Z}) \to H^3(M,\mathbb{Z})$ can be written as

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{pmatrix} B \to B & A \to B \\ B \to A & A \to A \end{pmatrix}.$$

Its action on a section ψ of $H^0(T,\mathcal{L}^N)$ is

$$S\psi(\tau|z) = \psi\left(S\tau|Sz\right) \exp\left(-\frac{N}{2}\gamma z \wedge Sz\right)$$

with

$$\begin{array}{rcl} \tau & \mapsto & S\tau = (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1} \\ z & \mapsto & Sz = (\gamma\tau + \delta)^{*-1} z. \end{array}$$

For the frame $\{\psi_{[a]}\}$ with $[a] \in \frac{1}{N}A/A$, one finds the automorphic transformation law

$$\psi_{[a]}(\tau|z) = \frac{\sqrt[8]{1}}{N^n} \sum_{[b]\in\frac{1}{N}B/B} S\psi_{[-\gamma b+\delta a]}(\tau|z)$$
$$\times \exp\left(-i\pi N \int_M \delta a \wedge \beta a + 2\beta a \wedge \gamma b + \gamma b \wedge \alpha b\right).$$

This defines a rank N^n vector bundle over

$$\mathcal{J} = \overline{\mathcal{J}}/\mathrm{Sp}_{2n}(\mathbb{Z}).$$

Summary

- The ADE-series of six-dimensional $(2,0)$ superconformal theories do not admit a Lagrangian formulation.

- Instead of a partition function, they have a 'partition vector' Z that takes its values in a finite dimensional vector space.

- As the six-dimensional geometric data on M are varied in their infinite dimensional moduli space, these vector spaces fit together to a 'partition bundle'.

- This bundle is the pullback of a holomorphic bundle over a finite-dimensional moduli space of data related to the complex geometry of the intermediate Jacobian torus $H^3(M,\mathbb{R})/H^3(M,\mathbb{Z})$.

Some further reading

- These results have been reported in my paper *'The partition bundle of type A_{N-1} $(2,0)$ theory'*.

- For more background on the $(2,0)$ theories, see e.g. E. Witten's papers
  - *'Some comments on string dynamics'*
  - *'AdS/CFT correspondence and topological field theory'*
  - *'Five brane effective action in M-theory'*
  - *'Geometric Langlands from six dimensions'*

- There is also related work by e.g. G. Moore et al.

Thank you!

Tack!

Спасибо!