Parent Lagrangians, unfolding, and covariant Hamiltonian formalism

Maxim Grigoriev

Lebedev Physical Institute, Moscow

SQS-11, July 18-23, Dubna, Russia

Based on:

M.G., arXiv:1012.1903, JHEP to appear

G. Barnich, M.G., arXiv:1009.0190, JHEP 2011

Preliminaries:

Batalin-Vilkovisky formalism: *Batalin, Vilkovisky (1981)*

Action $S_0[\phi]$, Gauge generators $R^i_\alpha(\phi)\frac{\partial}{\partial\phi^i}$, $[R_\alpha, R_\beta] = U^\gamma_{\alpha\beta}(\phi)R_\gamma$, etc.

Extended space of fields: $\{\psi^A\} = \{\phi^i, c^\alpha, \dots\}$,

Ghost degree $\mathrm{gh}(\phi^i) = 0, \mathrm{gh}(c^\alpha) = 1, \dots$

Gauge part of BRST differential:

$$\gamma = c^\alpha R^i_\alpha \frac{\partial}{\partial\phi^i} - \frac{1}{2}c^\alpha c^\beta U^\gamma_{\alpha\beta}\frac{\partial}{\partial c^\gamma} + \dots \qquad \gamma^2 \approx 0$$

Antifields: Λ_A, $\mathrm{gh}(\Lambda_A) = -\mathrm{gh}(\psi^A) - 1$, BV antibracket $\left(\psi^A, \Lambda_B\right) = \delta^A_B$

Master action: $S_{BV} = S_0 + \Lambda_A \gamma\psi^A + \dots$

Master equation:

$$\frac{1}{2}\left(S_{BV}, S_{BV}\right) = 0 \quad \Longleftrightarrow \quad s^2 = 0 \quad \text{for} \quad s = \left(\cdot, S_{BV}\right)$$

Total BRST differential: $s = \delta + \gamma + \ldots$, δ – Koszul-Tate differential (implements equations of motion)

BRST cohomology: $H^{\cdot}(s, \mathcal{F})$

$\mathcal{F}$ – local functionals, local forms, evolutionary (poly)vector fields etc.

BRST cohomology encode: conserved currents/global symmetries, anomalies, consistent deformations etc.

Example: YM theory

Fields: A_μ, C (with values in the Lie algebra)

Antifields: $A^{*\mu}$, C^*

Gauge part BRST differential: $\gamma A_\mu = \partial_\mu C + [A_\mu, C]$

Master action:

$$S_{BV} = S_0 + \int d^n x \mathrm{Tr}[A^{*\mu}(\partial_\mu C + [A_\mu, C]) + \frac{1}{2} C^* [C, C]]$$

- BV formalism makes perfect sense in the purely geometrical setting. So far there is no need to view the underling space as a space of maps from the “space-time” to the “target-space”.

- In applications: local gauge field theories. A consistent way: jet space BV (fields, antifields, and all their derivatives considered as independent variables)

  Barnich, Brandt, Henneaux

  . . . , Dubois-Vialette, Piguet, Stora, Stasheff,. . .

- Although jet-space BV is extremely useful it can be quite restrictive:
  – Boundary dynamics (e.g. AdS/CFT, asymptotic symmetries)
  – Coordinate-free formulation (e.g. for gravity)
  – Important structures such as generalized connections and curvatures are not realized in a manifest way *Brandt, 1996*

An alternative: *Vasiliev, 1988, . . . , 2005*

Unfolded formalism

Fields: differential forms Φ^a

Equations of motion: $d\Phi^a = Q^a(\Phi)$, $Q^a(\Phi)$ – wedge product function.

Consistency: $Q^2 = 0$ where $Q = Q^a(\Phi)\frac{\partial}{\partial \Phi^a}$

Free Differential Algebras, *Sullivan 1977, d'Auria, Fre, 1982. . .*

Advantages:

– manifestly coordinate free

– first order

– useful in analyzing global symmetries

– inevitable for nonlinear higher spin theories *Vasiliev, 1989,. . . , 2003*

Open issues:

1) No systematic procedure to "unfold" a given theory

2) Known unfolded forms for sufficiently general higher spin fields are quite involved

3) In spite of various algebraic similarities the relation between jet space BV and unfolded approaches remains unclear

4) Even for Lagrangian systems constructing unfolded Lagrangians is rather an art than a systematic procedure

For linear theories 1)-3) were mainly resolved within the first quantized BRST approach *Barnich, M.G., Semikhatov, Tipunin, 2004, Barnich, M.G. 2006*.

(Fedosov, Batalin-Fradkin, Fradkin-Linetsky, M.G.-Lyakhovich, Bordemann et all.)

BRST extension of unfolded systems *Barnich, M.G. 2005*

For 2) relevant works are *Alkalaev, M.G. 2009,2010, talk by Alkalaev*

The construction

Starting point theory:

Fields, ghosts, ghosts for ghosts, etc.: $\psi^A(x)$, (x^μ, $\mu = 0, \dots n-1$)

Gauge part of BRST differential: γ (for simplicity $\gamma^2 = 0$)

Lagrangian: $L[\psi, x]$, $\gamma L = \partial_\mu j^\mu$.

Parent formulation:

Fields: $\psi^A_{\lambda_1\dots\lambda_k|\nu_1\dots\nu_l}(x)$, $\quad$ gh$(\psi^A_{\lambda_1\dots\lambda_k|\nu_1\dots\nu_l})$ = gh$(\psi^A) - l$

Generating functions in terms of extra variables y^μ, θ^μ (gh$(\theta^\mu) = 1$)):

$$\tilde{\psi}^A(x, y, \theta) = \sum_{k,l \geqslant 0} \frac{1}{k!l!} \theta^{\nu_l} \dots \theta^{\nu_1} y^{\lambda_k} \dots y^{\lambda_1} \psi^A_{\lambda_1\dots\lambda_k|\nu_1\dots\nu_l}(x) \equiv \theta^{[\nu]} y^{(\lambda)} \psi^A_{(\lambda)[\nu]}(x)\,.$$

Parent differential γ:

$$\gamma^P \widetilde{\psi}^A = (d - \sigma + \bar{\gamma})\widetilde{\psi}^A\,, \quad \text{where} \quad d = \theta^\mu \frac{\partial}{\partial x^\mu}\,, \quad \sigma = \theta^\mu \frac{\partial}{\partial y^\mu}\,,$$

$$\bar{\gamma}\widetilde{\psi}^A = (\gamma\psi^A)\Big|_{\partial_{(\mu)}\psi^A \to \frac{\partial}{\partial y^{(\mu)}}\widetilde{\psi}^A}$$

If in addition one imposes algebraic constraints which are the starting point equations of motion expressed in terms of y-derivatives and their prolongations the formulation determined by γ^P is equivalent to the starting point one (or alternatively $\gamma \to s = \delta + \gamma + \ldots$) *Barnich, M.G., 2010*

This gives the general prescription for unfolding a given theory.

Parent BV action:

Lagrangian potential $\widehat{L}(\psi, x, \theta)$:

$$\widehat{L} = L_n + L_{n-1} + \ldots + L_0\,, \qquad \text{where} \quad L_n = \theta^{n-1} \ldots \theta^0 L|_{\partial_{(\mu)}\psi^A \to \psi^A_{(\mu)|[]}}$$

$L_{n-1}, \ldots, L_0$ through "Descent equation" $(-d + \bar{\gamma})\widehat{L} = 0$:

$$\begin{aligned} dL_n &= 0, \\ \gamma L_n &= dL_{n-1} \\ \gamma L_{n-1} &= dL_{n-2} \\ \ldots &= \ldots \\ \gamma L_0 &= 0 \end{aligned}$$

$\widehat{L}$ represents Lagrangian as a $\widetilde{\gamma} = -d + \gamma$ cohomology class.

Antifields $\Lambda_A^{(\mu)[\nu]}$, $\quad \mathrm{gh}(\Lambda_A^{(\mu)\nu_1\ldots\nu_l}) = -1 - \mathrm{gh}(\psi^A) + l$ and antibracket

$$\left(\psi^A_{(\mu)[\nu]}(x), \Lambda_B^{(\rho)[\sigma]}(x')\right) = \delta^A_B\, \delta^{(\rho)}_{(\mu)}\, \delta^{[\sigma]}_{[\nu]}\, \delta^{(n)}(x - x')$$

Parent BV action

$$S^P = \int d^n x [\Lambda_A^{(\mu)[\nu]} \gamma^P \psi^A_{(\mu)[\nu]} + \int d^n\theta \widehat{L}(\widetilde{\psi}, x, \theta)]$$

Satisfies: $\left(S^P, S^P\right) = 0$ and $\mathrm{gh}(S^P) = 0$.

In terms of antifields generating function

$$\widetilde{\Lambda}(x, \theta, p) = (\ldots) \sum_{k,l \geqslant 0} \frac{1}{k!l!} \theta^{\rho_{n-l}} \ldots \theta^{\rho_1} \epsilon_{\rho_1 \ldots \rho_n} \Lambda_A^{\mu_1 \ldots \mu_k | \rho_{n-l+1} \cdots \rho_n} p_{\mu_1} \ldots p_{\mu_k}$$

$$S^P = \int d^n x d^n \theta \Big[\langle \widetilde{\Lambda}_A, (d - \sigma + \bar{\gamma}) \widetilde{\psi}^A \rangle + \widehat{L}(\widetilde{\psi}, x, \theta) \Big].$$

$\langle , \rangle$ a natural inner product (contraction of indexes) e.g. $\langle p_\mu, y^\nu \rangle = \delta_\mu^\nu$

Diffeomorphism invariant theories

If the starting point theory is diffeomorphism invariant and diffeomorphisms are in the generating set of gauge transformations (there is a ghost field ξ^μ and $\gamma = \xi^\mu \partial_\mu + \ldots$) then by a field redefinition (originating from $\xi^\mu \to \xi^\mu + \theta^\mu$):

$$\langle \widetilde{\Lambda}_A, (d - \sigma + \bar{\gamma}) \widetilde{\psi}^A \rangle \quad \to \quad \langle \widetilde{\Lambda}_A, (d + \bar{\gamma}) \widetilde{\psi}^A \rangle$$

and $\widehat{L} = \widehat{L}(\widetilde{\psi})$

In this case: AKSZ sigma model

Alexandrov, Kontsevich, Schwartz, Zaboronsky, 1994

More geometrically: supermanifold $\mathcal{M}$ with coordinates $\psi^A_{(\mu)}$, $\Lambda^{(\mu)}_A$ equipped with the odd (Poisson) bracket

$$\left\{\psi^A_{(\mu)}, \Lambda^{(\nu)}_B\right\}_{\mathcal{M}} = \delta^A_B \delta^{(\nu)}_{(\mu)} \qquad \text{gh}(\{\ ,\ \}_{\mathcal{M}}) = -1 + n,$$

odd nilpotent vector field $\bar{\gamma}$, and the function $\widehat{L}(\psi^A)$, $\text{gh}(\widehat{L}) = n$.

Taking $\Pi T\mathcal{X}$ with coordinates x^μ, θ^μ as a source space the AKSZ sigma model has the following BV action

$$S^P_{AKSZ} = \int d^n x d^n \theta [\widetilde{\Lambda}^{(\mu)}_A (d + \bar{\gamma}) \widetilde{\psi}^A_{(\mu)} + \widehat{L}(\widetilde{\psi})]$$

In fact, $\mathcal{M} = \Pi T[n-1]$(Jet space/independent variables).

To arrive at AKSZ form for not necessarily diffeomorphism invariant theory one can keep independent coordinates x^μ and $dx^\mu \equiv \xi^\mu$. This sigma model describes parametrized version of the system.

Equivalence

At the level of EOM's parent formulation is equivalent to the starting point one through the elimination of EOM generalized auxiliary fields.

At the lagrangian level:

Generalized auxiliary fields: χ^i, χ^*_i are generalized auxiliary fields for S_{BV} if they are conjugate in the antibracket and equations $\frac{\delta S}{\delta \chi^i}\Big|_{\chi^*_i=0} = 0$ can be algebraically solved for χ^i. *Dresse, Grégoire, Hennessy, 1990*

(local) BRST cohomology is unchanged!

Truncation:

Degree: deg $\psi^A_{\mu_1\ldots\lambda_k|\nu_l\ldots\nu_l}$ = deg $\Lambda_A^{\mu_1\ldots\mu_k|\nu_1\ldots\nu_l} = k + l$. Eliminating all fields with the degree greater than a finite integer N gives the truncated action.

For any (high enough) N the truncated action is equivalent to the starting point one through the elimination of the generalized auxiliary fields.

Examples:

Relativistic particle

$$L[X,\lambda] = \lambda^{-1} g_{\mu\nu}(X)\partial X^\mu \partial X^\nu + \lambda m^2$$

If ξ is a ghost the BRST differential:

$$\gamma X^\mu = \xi \partial X^\mu\,, \quad \gamma\lambda = \partial(\xi\lambda)\,, \quad \gamma\xi = \partial\xi\xi\,,$$

Supermanifold $\mathcal{M}$:

$X^\mu, \xi, \lambda,$ derivatives $X^\mu_{(l)}, \xi_{(l)}, \lambda_{(l)},$ conjugate momenta $p^{(l)}_\mu, \xi^{(l)}_*, \lambda^{(l)}_*$.

Function $S_\mathcal{M}$:

$$S_\mathcal{M} = \xi[p_\mu X^\mu_{(1)} + \xi^* \xi_{(1)} - \lambda^* \lambda_{(1)} + \lambda^* \xi_{(1)} \lambda + \frac{1}{2}(\lambda^{-1} g_{\mu\nu} X^\mu_{(1)} X^\nu_{(1)} + \lambda m^2) + \ldots]$$

Reducing and redefining ghosts give

$$S^{\text{red}}_\mathcal{M} = -\frac{1}{2}\xi(g^{\mu\nu} p_\mu p_\nu - m^2) \qquad \{X^\mu, p_\nu\}^{\text{red}}_\mathcal{M} = \delta^\mu_\nu\,, \quad \{C, \mathcal{P}\}^{\text{red}}_\mathcal{M} = 1$$

Recall $n = 1$ so that $\text{gh}(\{,\}_\mathcal{M}) = 0$. We arrived at BFV phase space.

Scalar field

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,, \qquad \gamma = 0$$

Variables:

$$\widehat{\phi}(x,y,\theta) = \phi(x,y) + \theta^\mu\phi^1_\mu(x,y) + \dots$$

$$\widehat{\Lambda}(x,p,\theta) = \pi(x,p)\theta^n + \pi^\mu(x,p)\theta^n_\mu + \dots$$

where

$$\theta^n = \theta^0 \dots \theta^{n-1}\,, \qquad \theta^n_\mu = \frac{\partial}{\partial\theta^\mu}\theta^n\,, \qquad \dots$$

Note gh(ϕ) = gh(π^μ) = 0 and gh(ϕ^1_μ) = gh(π) = -1

and π, ϕ and π^μ, ϕ_μ are conjugate in the antibracket.

Elimination of all the fields but $\phi(x)$, $\phi_\mu(x)$, $\pi^\mu(x)$ gives

$$S^{red}_0 = \int d^nx\,[\pi^\mu(\partial_\mu\phi - \phi_\mu) + L(\phi,\phi_\mu)]$$

The familiar elimination of ϕ_μ gives

$$S^{red-1}_0 = \int d^nx\left[\pi^\mu\partial_\mu\phi - (\frac{1}{2}\pi^\mu\pi_\mu + V(\phi))\right]$$

Relation to De Donder-Weyl covariant Hamiltonian formalism.

In this example:

$$H_{DW} = \frac{1}{2}\pi^{\mu}\pi_{\mu} + V(\phi) \quad \text{DW Hamiltonian}$$

The parent formulation odd symplectic structure:

$$\omega = \int_{\Pi T\mathcal{X}} \delta\widetilde{\phi} \wedge \delta\widetilde{\pi}$$

Its integrand in the sector of ghost degree zero fields ϕ and π^{μ} is

$$d\phi \wedge d\pi^{\mu} \wedge (d^{n-1}x)_{\mu}$$

which is a known polysymplectic form.

The polysymplectic form is merely an $n-1$-form component of the canonical odd symplectic BV form.

Given parent form the usual BFV Hamiltonian formulation is obtained by

$$\Omega^{P} = \int d^{n-1}x d^{n-1}\theta [\widetilde{\Lambda}_{A}^{(\mu)}(d+\bar{\gamma})\widetilde{\psi}^{A}_{(\mu)} + \widehat{L}(\widetilde{\psi})] , \quad \{\cdot\, ,\, \cdot\} = \int d^{n-1}x d^{n-1}\theta \{,\}_{\mathcal{M}}$$

Metric Gravity:

Fields: metric g^{ab}, diffeomorphism ghost ξ^a.

BRST differential:

$$\gamma g^{ab} = L_\xi g^{ab} = \xi^c \partial_c g^{ab} - g^{cb}\partial_c \xi^a - g^{ac}\partial_c \xi^b \,, \qquad \gamma \xi^c = (\partial_a \xi^c)\xi^a \,.$$

Auxiliary variables y^a, p_b, auxiliary bracket $\{y^a, p_b\} = \delta^a_b$.

Generating functions for $\mathcal{M}$ coordinates

$$G = \frac{1}{2} g^{ab}_{(c)} y^{(c)} p_a p_b \,, \qquad \Xi = \xi^a_{(c)} y^{(c)} p_a$$
$$\Pi = \pi_a^{(b)} p_{(b)} y^a \,, \qquad U = \frac{1}{2} u^{(c)}_{ab} p_{(c)} y^a y^b \,.$$

$\bar{\gamma}$ compactly written as

$$\bar{\gamma}\Xi = \frac{1}{2}\{\Xi, \Xi\} \,, \qquad \bar{\gamma} G = \{G, \Xi\} \,.$$

Ghost degree zero fields (besides the antifields momenta)

$$\widetilde{G}(x, \theta | y, p) = F(x, y, p) + \ldots \,, \quad \widetilde{\Xi}(x, \theta | y, p) = \Xi(x|y, p) + A_\mu(x|y, p)\theta^\mu + \ldots \,.$$

Action for ghost degree fields

$$S_0^P = \int d^nx d^n\theta \Big[\langle P, dF + \{F, A\}\rangle + \langle \pi, dA + \frac{1}{2}\{A, A\}\rangle + e^0 \ldots e^{n-1} L_0[F]\Big],$$

Here $e^a = e^a_\mu(x)\theta^\mu$ enters $A(x, \theta | y, p)$ as $A = \theta^\mu e^a_\mu(x) p_a + \ldots$.

Implicitly *Vasiliev, 2005*

Related approaches *Borisov, Ogievetsky 1974, Pashnev 97, Ivanov 1979*

Further reduction: all fields except those originating from ξ^a, ξ^a_b and (the independent components of the covariant derivatives of) the curvature are contractible pairs for $\bar{\gamma}$.

The reduced differential: "Russian formula" *e.g. Stora, 1983, cf. Vasiliev*

$$\bar{\gamma}\xi^a = \xi^a_c \xi^c, \qquad \bar{\gamma}\xi^a_b = \xi^a_c \xi^c_b - \frac{1}{2}\xi^c \xi^d R^a_{b\,cd},$$

$$\bar{\gamma} R^b_{c_1 \ldots c_k a_1 a_2 a_3} = \xi^{c_0} R^b_{c_0 c_1 \ldots c_k a_1 a_2 a_3} - \xi^b_d R^d_{c_1 \ldots c_k a_1 a_2 a_3} + \ldots$$

All derivatives of the curvature are also auxiliary so that:

$$S_0^{\text{red}}[\pi^a, \pi^a_b, e^a, \omega^{ab}] = \int d^n x d^n \theta \Big[\pi_a (de^a + \omega^a_b e^b) + \\ + \pi^b_a (d\omega^a_b + \omega^a_c \omega^c_b - \frac{1}{2} e^c e^d R^a_{bcd}) + e^0 \dots e^{n-1} L_0(R) \Big] ,$$

Reducing further:

$$S_0^{\text{red}-1}[\pi^a, e^a, \omega^{ab}] = \int d^n x \pi^{\mu\nu}_a (\partial_{[\nu} e^a_{\mu]} + \omega^{ac}_{[\nu} e^c_{\mu]}) + \int d^n x d^n \theta \; e^0 \dots e^{n-1} L_0[e, \omega] .$$

Finally, standard frame-type action

$$S_{frame}[e^a, \omega^{ab}] = \int d^n x d^n \theta \; \epsilon_{a_1 \dots a_{n-2} a_{n-1} a_n} e^{a_1} \dots e^{a_{n-2}} (d\omega^{a_{n-1} a_n} + \omega_c^{a_{n-1}} \omega^{c a_n}) .$$

Conclusions

- Fruitful exchange of ideas and methods between the local BV-BRST cohomology methods, unfolded formalism, and various approaches to covariant Hamiltonian formalism.

- Provides set-up for quantization problem along the BV quantization method. However, gauge-fixing fermion, integration measure is still to be studied, chance for well-defined Δ-operator?

- Systematic way to construct unfolded formulation starting from the usual form. In particular, to generate frame-type action. It is likey, that frame type free HS actions for symmetric *Vasiliev* and mixed-symmery *Skvortsov* fields starting from respectively *Fronsdal* and *Labastida* actions or better BRST triplet actions *Bengtsson, Stern, Ouvry* and *Alkalaev, M.G., Tipunin*.

- To get McDowell-Mansouri-Stelle-West type formulation starting from ambient space description. At the linear level quite successful for general AdS gauge fields and some conformal fields.

  Alkalaev, M.G. 2009,2010, (Alkalaev talk), Bekaert, M.G. 2009

- Generating procedure for new formulations. In particular, those that manifest one or another structure. In some sense parent formulation and its reductions make the gauge and the BRST cohomology structure manifest. For instance, gravity as a gauge theory of diffeomorphism algebra or bosonic string as a gauge theory for (regular part of) Virasoro algebra.

- As a tool to find a relevant geometry. For instance starting from metric gravity one end up with the Cartan formulation and finds relevant connections and curvatures just by trying to compute BRST cohomology.