On S-brane and flux-brane solutions related to simple finite-dimensional Lie algebras

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The International Workshop Supersymmetries and Quantum Symmetries", 2011

Outline

1. The p-brane bosonic model
   - The model
   - The manifold
   - The p-brane set
   - The field equations

Outline

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The model

The action

$$S = \frac{1}{2\kappa^2} \int\limits_M d^D z \sqrt{|g|} \Big\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \triangle} \frac{\theta_a}{n_a!} \exp[2\lambda_a(\varphi)](F^a)^2_g \Big\} \quad (1.1)$$

where

- $g = g_{MN}(z) dz^M \otimes dz^N$ - a metric defined on M , $|g| = |\det(g_{MN})|$
- $\varphi = (\varphi^\alpha) \in \mathbf{R}^l$ - a set of scalar fields,
- $h_{\alpha\beta}$ - a constant non-degenerate symmetric matrix, $l \times l$, $(l \in \mathbf{N})$,
- $F^a = dA^a = \frac{1}{n_a!} F^a_{M_1 \ldots M_{n_a}} dz^{M_1} \wedge \ldots \wedge dz^{M_{n_a}}$ is an n_a -form
- λ_a - is a 1-form on $\mathbf{R}^l$

The manifold and spaces

The manifold

$$M = (u_-, u_+) \times M_1 \times M_2 \times \ldots \times M_n, \qquad (u_-, u_+) \in \mathbb{R} \tag{1.2}$$

The ansatz for the metric

$$g = we^{2\gamma(u)} du \otimes du + \sum_{i=1}^{n} e^{2\beta^i(u)} \hat{g}^i, \tag{1.3}$$

$w = \pm 1$,
$g^i = g^i_{m_i n_i}(y_i) dy_i^{m_i} \otimes dy_i^{n_i}$ is a metric on M_i , $i = 1, \ldots, n$.
$\hat{g}^i = p_i^* g^i$, $p_i : M \to M_i$, $i = 1, \ldots, n$.
β^i is a scale factor corresponding to M_i .

The factor spaces

All manifolds M_i are Ricci flat spaces.

$$Ric[g^i] = 0, \quad i = 1, \ldots, n.$$

The volume form

The volume d_i -form on the manifold M_i , and a signature factor

$$\tau_i \equiv \sqrt{|g^i(y_i)|} dy_i^1 \wedge \ldots \wedge dy_i^{d_i}. \tag{1.4}$$

$$\varepsilon(i) = sign\left(\det\left(g^i_{m_i n_i}\right)\right) = \pm 1, \qquad for \quad \forall \quad i = 1, \ldots, n. \tag{1.5}$$

The p -brane set S

Let $\Omega = \Omega(n)$ be a set of all ordered (non-empty) subsets of $\{1, \ldots, n\}$ and $I = \{i_1, \ldots, i_k\} \in \Omega$, $i_1 < \ldots < i_k$.

The composite form-fields

$$F^a = \sum_{s \in S} \delta_{a_s} \mathcal{F}^s, \quad S = S_e \sqcup S_m, \quad s = (a_s, v_s, I_s) \forall s \in S$$

$$\mathcal{F}^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I), \quad \mathcal{F}^{(a,m,I)} = e^{-2\lambda_a(\varphi)} * \left(d\Phi^{(a,m,J)} \wedge \tau(J) \right).$$

where

- $a_s \in \triangle$ is a color index,
- $v_s = e, m$ is an electro-magnetic index
- $\Omega_{a,e}, \Omega_{a,m} \subset \Omega$, $I_s \in \Omega_{a_s, v_s}$ describes the location of p -brane worldvolume.

The field equations

$$R_{MN} - \frac{1}{2} g_{MN} R = T_{MN} \quad (1.6)$$

$$\triangle[g]\varphi^{\alpha} - \sum_{a \in \triangle} \theta_a \frac{\lambda_a^{\alpha}}{n_a!} e^{2\lambda_a(\varphi)} (F^a)^2_g = 0 \quad (1.7)$$

$$\nabla_{M_1}[g](e^{2\lambda_a(\varphi)} F^{a, M_1 \dots M_{n_a}}) = 0 \quad (1.8)$$

The stress-energy tensor splits to two components

$$T_{MN} = T_{MN}[g, \varphi] + \sum_{a \in \triangle} \theta_a e^{2\lambda_a(\varphi)} T_{MN}[F^a, g] \quad (1.9)$$

$$T_{MN}[g, \varphi] = h_{\alpha\beta}(\partial_M \varphi^{\alpha} \partial_N \varphi^{\beta} - \frac{1}{2} g_{MN} \partial_P \varphi^{\alpha} \partial^P \varphi^{\beta})$$

$$T_{MN}[F^a, g] = \frac{1}{n_a!}[-\frac{1}{2} g_{MN} (F^a)^2_g + n_a F^a_{M M_2 \dots M_{n_a}} F^{a, M_2 \dots M_{n_a}}]$$

Non-linear σ -model

Enough space-time symmetries

- black-brane solutions
- S-brane solutions
- Flux-brane solutions

Assumption

$\varphi^\alpha = \varphi^\alpha(u)$, $\Phi^s = \Phi^s(u)$, for $u \in (u_-, u_+)$.

- Block-diagonal structure of the stress energy tensor

$$\sigma^A = (\beta^i, \varphi^\alpha) \in \mathbb{R}^N, \qquad N = n + l \qquad (2.1)$$

$$U^s_A = U^s_A \sigma^A = \sum_{i \in I} d_i \beta^i - \chi_s \lambda_a(\varphi), \quad \chi_s = \pm 1, \quad s = e, m. \qquad (2.2)$$

V.D. Ivashchuk, V.N. Melnikov, Class.Quant.Grav. **14**, 1997.

Non-linear σ -model

The action of σ -modenic harmonic gauge

$$S_\sigma = \frac{\mu}{2} \int du \left\{ \hat{G}_{\hat{A}\hat{B}}(X) \partial X^{\hat{A}} \partial X^{\hat{B}} \right\},$$

where $X = (X^{\hat{A}}) = (\beta^i, \varphi^\alpha, \Phi^s)$.

The minisupermetric of the target space

$$\mathcal{G} = \hat{G}_{\hat{A}\hat{B}} X^{\hat{A}} \otimes X^{\hat{B}} = \hat{G} + \sum_{s \in S} \varepsilon_s e^{-2U^s(\sigma)} d\Phi^s \otimes d\Phi^s,$$

$$\hat{G} = G_{ij} d\beta^i \otimes \beta^j + h_{\alpha\beta} d\varphi^\alpha \otimes d\varphi^\beta .$$

$$\hat{G}_{\hat{A}\hat{B}} = \begin{pmatrix} G_{ij} & 0 & 0 \\ 0 & h_{\alpha\beta} & 0 \\ 0 & 0 & \varepsilon_s(I) \exp(-2U^s(\sigma)) \delta_{ss'} \end{pmatrix}$$

Toda-like system

Equations of motion for Φ^s

$$\frac{d}{du}\left(\exp\left(-2U^s(\sigma)\right)\dot{\Phi}^s\right) = 0 \Longleftrightarrow \dot{\Phi}^s = Q_s \exp\left(2U^s(\sigma)\right) \qquad (2.3)$$

Toda-like system Lagrangian and energy

$$L_Q = \frac{1}{2} G_{AB} \dot{\sigma}^A \dot{\sigma}^B - V_Q, \quad V_Q = \frac{1}{2} \sum_{s \in S} \varepsilon_s Q_s^2 \exp 2U^s(\sigma) \qquad (2.4)$$

$$E_Q = \frac{1}{2} G_{AB} \dot{\sigma}^A \dot{\sigma}^B + V_Q, \quad A = (i, \alpha). \qquad (2.5)$$

The solutions of Toda-like system

$$\sigma^A = \sum_{s \in S} \frac{U^{sA}}{(U^s, U^s)} q^s + c^A u + \bar{c}^A, \quad A = (\alpha, i). \tag{2.6}$$

$$G^{AB} = \begin{pmatrix} G^{ij} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix}, \qquad G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}. \tag{2.7}$$

$$U^{si} = G^{ij} U^s_j = \delta_{iI_s} - \frac{d(I_s)}{D-2}, \quad U^{s\alpha} = -\chi_s \lambda^{\alpha}_{a_s}, \tag{2.8}$$

$$(U^s, U^{s'}) = G_{AB} U^{sA} U^{s'B}. \tag{2.9}$$

Notation

$$(U^s, U^{s'}) \equiv d(I_s \cap I_{s'}) + \frac{d(I_s)d(I_{s'})}{2-D} + \chi_s \chi_{s'} \lambda_{\alpha a_s} \lambda_{\beta a_{s'}} h^{\alpha\beta}, \qquad (2.10)$$

$s, s' \in S$, with $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$.

Assume that

$$\textbf{(i)} \qquad (U^s, U^s) \neq 0, \qquad (2.11)$$

for all $s \in S$, and

$$\textbf{(ii)} \qquad \det(U^s, U^{s'}) \neq 0, \qquad (2.12)$$

i.e. the matrix $(U^s, U^{s'})$ is a non-degenerate one.

$$(A_{ss'}) = \left(2(U^s, U^{s'})/(U^{s'}, U^{s'})\right) \qquad (2.13)$$

is "quasi-Cartan" matrix.

Toda molecule and its Lagrangian

$$\ddot{q}^s = -B_s \exp\Big(\sum_{s' \in S} A_{ss'} q^{s'}\Big), \qquad (2.14)$$

$$L_{TL} = \frac{1}{4} \sum_{ss' \in S} h_s A_{ss'} \dot{q}^s \dot{q}^{s'} - \sum_{s \in S} A_s \exp\Big(\sum_{s' \in S} A_{ss'} q^{s'}\Big), \qquad (2.15)$$

where h_s satisfy the relations

$$h_s = K_s^{-1}, \qquad K_s = (U^s, U^s). \qquad (2.16)$$

$$B_s = \varepsilon_s K_s Q_s^2, \quad A_s = B_s h_s / 2, \qquad (2.17)$$

$s \in S$.

The general solutions

$$g = \left(\prod_{s \in S} [f_s(u)]^{2d(I_s)h_s/(D-2)} \right) \Big\{ \exp(2c^0 u + \bar{c}^0) w du \otimes du + \sum_{i=1}^{n} \left(\prod_{s \in S} [f_s(u)]^{-2h_s \delta_{iI_s}} \right) \exp\left(2c^i u + 2\bar{c}^i\right) g^i \Big\} \quad (3.1)$$

$$\exp\left(\varphi^\alpha\right) = \left(\prod_{s \in S} f_s^{h_s \chi_s \lambda^\alpha_{a_s}} \right) \exp\left(c^\alpha u + \bar{c}^\alpha\right) \quad (3.2)$$

$$F^a = \sum_{s \in S} \delta^a_{a_s} \mathcal{F}^s, \quad \mathcal{F}^s = Q_s \Big(\prod_{s' \in S} f_{s'}^{-A_{ss'}} \Big) du \wedge \tau(I_s), \quad s \in S_e \quad (3.3)$$

$$\mathcal{F}^s = Q_s \tau(\bar{I}_s), \quad s \in S_m \quad (3.4)$$

where $f_s = \exp(-q^s)$

Flux and S - brane solutions

Flux-brane solutions

$M_1 = S^1$, $g^1 = d\phi \otimes d\phi$,
$0 < \phi < 2\pi$, - a family of composite fluxbrane solutions.

S-brane solutions

$M_1 = \mathbb{R}$, $g^1 = -dt \otimes dt$,
$-\infty < t < +\infty$ (t is time variable).

The solutions related to Lie algebras

Toda chains related to A_m

$$C_s e^{-q^s(u)} = \sum_{r_1<\ldots<r_s}^{m+1} v_{r_1}\ldots v_{r_s}\Delta^2(w_{r_1},\ldots,w_{r_s})e^{(w_{r_1}+\ldots+w_{r_s})u}, \quad (3.5)$$

where $\Delta^2(w_{r_1}\ldots w_{r_s})$ is the square of the Vandermonde determinant

$$\Delta^2(w_{r_1}\ldots w_{r_s}) = \prod_{r_i<r_j}(w_{r_i}-w_{r_j})^2, \quad s=1,\ldots,m \quad (3.6)$$

and w_r and v_r are integrating constants, satisfying

$$\prod_{r=1}^{m+1} v_r = \Delta^{-2}(w_1,\ldots,w_{m+1}), \qquad \sum_{r=1}^{m+1} w_r = 0. \quad (3.7)$$

$$C_s = \prod_{s'=1}^{m} B_{s'}^{-A^{ss'}}, \quad A^{ss'} = (A_{ss'})^{-1}. \quad (3.8)$$

Polynomial solutions

linear asymptotics at infinity

$$q^s = -\beta^s u + \bar{\beta}^s + o(1), \quad u \to +\infty, \tag{3.9}$$

where $\beta^s, \bar{\beta}^s$ are constants, $s \in S$.

$$\exp(-u) = \rho, \quad H_s = f_s e^{-\beta^s u} = e^{-q^s - \beta^s u} \tag{3.10}$$

V.D. Ivashchuk, Composite fluxbranes with general intersections, *Class. Quantum Grav.*, **19**, 3033-3048 (2002); hep-th/0202022.

The solutions

$$g = \left(\prod_{s \in S} H_s^{2h_s d(I_s)/(D-2)}\right) \left\{ d\rho \otimes d\rho + \left(\prod_{s \in S} H_s^{-2h_s}\right) \rho^2 g^1 + \sum_{i=2}^{n} \left(\prod_{s \in S} H_s^{-2h_s \delta_{iI_s}}\right) g^i \right\}, \qquad (3.11)$$

$$\exp(\varphi^\alpha) = \prod_{s \in S} H_s^{h_s \chi_s \lambda_{a_s}^\alpha}, \qquad (3.12)$$

$$\mathcal{F}^s = -Q_s \left(\prod_{s' \in S} H_{s'}^{-A_{ss'}} \right) \rho d\rho \wedge \tau(I_s), \quad s \in S_e, \qquad (3.13)$$

$$\mathcal{F}^s = Q_s \tau(\bar{I}_s), \quad s \in S_m. \qquad (3.14)$$

Polynomial structure of H_s for Lie algebras

We introduce $z = \rho^2$, $H_s(z) > 0$ and $H(z = +0) = 1$.

Non-linear differential equations for H_s

$$\frac{d}{dz}\left(\frac{z}{H_s}\frac{d}{dz}H_s\right) = \frac{1}{4}B_s\prod_{s'\in S}H_{s'}^{-A_{ss'}}, \tag{3.15}$$

Conjecture.

Let $(A_{ss'})$ be a Cartan matrix for a semisimple finite-dimensional Lie algebra $\mathcal{G}$. Then the solution to eqs. (3.15), (if exists) is a polynomial

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k,$$

where $P_s^{(k)}$ *are constants,* $k = 1, \ldots, n_s$, $\beta^s = 2\sum_{s'\in S} A^{ss'} = n_s \in \mathbf{N}$,

$P_s^{(n_s)} \neq 0$, $s \in S$.

Example: B_3 -polynomials

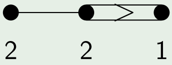
$$B_3 \cong so(7)$$

The Cartan matrix

$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$\Longleftrightarrow$

The Dynkin diagram



The twice components for Weyl vector in dual basis

$$n_1 = 6, \quad n_2 = 10, \quad n_3 = 6. \tag{3.16}$$

A.A.Golubtsova, V.D. Ivashchuk, On calculation of fluxbrane polynomials corresponding to classical series of Lie algebras, nlin.SI:0804.0757

B_3 -polynomials

$$H_1 = 1 + P_1 z + \frac{1}{4} P_1 P_2 z^2 + \frac{1}{18} P_1 P_2 P_3 z^3 + \frac{1}{144} P_1 P_2 P_3^2 z^4 + \frac{1}{3600} P_1 P_2^2 P_3^2 z^5 + \frac{1}{129600} P_1^2 P_2^2 P_3^2 z^6,$$

$$H_2 = 1 + P_2 z + \left(\frac{1}{4} P_1 P_2 + \frac{1}{2} P_2 P_3\right) z^2 + \left(\frac{1}{9} P_2 P_3^2 + \frac{2}{9} P_1 P_2 P_3\right) z^3 + \left(\frac{1}{144} P_2^2 P_3^2 + \frac{1}{72} P_1 P_2^2 P_3 + \frac{1}{16} P_1 P_2 P_3^2\right) z^4 + \frac{7}{600} P_1 P_2^2 P_3^2 z^5 + \left(\frac{1}{1600} P_1 P_2^3 P_3^2 + \frac{1}{5184} P_1^2 P_2^2 P_3^2 + \frac{1}{2592} P_1 P_2^2 P_3^3\right) z^6 + \left(\frac{1}{16200} P_1 P_2^3 P_3^3 + \frac{1}{32400} P_1^2 P_2^3 P_3^2\right) z^7 + \left(\frac{1}{518400} P_1 P_2^3 P_3^4 + \frac{1}{259200} P_1^2 P_2^3 P_3^3\right) z^8 + \frac{1}{4665600} P_1^2 P_2^3 P_3^4 z^9 + \frac{1}{466560000} P_1^2 P_2^4 P_3^4 z^{10},$$

$$H_3 = 1 + P_3 z + \frac{1}{4} P_2 P_3 z^2 + \left(\frac{1}{36} P_1 P_2 P_3 + \frac{1}{36} P_2 P_3^2\right) z^3 + \frac{1}{144} P_1 P_2 P_3^2 z^4 + \frac{1}{3600} P_1 P_2^2 P_3^2 z^5 + \frac{1}{129600} P_1 P_2^2 P_3^3 z^6.$$

The examples: A_1 -solutions, $D = 11$ supergravity

F is a 4-form in the bosonic sector.
Let $n = 3$, M_3 be 7-dimensional (Ricci-flat) manifold with the metric $g^3 = g^3_{\mu\nu} dx^\mu \otimes dx^\nu$ of signature $(-, +, \dots, +)$ and M_2 be 2-dimensional (flat) manifold of signature $(+, +)$ with the metric $g^2 = g^2_{mn} dy^m \otimes dy^n$ and $I_s = \{1, 2\}$.

The solution reads

$$g = H^{1/3} \Big\{ d\rho \otimes d\rho + H^{-1} (\rho^2 d\phi \otimes d\phi + g^2) + g^3 \Big\}, \qquad (4.1)$$

$$F = -Q H^{-2} \rho d\rho \wedge d\phi \wedge \tau_2, \qquad (4.2)$$

where $H = 1 + \frac{1}{2} Q^2 \rho^2$.

The examples: A_2-solutions, $D = 11$

$F6 \cap F3$ fluxbrane configuration with (a non-standard) A_2 intersection rules defined on the manifold

$$M = (0, +\infty) \times M_1 \times M_2 \times M_3 \times M_4, \qquad (4.3)$$

where $d_2 = 2$, $d_3 = 5$, $d_4 = 2$.

$$g = H_e^{1/3} H_m^{2/3} \Big\{ d\rho \otimes d\rho + H_e^{-1} H_m^{-1} \rho^2 d\phi \otimes d\phi + H_e^{-1} g^2 + H_m^{-1} g^3 + g^4 \Big\}, \qquad (4.4)$$

$$F = -Q_e H_e^{-2} H_m \rho d\rho \wedge d\phi \wedge \tau_2 + Q_m \tau_2 \wedge \tau_4, \qquad (4.5)$$

where metrics g^2 and g^3 are (Ricci-flat) metrics of Euclidean signature, g^4 is the (flat) metric of the signature $(-,+)$ and

$$H_s = 1 + P_s \rho^2 + \frac{1}{4} P_1 P_2 \rho^4, \qquad (4.6)$$

where $P_s = \frac{1}{2} Q_s^2$, $s = e, m$.

The examples: $S0$-branes related to Lie algebras (rank 3)

$$M = (0,t) \times M_1 \times M_2$$

Let $P_s = n_s P$, $\Longrightarrow$ $H_s = (1+Pt)^{n_s} = X^{n_s}$, where $X = 1 + Pt$

$$g = X^{2A}\left\{ -dt \otimes dt + X^{-2B} t^2 d\phi \otimes d\phi + g^2 \right\},$$

$$\exp(\varphi^\alpha) = X^{B_1\lambda_1^\alpha + B_2\lambda_2^\alpha + B_3\lambda_3^\alpha},$$

$$F^1 = -Q_1 X^{n_2 - 2n_1} t dt \wedge d\phi,$$

$$F^2 = -Q_2 X^{n_1 - 2n_2 + k_1 n_3} t dt \wedge d\phi,$$

$$F^3 = -Q_3 X^{k_2 n_2 - 2n_3} t dt \wedge d\phi,$$

where

$$A = \frac{B}{D-2}, \quad B = \sum_{s=1}^{3} B_s, \quad B_s = n_s K_s^{-1},$$

$k_1 = (1,2,1)$, $k_2 = (1,1,2)$, for A_3 , B_3 and C_3 , respectively.

THANK YOU FOR YOU ATTENTION!