Integrable string model of WZNW type with constant $SU(2)$ torsion

V. D. Gershun

ITP NSC KIPT, Ukraine

International Workshop "Supersymmetries & Quantum Symmetries – 2011"
July 18–23, 2011

$$L = \frac{1}{2}\int_0^{2\pi}\left[\delta^{\alpha\beta}g_{ab}(X)\frac{\partial X^a}{\partial x^\alpha}\frac{\partial X^b}{\partial x^\beta} + \epsilon^{\alpha\beta}B_{ab}\frac{\partial X^a}{\partial x^\alpha}\frac{\partial X^b}{\partial x^\beta}\right]dx. \qquad (1)$$

$$g_{ab}(X(x)) = g_{ba}(X(x)), \quad B_{ab}(X(x)) = -B_{ba}(X(x)) \qquad (2)$$

The string equations of motion have the form:

$$g_{ab}(\ddot{X}^b - X^{''b}) + \Gamma_{abc}(\dot{X}^b\dot{X}^c - X'^b X"^c) + H_{abc}\dot{X}^b X^{'c} = 0, \qquad (3)$$

$$\Gamma_{abc} = \frac{1}{2}\left(\frac{\partial g_{ab}}{\partial X^c} + \frac{\partial g_{ac}}{\partial X^b} - \frac{\partial g_{bc}}{\partial X^a}\right),$$
$$H_{abc} = \frac{\partial B_{ab}}{\partial X^c} + \frac{\partial B_{ca}}{\partial X^b} + \frac{\partial B_{bc}}{\partial X^a}. \qquad (4)$$

H_{abc} is total antisymmetric tensor.

Let us introduce repers e^a_μ such that

$$g_{ab}(X) = \delta_{\mu\nu} e^\mu_a(X) e^\nu_b(X), \;\; e^\mu_a e^b_\mu = \delta^b_a, \;\; e^\mu_a e^{a\nu} = \delta^{\mu\nu}. \tag{5}$$

where $\mu, \nu = 1, 2, ..24$
The Hamiltonian has form:

$$H = \frac{1}{2} \int\limits_0^{2\pi} [\delta^{\mu\nu} J_{0\mu} J_{0\nu} + \delta_{\mu\nu} J^\mu_1 J^\nu_1] dx, \tag{6}$$

Canonical currents have form

$$\begin{aligned} J_{0\mu}(X) &= e^a_\mu(X)[p_a - B_{ab}(X) X'^b], \\ J^\mu_1(X) &= e^\mu_a X'^a, \qquad p_a(t, x) = g_{ab} \dot{X}^b + B_{ab} X'^b, \end{aligned} \tag{7}$$

Canonical currents satisfy to equation of motion

$$\partial_0 J^\mu_1 - \partial_1 J^\mu_0 = C^\mu_{\nu\lambda} J^\nu_0 J^\lambda_1, \qquad \partial_0 J^\mu_0 - \partial_1 J^\mu_1 = -H^{\mu\nu\lambda} J^\nu_0 J^\lambda_1. \tag{8}$$

Last term in the equation of motion describe the anomaly.

The canonical currents satisfy to following relations:

$$\{J_0^\mu(x), J_0^\nu(y)\} = C_\lambda^{\mu\nu} J_0^\lambda(x)\delta(x-y) + H^{\mu\nu\lambda} J_1^\lambda \delta(x-y),$$
$$\{J_0^\mu(x), J_1^\nu(y)\} = C_\lambda^{\mu\nu} J_1^\lambda(x)\delta(x-y) + \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{J_1^\mu(x), J_1^\nu(y)\} = 0.$$

Here $C^{\mu\nu\lambda}$ is $SU(2)$ torsion.

$$C_{\nu\lambda}^\mu = \frac{\partial e_a^\mu}{\partial x^b}\left(e_\nu^b e_\lambda^a - e_\nu^a e_\lambda^b\right) = \left(\frac{\partial e_a^\mu}{\partial x^b} - \frac{\partial e_b^\mu}{\partial x^a}\right) e_\nu^b e_\lambda^a. \tag{9}$$

Let us introduce chiral currents:

$$U^\mu = \delta^{\mu\nu} J_{0\nu} + J_1^\mu, \qquad V^\mu = \delta^{\mu\nu} J_{0\nu} - J_1^\mu. \tag{10}$$

Equations of motion in light-cone coordinates

$$x^{\pm} = \frac{1}{2}(t \pm x), \qquad \frac{\partial}{\partial x^{\pm}} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} \tag{11}$$

have form:

$$\partial_{-} U^{\mu} = -[C^{\mu}_{\nu\lambda}(X) + H^{\mu}_{\nu\lambda}(X)]U^{\nu}V^{\lambda},$$
$$\partial_{-} V^{\mu} = [C^{\mu}_{\nu\lambda}(X) - H^{\mu}_{\nu\lambda}(X)]U^{\nu}V^{\lambda}.$$

The chiral currents satisfy to following relations:

$$\{U^{\mu}(x), U^{\nu}(y)\} = \frac{1}{2}[(3C^{\mu\nu}_{\lambda} + H^{\mu\nu}_{\lambda})U^{\lambda} - (C^{\mu\nu}_{\lambda} + H^{\mu\nu}_{\lambda})V^{\lambda}]\delta(x-y) + $$
$$+ \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{V^{\mu}(x), V^{\nu}(y)\} = \frac{1}{2}[(3C^{\mu\nu}_{\lambda} - H^{\mu\nu}_{\lambda})V^{\lambda} - (C^{\mu\nu}_{\lambda} - H^{\mu\nu}_{\lambda})U^{\lambda}]\delta(x-y) - $$
$$- \delta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{U^{\mu}(x), V^{\nu}(y)\} = \frac{1}{2}[(C^{\mu\nu}_{\lambda} + H^{\mu\nu}_{\lambda})U^{\lambda} + (C^{\mu\nu}_{\lambda} - H^{\mu\nu}_{\lambda})V^{\lambda}]\delta(x-y). \tag{12}$$

Here $H_{\mu\nu\lambda}$ is additional external torsion.

This relations form algebra if both internal and external torsions are constant. Here are two possibility to simplify this algebra:

1. $H_{\lambda}^{\mu\nu} = -C_{\lambda}^{\mu\nu}$

$$\{U^{\mu}(x), U^{\nu}(y)\} = C_{\lambda}^{\mu\nu} U^{\lambda} \delta(x-y) + \delta^{\mu\nu} \frac{\partial}{\partial x} \delta(x-y), \tag{13}$$

$$\{V^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu} (2V^{\lambda} - U^{\lambda}) \delta(x-y) - \delta^{\mu\nu} \frac{\partial}{\partial x} \delta(x-y), \tag{14}$$

$$\{U^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu} V^{\lambda} \delta(x-y). \tag{15}$$

2. $H_{\lambda}^{\mu\nu} = C_{\lambda}^{\mu\nu}$

$$\{U^{\mu}(x), U^{\nu}(y)\} = C_{\lambda}^{\mu\nu} (2U^{\lambda} - V^{\lambda}) \delta(x-y) + \delta^{ab} \frac{\partial}{\partial x} \delta(x-y), \tag{16}$$

$$\{V^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu} V^{\lambda} - \delta^{\mu\nu} \frac{\partial}{\partial x} \delta(x-y), \tag{17}$$

$$\{U^{\mu}(x), V^{\nu}(y)\} = C_{\lambda}^{\mu\nu} U^{\lambda} \delta(x-y). \tag{18}$$

The chiral currents U^μ in first case and V^μ in second case form Kac-Moody algebras. Equations of motion in light-cone coordinates have form

$$\begin{aligned} &H^\mu_{\nu\lambda} = -C^\mu_{\nu\lambda}, && \partial_- U^\mu = 0, && \partial_+ V^\mu = 2C^{\mu\nu\lambda}U^\nu V^\lambda, \\ &H^\mu_{\nu\lambda} = C^\mu_{\nu\lambda}, && \partial_+ V^\mu = 0, && \partial_- U^\mu = -2C^{\mu\nu\lambda}U^\nu V^\lambda. \end{aligned} \tag{19}$$

Integrable string models with constant $SU(2)$ torsion

We use Latin letters in this section instead of Greek for simplicity. Let torsion $C^a_{bc}(X(x)) \neq 0$ and $C_{abc} = \epsilon_{abc}$ are structure constants of $SU(2)$ Lie algebra. This model coincides to principal chiral model on compact simple Lie group (Gershun V.D. *Integrable string models with constant torsion in terms of chiral invarians of $SU(n)$, $SO(n)$, $SP(n)$ groups*, Ядерная Физика, Том 73, p.325-331, 2010; Phys. Atom. Nucl. Vol. 73, 304-310. , Gershun V.D. *Integrable string models with constant $SU(3)$ torsion*, Physics of particles and nuclti letters, vol. 8, № 3, p. 293-298, 2011; Письма в ЭЧАЯ, т. 8, № 3, с. 293-298,2011 .

To construct integrable dynamical system we must to have hierarchy of PS brackets and to find hierarchy of Hamiltonians through bi-Hamiltonity condition. Another way we must have hierarchy of Hamiltonians and to find hierarchy of PS brackets. This way is more simple if the dynamical system have some group structure.

In bi-Hamiltonian approach to integrable string models with constant torsion we considered the conserved primitive chiral currents currents $H_n(U(x))$, as local fields of the Riemann manifold. The non-primitive local charges of invariant chiral currents form the hierarchy of new Hamiltonians.

We have considered total symmetrical invariant chiral currents of $SU(n)$ group. We shown that the infinite set of non-primitive invariant charges are not commuting and they can not consider as Hamiltonians in a bi-Hamiltonian approach to integrable systems. The consistent systems have $SU(3)$ and $SU(2)$ torsion.
Commutation relations (13) show that currents U^{μ} form closed algebra.
Therefore, we will consider PS right chiral currents and Hamiltonians constructed from right currents.

Let t_a are $2 \otimes 2$ traceless hermitian matrix representations of generators $SU(2)$ Lie algebra:

$$[t_a, t_b] = 2i\epsilon_{abc}t_c,\ Tr(t_a t_b) = 2\delta_{ab}, a,\ b\ = 1, 2, 3. \tag{20}$$

Non primitive invariant tensors may to construct as invariant symmetric polynomials on $su(2)$:

$$d_n = d_{(a_1 \ldots a_n)} = \frac{1}{n!} STr(t_{a_1} \ldots t_{a_{2n}}) = \delta_{(a_1 a_2} \delta_{a_3 a_4} \ldots \delta_{a_{2(n-1)} a_{2n})} \tag{21}$$

where $n = 1, 2, 3, \ldots$ Here is the 1 primitive invariant tensor on $SU(2)$. The invariant non primitive tensors for $n \geq 2$ are functions of primitive tensor. Let us introduce the local chiral currents based on the invariant symmetric polynomials on $SU(2)$ Lie group:

$$H_0(U) = \frac{1}{2}\eta_{ab}U^a U^b, \ldots, H_n(U) = \frac{1}{2(n+1)}(\eta_{ab}U^a U^b)^{n+1} \tag{22}$$

where $n = 0, 1, 2, \ldots$ $H_n(x)$ are the density of Hamiltonians.

The flat PB is

$$\{U^a(x), U^b(x)\}_0 = C_c^{ab}(x) U^c(x) \delta(x-y) + \frac{\partial}{\partial x} \delta^{ab}(x-y) \tag{23}$$

We forgot that C_c^{ab} was torsion of space string coordinates. Here C_c^{ab} is torsion of space chiral currents.
Local PB's on these spaces have form:

$$\begin{aligned} \{U^a(x), U^b(y)\}_n &= C_c^{ab} U^c(x) \delta(x-y) + \\ &+ \frac{1}{2(n+1)} [W_n^{ab}(x) \frac{\partial}{\partial x} \delta(x-y) - W_n^{ab}(y) \frac{\partial}{\partial y} \delta(y-x)] \\ \{U^a(x), U^b(y)\}_n &= -\{U^a(x), U^b(y)\}_n, \qquad W^{ab} = W^{ba} \end{aligned} \tag{24}$$

The weak Jacobi identity

$$\int_0^{2\pi} dx \int_0^{2\pi} dy \int_0^{2\pi} dz A(x)B(y)C(z)\{U^a(x), U^b(y)\}_n, U^c(z)\}_n + (\text{cyclic permutation}) =$$

satisfied for arbitrary W^{ab} except term

$$C_k^{ab}\left[W^{kc}(x)\frac{\partial}{\partial x}\delta(x-y) - W^{kc}(z)\frac{\partial}{\partial z}\delta(z-y)\right]\delta(x-y) + (\text{cyclic } x, y, z). \quad (25)$$

We choose following form of function W^{ab}:

$$W_n^{ab} = \delta^{ab}(UU)^n + 2nU^aU^b(UU)^{n-1}, \;\; UU = \eta_{ab}U^aU^b \quad (26)$$

These PB's satisfy to bi-Hamiltonity condition:

$$\frac{\partial U^a}{\partial t_n} = \{U^a(x), \int_0^{2\pi} H_n(y)dy\}_0 = \{U^a(x), \int_0^{2\pi} H_0(y)dy\}_n,$$

$$\frac{\partial U^a}{\partial t_n} = \frac{\partial}{\partial x}[U^a(UU)^n] \;\; n = 0, 1, 2.... \quad (27)$$

These PB's are PB's of hydrodynamic type with the exception first term. The ultra local term with antisymmetric structure constant C_{abc} in commutation relation of chiral currents U^a does not contribution to equation of motion. We could not obtain equation for current U^a in σ-model without background torsion H_{abc}. We can obtain equation of moution only for invariant currents. The family of invariant chiral currents $H_n(U(x))$ satisfy to conservation equations

$$\partial_- H_n(U(x)) = 0. \tag{28}$$

Notation:

$$f(x) = H_0(U(x)) = \frac{1}{2}(UU). \tag{29}$$

The new nonlinear equations of motion for invariant chiral currents (densities of Hamiltonians) are the following:

$$\frac{\partial f}{\partial t_n} + f^{n-1}\frac{\partial f}{\partial x} = 0, \qquad \text{(not sum)} \tag{30}$$

This equation is generalized unviscid Burgers' equations. They have following solutions:

$$f(x, t_n) = h_n[x - t_n f^{n-1}(x, t_n)] \tag{31}$$

and $h(x)$ is periodical arbitrary function of x.

For simplicity we introduce new variables

$$f^n = y_n, \quad n = 2, ..., \infty. \tag{32}$$

New equations of motion for functions $y(x)_n$ are coincide and they do not depend of n:

$$\frac{\partial y_n}{\partial t_n} + y_n \frac{\partial y_n}{\partial x} = 0, \qquad \text{(no sum)}. \tag{33}$$

These equations are n Burgers equations. General solutions are:

$$y_n = h_n[x - t_n y_n(x, t_n)]. \tag{34}$$

Let us introduce periodical initial conditions

$$\begin{aligned} y_n(x, 0) &= h_0(x, 0), \qquad h_0(x) = h_0(x + 2\pi), \\ y_n(x, t) &= h_0[x - t_n h_0(x)] + O(t_n). \end{aligned} \tag{35}$$

For example, in first approximation on t we obtained following solutions:

$$h_0(x) = e^{ix}, \qquad y_n = e^{ix} - t_n e^{2ix},$$

$$h_0(x) = \sin x, \qquad y_n = \sin x - \frac{1}{2} t_n \sin 2x.$$

(and so on).
Here is exact solution of Burgers equation with periodical bound conditions in term of the Lambert function:

$$y_n(t, x) = h_n[x - t_n y_n(x, t_n)], \qquad h_n = \exp\{a + i(x - t_n y_n)\}, \tag{36}$$

where a is arbitrary parameter. Burgers equation can be rewritten in the following form:

$$Y_n = Z_n e^{Z_n}, \qquad Y_n = i t_n e^{a+ix}, \qquad Z_n = i t_n y_n. \tag{37}$$

The inverse transformation $Z_n = Z_n(Y_n)$ define by the W Lambert function $Z_n = W(Y_n)$:

$$y_n(x, t) = \frac{-i}{t_n} W(i t_n e^{a+ix}), \;\; y_n(x) = (U^a U^a)^n \tag{38}$$

We obtained part of classical solution for 3 dimensional string, last 21 dimensions can be divided to 7 string with $SU(2)$ torsion or can be considered as string with null torsion. Hydrodynamic approach of Dubrovin-Novikov describe this part of string.
The construction of integrable equations with $SU(n)$ symmetries for $SU(n)$ has difficulties of reduction non-primitive invariant currents to primitive currents. We obtained following expressions for non-primitive chiral currents H_n:

$$
\begin{aligned}
SU(4): & H_5 \to \frac{2}{3}H_2H_3, \qquad H_6 \to \frac{1}{6}H_3^2 + \frac{1}{2}H_2H_4, \\
& H_7 \to \frac{1}{3}H_2^2H_3 + \frac{1}{6}H_3H_4, \qquad H_8 \to \frac{7}{36}H_2H_3^2 + \frac{1}{4}H_2^2H_4, \\
& H_9 \to \frac{1}{6}H_2^3H_3 + \frac{1}{36}H_3^3 + \frac{1}{6}H_2H_3H_4. \\
SU(5): & H_6 \to -\frac{3}{50}H_2^3 + \frac{4}{15}H_3^2 + \frac{7}{10}H_2H_4, \\
& H_7 \to -\frac{3}{50}H_2^2H_3 + \frac{11}{30}H_3H_4 + \frac{3}{5}H_2H_5, \\
& H_8 \to -\frac{9}{250}H_2^4\frac{4}{25}H_2H_3^2 + \frac{9}{25}H_2^2H_4 + \frac{1}{10}H_4^2 + \frac{4}{15}H_3H_5.
\end{aligned}
$$

Here H_n was constructed from symmetric structure constant $SU(n)$ algebra.

However, the non-primitive charges are not commuting. They are not Casimirs and we can not consider them as Hamiltonians. This procedure decomposition non primitive invariants to primitive its in the formalism of Poisson brackets is procedure of introduction of infinite number of second kind constraints.
The nonlinear integrable string equation can be obtained for infinite number coordinates of Riemannian space $U^a(x)$ $a = 1, 2, ..\infty$ for system with $SU(\infty)$ torsion. Also, we can obtaine linear infinite chains for string with $SO(\infty)$, $SP(\infty)$ constant torsion.