Black Holes Tits -Satake Universality classes and Nilpotent Orbits

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Based on common work with Aleksander S. Sorin & Mario Trigiante

A well defined mathematical problem

Our goal is just to find and classify all spherical symmetric solutions of Supergravity with a static metric of Black Hole type

The solution of this problem is found by reformulating it into the context of a very rich mathematical framework which involves:

1. The Geometry of COSET MANIFOLDS
2. The theory of **Liouville Integrable systems** constructed on Borel-type subalgebras of SEMISIMPLE LIE ALGEBRAS
3. A very topical issue in conyemporary ADVANCED LIE ALGEBRA THEORY namely:
    1. ***THE CLASSIFICATION OF ORBITS OF NILPOTENT OPERATORS***

The N=2 Supergravity Theory

$$\mathcal{L}^{(4)} = \sqrt{\det g}\left[-2R[g] - \frac{1}{6}\partial_{\hat{\mu}}\phi^a\partial^{\hat{\mu}}\phi^b h_{ab}(\phi) + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\, F^{\Lambda}_{\hat{\mu}\hat{\nu}} F^{\Sigma|\hat{\mu}\hat{\nu}}\right] + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\, F^{\Lambda}_{\hat{\mu}\hat{\nu}} F^{\Sigma}_{\hat{\rho}\hat{\sigma}}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$$

We have gravity
and
n vector multiplets

⇒

2 n scalars yielding n complex scalars z^i

and n+1 vector fields $\mathbf{A}^\Lambda$

The matrix $N_{\Lambda\Sigma}$ encodes together with the metric h_{ab} Special Geometry

Special Kahler Geometry

Let $\mathcal{L} \longrightarrow \mathcal{M}$ complex line bundle such that first Chern class equals Kähler form K. Let $\mathcal{SV} \longrightarrow \mathcal{M}$ be a holomorphic flat vector bundle of rank $2n+2$ with structural group $\mathrm{Sp}(2n+2,\mathrm{R})$

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \quad \Lambda, \Sigma = 0, 1, \dots, n$$

symplectic section

$$\mathrm{i}\langle \Omega \,|\, \bar{\Omega} \rangle \equiv \mathrm{i}\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \bar{\Omega}$$

$$\mathrm{K} = \frac{\mathrm{i}}{2\pi} \partial \bar{\partial} \log \left(\mathrm{i}\langle \Omega \,|\, \bar{\Omega} \rangle \right)$$

Special Geometry identities

$$V = \begin{pmatrix} L^{\Lambda} \\ M_{\Sigma} \end{pmatrix} \equiv e^{\mathcal{K}/2}\Omega = e^{\mathcal{K}/2} \begin{pmatrix} X^{\Lambda} \\ F_{\Sigma} \end{pmatrix}$$

$$U_i = \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i \mathcal{K}\right) V \equiv \begin{pmatrix} f_i^{\Lambda} \\ h_{\Sigma|i} \end{pmatrix}$$

$$\bar{U}_{i^\star} = \nabla_{i^\star} \bar{V} = \left(\partial_{i^\star} + \frac{1}{2}\partial_{i^\star} \mathcal{K}\right) \bar{V} \equiv \begin{pmatrix} \bar{f}_{i^\star}^{\Lambda} \\ \bar{h}_{\Sigma|i^\star} \end{pmatrix}$$

$$\nabla_i V = U_i$$

$$\nabla_i U_j = \mathrm{i} C_{ijk} g^{k\ell^\star} U_{\ell^\star}$$

$$\nabla_{i^\star} U_j = g_{i^\star j} V$$

$$\nabla_{i^\star} V = 0$$

The matrix $N_{\Lambda\Sigma}$

the two $(n+1) \times (n+1)$ vectors

$$f_I^{\Lambda} = \begin{pmatrix} f_i^{\Lambda} \\ \bar{L}^{\Lambda} \end{pmatrix} \quad ; \quad h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \bar{M}_{\Lambda} \end{pmatrix}$$

$$\bar{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda|I} \circ \left(f^{-1}\right)^{I}{}_{\Sigma}$$

When the special manifold is a symmetric coset ..

$$\mathcal{SK}_n = \frac{\mathsf{U}_{D=4}}{\mathsf{H}_{D=4}}$$

$$U_{D=4} \ni \mathbb{L}(\phi) \mapsto \left(\begin{array}{c|c} A(\phi) & B(\phi) \\ \hline C(\phi) & D(\phi) \end{array}\right) \in \mathrm{Sp}(2n+2, \mathrm{R})$$

Symplectic embedding

$$\mathbf{f} = \frac{1}{\sqrt{2}} \left(A(\phi) - \mathrm{i}\, B(f)\right)$$

$$\mathbf{h} = \frac{1}{\sqrt{2}} \left(C(\phi) - \mathrm{i}\, D(f)\right)$$

$$\overline{\mathcal{N}}(\phi) = \mathbf{h}\,\mathbf{f}^{-1}$$

The main point

1) space-like p-branes as the cosmic billiards, or

2) time-like p-branes as several rotational invariant black-holes in $D = 4$ and more general solitonic branes in diverse dimensions

reduce to geodesic equations on coset manifolds of the type

$$\mathcal{M} = \frac{\mathrm{U}}{\mathrm{H}} \quad or \quad \mathcal{M}^\star = \frac{\mathrm{U}}{\mathrm{H}^\star} \simeq \exp\left[\mathrm{Solv}_{\mathcal{M}}\right]$$

Dimensional Reduction to D=3

THE C-MAP

D=4 SUGRA with SK_n → D=3 σ-model on Q_{4n+4}

$$ds^2_Q = \frac{1}{4}\Big[dU^2 + g_{i\bar{\jmath}}\, dz^i\, d\bar{z}^{\bar{\jmath}} + e^{-2U}\,(da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z})^2 \mp 2\, e^{-U}\, d\mathbf{Z}^T\, \mathcal{M}_4(z,\bar{z})\, d\mathbf{Z}\Big]$$

Space red. / Time red.
Cosmol. / Black Holes

$$\underbrace{\{U, a\}}_{2} \bigcup \underbrace{\{z^i\}}_{2n} \bigcup \underbrace{\mathbf{Z} = \{Z^\Lambda, Z_\Sigma\}}_{2n+2}$$

4n + 4 coordinates

Gravity **scalars** **From vector fields**

$$\mathcal{M}_4 = \left(\begin{array}{c|c} \mathrm{Im}\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} \\ \hline \mathrm{Re}\mathcal{N}\,\mathrm{Im}\,\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\,\mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} \end{array}\right)$$

SUGRA BH.s = one-dimensional Lagrangian model

Evolution parameter $\tau \sim \frac{1}{r}$ $\dot{f} \equiv \frac{d}{d\tau} f$

$$\mathcal{L} = \dot{U}^2 + h_{rs}\,\dot{\phi}^r\,\dot{\phi}^s + e^{-2U}\,(\dot{a} + \mathbf{Z}^T\mathbb{C}\dot{\mathbf{Z}})^2 + 2\,e^{-U}\,\dot{\mathbf{Z}}^T\,\mathcal{M}_4\,\dot{\mathbf{Z}}$$

$$\mathcal{L} = \begin{cases} v^2 > 0 \\ v^2 = 0 \\ -v^2 < 0 \end{cases}$$

Time-like geodesic = non-extremal Black Hole
Null-like geodesic = extremal Black Hole
Space-like geodesic = naked singularity

A Lagrangian model can always be turned into a Hamiltonian one by means of standard procedures.

SO BLACK-HOLE PROBLEM = DYNAMICAL SYSTEM

FOR SK_n = symmetric coset space THIS DYNAMICAL SYSTEM is LIOUVILLE INTEGRABLE, always!

When homogeneous symmetric manifolds

$$\frac{\mathsf{U}_{D=4}}{\mathsf{H}_{D=4}} \to \frac{\mathsf{U}_{D=3}}{\mathsf{H}_{D=4}}$$

C-MAP

$$\mathsf{U}_{D=3} \supset \mathsf{U}_{D=4}$$

General Form of the Lie algebra decomposition

$$\mathrm{adj}(\mathbb{U}_{D=3}) = \mathrm{adj}(\mathbb{U}_{D=4}) \oplus \mathrm{adj}(\mathsf{SL}(2,\mathbb{R})_{\mathsf{E}}) \oplus W_{(2,\mathbf{W})}$$

$$[T^a, T^b] = f^{ab}{}_c \, T^c$$
$$[L^x, L^y] = f^{xy}{}_z \, L^z,$$
$$[T^a, W^{iM}] = (\Lambda^a)^M{}_N \, W^{iN},$$
$$[L^x, W^{iM}] = (\lambda^x)^i{}_j \, W^{jM},$$
$$[W^{iM}, W^{jN}] = \epsilon^{ij} \, (K_a)^{MN} \, T^a + \mathbb{C}^{MN} \, k_x^{ij} \, L^x$$

Relation between $\dfrac{U}{H}$ and $\dfrac{U}{H^\star}$

One just changes the sign of the scalars coming from $W_{(2,R)}$ part in:

$$\mathrm{adj}(G_{D=3}) = \mathrm{adj}(G_{D=4}) \oplus \mathrm{adj}(SL(2,\mathbb{R})) \oplus W_{(2,R)}$$

where R is a **symplectic** representation of $G_{D=4}$

Examples

$$\frac{E_{8(8)}}{SO(16)} \longrightarrow \frac{E_{8(8)}}{SO(16)^\star}$$

$$\frac{SO(4,4)}{SO(4)\times SO(4))} \longrightarrow \frac{SO(4,4)}{SO(2,2)\times SO(2,2)}$$

$$\frac{G_{(2,2)}}{SU(2)\times SU(2)} \longrightarrow \frac{G_{(2,2)}}{SU(1,1)\times SU(1,1)}$$

The solvable parametrization

There is a fascinating theorem which provides an identification of the geometry of moduli spaces with Lie algebras for (almost) all supergravity theories.

THEOREM: All **non compact** (symmetric) coset manifolds are ***metrically equivalent*** to a **solvable group manifold**

$$\mathsf{U}/\mathsf{H} \cong \exp\left[Solv\left(\mathsf{U}/\mathsf{H}\right)\right]$$

Splitting the Lie algebra **U** *into the maximal compact subalgebra* **H** *plus the orthogonal complement* **K**

$$\mathbb{U} = \mathbb{H} \oplus \mathbb{K}$$

- There are precise rules to construct **Solv(U/H)**
- Essentially **Solv(U/H)** is made by
  - the non-compact Cartan generators $\mathbf{H_i} \in \mathbf{CSA} \cap \mathbf{K}$ **and**
  - those positive root step operators $\mathbf{E}^{\alpha}$ which are not orthogonal to the non compact Cartan subalgebra $\mathbf{CSA} \cap \mathbf{K}$

The simplest example $G_{2(2)}$

One vector multiplet

$$\text{adj}\ \mathfrak{g}_{2(2)} = (\text{adj}\,[\mathfrak{sl}(2,\mathbb{R})_E]\ \mathbf{1}) \oplus (\mathbf{1}\,,\ \text{adj}\,[\mathfrak{sl}(2,\mathbb{R})]) \oplus (\mathbf{2}\,,\ \mathbf{4})$$

$$g_{z\bar{z}} dz\, d\bar{z} = \frac{3}{4}\frac{1}{(\text{Im}z)^2}\partial^\mu z\, \partial_\mu \bar{z}$$ **Poincaré metric**

$$\Omega(z) = \begin{pmatrix} -\sqrt{3}z^2 \\ z^3 \\ \sqrt{3}z \\ 1 \end{pmatrix}$$ **Symplectic section**

$$\overline{\mathcal{N}}_{\Lambda\Sigma}(z) = \begin{pmatrix} -\frac{3z+\bar{z}}{2z\bar{z}} & -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} \\ -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} & -\frac{z+3\bar{z}}{2z\bar{z}^3} \end{pmatrix}$$ **Matrix $N_{\Lambda\Sigma}$**

OXIDATION 1

The metric

$$ds^2_{(4)} = -e^{U(\tau)}\,(dt + A_{KK})^2 + e^{-U(\tau)}\left[e^{4A(\tau)}\,d\tau^2 + e^{2A(\tau)}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

where $A_{KK} = 2\,\mathbf{n}\cos\theta\,d\varphi$

Taub-NUT charge

$$\underbrace{\left[e^{-2U}\left(\dot{a} + Z^\Lambda \dot{Z}_\Lambda - Z_\Sigma \dot{Z}^\Sigma\right)\right]}_{\mathbf{n}=\text{Taub NUT charge}}$$

The electromagnetic charges

$$\mathcal{Q}^M = \sqrt{2}\left[e^{-U}\mathcal{M}_4\dot{Z} - \mathbf{n}\,\mathbb{C}\,Z\right]^M = \begin{pmatrix} p^\Lambda \\ e_\Sigma \end{pmatrix}$$

From the σ-model viewpoint all these first integrals of the motion

$$e^{2A(\tau)} = \begin{cases} \dfrac{v^2}{\sinh^2(v\tau)} & \text{if } v^2 > 0 \\ \dfrac{1}{\tau^2} & \text{if } v^2 = 0 \end{cases}$$

Extremality parameter

OXIDATION 2

The electromagnetic field-strenghts

$$F^{\Lambda} = 2\,p^{\Lambda}\,\sin\theta\,d\theta \wedge d\varphi + \dot{Z}^{\Lambda} d\tau \wedge (dt + 2\mathbf{n}\,\cos\theta\,d\varphi)$$

U, a, $\phi \sim$ z, Z^A *parameterize in the G/H case the coset representative*

$$\mathbb{L}(\Phi) = \exp\left[-a\,L^E_+\right]\,\exp\left[\sqrt{2}\,Z^M\,\mathcal{W}_M\right]\,\mathbb{L}_4(\phi)\,\exp\left[U\,L^E_0\right]$$

gen. in **(2,W)**

Coset repres. in D=4

Element of Borel $(U_{D=3})$

Ehlers SL(2,R)

From coset rep. to Lax equation

$$\Sigma(\tau) \equiv \mathbb{L}^{-1}(\tau)\,\frac{d}{d\tau}\mathbb{L}(\tau)$$ **From coset representative**

$$\begin{aligned} \Sigma(\tau) &= L(\tau) \oplus W(\tau) \\ W(\tau) \in \mathbb{H}^{\star} \;\Rightarrow\; & \eta\, W^{T}(\tau) + W(\tau)\eta = 0 \\ L(\tau) \in \mathbb{K} \;\Rightarrow\; & \eta\, L^{T}(\tau) - L(\tau)\eta = 0 \end{aligned}$$ **decomposition**

$$W(\tau) = L_{>}(\tau) - L_{<}(\tau)$$ **R-matrix**

$$\frac{d}{d\tau}L(\tau) = [W(\tau)\,,\,L(\tau)]$$ **Lax equation**

Integration algorithm

Initial conditions $L_0 = L(0) \quad , \quad \mathbb{L}_0 = \mathbb{L}(0)$

Building block $\mathcal{C}(\tau) := \exp\left[-2\,\tau\,L_0\right]$

$$\mathfrak{D}_i(\mathcal{C}) := \text{Det}\begin{pmatrix} \mathcal{C}_{1,1}(\tau) & \dots & \mathcal{C}_{1,i}(\tau) \\ \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(\tau) & \dots & \mathcal{C}_{i,i}(\tau) \end{pmatrix}, \quad \mathfrak{D}_0(\tau) := 1\,.$$

$$\left(\mathbb{L}(\tau)^{-1}\right)_{ij} \equiv \frac{1}{\sqrt{\mathfrak{D}_i(\mathcal{C})\mathfrak{D}_{i-1}(\mathcal{C})}}\text{Det}\begin{pmatrix} \mathcal{C}_{1,1}(\tau) & \dots & \mathcal{C}_{1,i-1}(\tau) & (\mathcal{C}(\tau)\mathbb{L}(0)^{-1})_{1,j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(\tau) & \dots & \mathcal{C}_{i,i-1}(\tau) & (\mathcal{C}(\tau)\mathbb{L}(0)^{-1})_{i,j} \end{pmatrix}$$

Found by Fre & Sorin 2009 - 2010

Key property of integration algorithm

$$L(\tau) = \mathcal{Q}(\mathcal{C})\, L_0\, (\mathcal{Q}(\mathcal{C}))^{-1}$$

$$\mathcal{Q}(\mathcal{C}) \in \mathsf{H}^\star$$

Hence all LAX evolutions occur within distinct orbits of H*

Fundamental Problem: classification of ORBITS

The role of H*

$$\mathsf{U}_{D=3} \supset \begin{cases} \mathsf{H} \\ \text{and} \\ \mathsf{H}^\star \end{cases}$$

H — **Max. comp. subgroup** — **COSMOL.**

H* — **Different real form of H** — **BLACK HOLES**

In our simple $G_{2(2)}$ model

$$\mathbb{H}^\star = \mathfrak{sl}(2,R) \oplus \mathfrak{sl}(2,R)$$

The algebraic structure of Lax

For the simplest model ,the Lax operator, is in the representation

$$\left(j = \frac{1}{2}\right) \times \left(j = \frac{3}{2}\right)$$

of

$$\mathfrak{sl}(2, R) \times \mathfrak{sl}(2, R)$$

$$L \sim \Delta^{\alpha|A}$$

We can construct invariants and tensors with powers of L

Invariants & Tensors

$$\mathfrak{h}_6 = \frac{1}{6}\mathrm{Tr}L^6 + \frac{1}{96}\left(\mathrm{Tr}L^2\right)^3$$

$$\mathfrak{h}_2 = \frac{1}{4}\mathrm{Tr}\,L^2$$

$$\left[\left(\mathrm{j}=\frac{3}{2}\right)\otimes\left(\mathrm{j}=\frac{3}{2}\right)\right]_{symm} = \underbrace{(\mathbf{j}=\mathbf{3})}_{7}\oplus\underbrace{(\mathbf{j}=\mathbf{1})}_{1}$$

$$\left[\left(\mathrm{j}=\frac{3}{2}\right)\otimes\left(\mathrm{j}=\frac{3}{2}\right)\right]_{antisym} = \underbrace{(\mathbf{j}=\mathbf{2})}_{5}\oplus\underbrace{(\mathbf{j}=\mathbf{0})}_{1}$$

Quadratic Tensor

$$\mathcal{T}^{xy} \equiv \frac{128}{\sqrt{3}}\, t^{xy}{}_{AB}\,\Delta^{\alpha|A}\,\Delta^{\beta|B}\,\epsilon_{\alpha\beta}$$

Tensors 2

QUADRATIC BIVECTOR

$$\mathcal{W}^{a|x} \equiv 1280\, \Pi^{a}_{\alpha\beta}\, \Sigma^{z}_{AB}\, \Delta^{\alpha|A}\, \Delta^{\beta|B}$$

$$\left[\left(\mathrm{j}=\frac{3}{2}\right)\otimes\left(\mathrm{j}=\frac{3}{2}\right)\right]_{symm} = \underbrace{(\mathbf{j}=\mathbf{3})}_{7}\oplus\underbrace{(\mathbf{j}=\mathbf{1})}_{1}$$

$$\left[\left(\mathrm{j}=\frac{3}{2}\right)\otimes\left(\mathrm{j}=\frac{3}{2}\right)\right]_{antisym} = \underbrace{(\mathbf{j}=\mathbf{2})}_{5}\oplus\underbrace{(\mathbf{j}=\mathbf{0})}_{1}$$

Tensors 3

Hence we are able to construct quartic tensors

$$\mathfrak{T}^{xy} = \mathcal{W}^{a|x}\,\mathcal{W}^{b|y}\,\eta_{ab}$$

$$\mathbb{T}^{ab} = \mathcal{W}^{a|x}\,\mathcal{W}^{b|y}\,\eta_{xy}$$

ALL TENSORS, QUADRATIC and QUARTIC are symmetric

Their signatures classify orbits, both regular and nilpotent!

Tensor classification of orbits

Orbit	Order Nilp.	Stand. Repr.	Stab. subg.	Sign. $\mathcal{T}^{xy}$	Sign. $\mathfrak{T}^{xy}$	Sign. $\mathbb{T}^{ab}$	Bivect. $W^{a\|x}$	$\mathfrak{I}_4$ at $\mathfrak{n}=0$	Dim. $\mathfrak{n}=0$ shell
Schw.	∞	$\mathfrak{S}$	$O(2)$	$\{+,+,+\}$	$\{+,0,0\}$	$\{+,0,0\}$	$\neq 0$	$\neq 0$	4
Dil.	∞	$\mathfrak{D}$	$O(1,1)$	$\{-,-,+\}$	$\{-,0,0\}$	$\{-,0,0\}$	$\neq 0$	$\neq 0$	4
N01	2	$\mathcal{L}_{NO_1}$	$O(1,1) \ltimes \mathbb{R}^2$	$\{0,0,0\}$	$\{0,0,0\}$	$\{0,0,0\}$	0	0	2
N02	3	$\mathcal{L}_{NO_2}$	$O(1,1) \ltimes \mathbb{R}$	$\{0,0,0\}$	$\{0,0,0\}$	$\{0,0,0\}$	$\neq 0$	0	3
N03	3	$\mathcal{L}_{NO_3}$	$\mathbb{R}$	$\{0,0,+\}$	$\{0,0,0\}$	$\{0,0,0\}$	$\neq 0$	< 0	4
N03′	3	$\mathcal{L}'_{NO_3}$	$\mathbb{R}$	$\{0,0,-\}$	$\{0,0,0\}$	$\{0,0,0\}$	$\neq 0$	< 0	4
N04	3	$\mathcal{L}_{NO_4}$	$\mathbb{R}$	$\{0,0,0\}$	$\{0,0,0\}$	$\{0,0,+\}$	$\neq 0$	> 0	4
N04′	3	$\mathcal{L}'_{NO_4}$	$\mathbb{R}$	$\{0,0,0\}$	$\{0,0,0\}$	$\{0,0,-\}$	$\neq 0$	> 0	4
N05	7	$\mathcal{L}_{NO_5}$	0	$\{0,+,-\}$	$\{0,0,-\}$	$\{0,+,-\}$	$\neq 0$	< 0	5

How do we get to this classification? The answer is the following: by choosing a new Cartan subalgebra inside H* and recalculating the step operators associated with roots in the new Cartan Weyl basis! Then applying the technique of **standard triples…!**

Relation between old and new Cartan Weyl bases

New Cartan Weyl generators	their form in the HK-basis
H_1	h_3
H_2	h_5
E_1	$\frac{1}{2\sqrt{2}}\left(k_3 - 3\mathcal{H}_1 - \mathcal{H}_2\right)$
E_2	$-\frac{1}{4}\sqrt{\frac{3}{2}}\left(k_1 + k_2 + k_4 - k_6\right)$
E_3	$\frac{1}{4\sqrt{2}}\left(-3h_1 + h_2 - h_4 + 3h_6\right)$
E_4	$\frac{1}{4\sqrt{2}}\left(-3k_1 + k_2 + k_4 + 3k_6\right)$
E_5	$\frac{1}{4}\sqrt{\frac{3}{2}}\left(-h_1 - h_2 + h_4 + h_6\right)$
E_6	$\frac{1}{2}\sqrt{\frac{3}{2}}\left(k_5 - \mathcal{H}_1 - \mathcal{H}_2\right)$

$\in \mathbb{K}$

$$
\begin{aligned}
h_1 &= e_2 + f_2 & k_1 &= e_2 - f_2 \\
h_2 &= e_1 - f_1 & k_2 &= e_1 + f_1 \\
h_3 &= e_3 + f_3 & k_3 &= e_3 - f_3 \\
h_4 &= e_4 + f_4 & k_4 &= e_4 - f_4 \\
h_5 &= e_5 + f_5 & k_5 &= e_5 - f_5 \\
h_6 &= e_6 - f_6 & k_6 &= e_6 + f_6
\end{aligned}
$$

The method of standard triplets

The basic theorem proved by mathematicians is that any nilpotent element of a Lie algebra $X \in \mathfrak{g}$ can be regarded as belonging to a triplet of elements $\{x, y, h\}$ satisfying the standard commutation relations of the $\mathfrak{sl}(2)$ Lie algebra, namely:

$$[h\,,\,x] = x \quad ; \quad [h\,,\,y] = -\,y \quad ; \quad [x\,,\,y] = 2\,h$$

Hence the classification of nilpotent orbits is just the classification of embeddings of an $\mathfrak{sl}(2)$ Lie algebra in the ambient one, modulo conjugation by the full group $\mathrm{G}_{\mathbb{R}}$ or by one of its subgroups. In our case the relevant subgroup is $\mathrm{H}^\star \subset \mathrm{G}_{\mathbb{R}}$.

Angular momenta

Embeddings of subalgebras $\mathfrak{h} \subset \mathfrak{g}$ are characterized by the branching law of any representation of $\mathfrak{g}$ into irreducible representations of $\mathfrak{h}$. In the case of the $\mathfrak{sl}(2) \sim \mathfrak{so}(1,2)$ algebra the branching law is expressed by listing the angular momenta $\{j_1, j_2, \ldots j_n\}$ of the irreducible blocks into which the fundamental representations decomposes.

$$\sum_{i=1}^{n} (2j_i + 1) = N$$

Partitions

- (j=3) → The largest orbit NO5
- (j=1, j=1/2, j=1/2) → The orbit NO2
- (j=1, j=1,j=0) → Splits into NO3 and NO4 orbits and their primed versions
- (j=1/2, j=1/2, j=0, j=0, j=0) → The smallest orbit NO1

Results to appear in short

- Mario Trigiante will describe the classification of nilpotent orbits by means of α , β and γ – labels in the next talk.
- In a paper to appear in a couple of weeks we (Fre, Trigiante & Sorin) will:
  - Show that the classification of nilpotent orbits is a universality property, namely depends only on the **Tits Satake universality class** of the considered homogeneous special geometry.
  - Describe a computational algorithm of H* orbits based on the **Weyl group** and certain **appropriate subgroups** thereof so far not yet introduced in the math literature

Спосибо за внимание

Thank you for your attention