SQS'2011

GENERALIZED EINSTEIN – EDDINGTON AFFINE THEORIES OF GRAVITY,

THEIR SPHERICAL / CYLINDRICAL SOLUTIONS AND DUALITY BETWEEN VECTON AND SCALARON

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Based on:

arXiv:1011.2445 v1 (gr-qc) includes interpretation of 1-dim. reduced theory as a relativistic particle in a potential

+ **new paper, in preparation**.

arXiv:1008.2333 v1 (hep-th) attempt at a new general formulation of geom.
arXiv:1003.0782 v3 (hep-th) further generalizations, cosmological solutions.
arXiv:0812.2616 v2 (gr-qc) the first paper on new interpretation of **Einstein**
3 papers of **1926**; simplified model, static solutions, existence of horizons,
non-integrability, approximate solutions by various power series expansions.

A selection of relevant earlier papers

A.T. Filippov, Some unusual dimensional reductions of gravity: geometric potentials, separation of variables, and static - cosmological duality, arXiv:hep th/0605276v2, 2006;
Proc. of the workshop ‘Supersymmaetries and Quantum Symmetries, Dubna, July 27-31, JINR, Dubna 2006.

A.T. Filippov and D. Maison, Class. Quant. Grav.**20** (2003) 1779

V. de Alfaro and A.T. Filippov, Integrable low dimensional theories describing high dimensional branes, black holes and cosmologies, hep-th/0307269 (2003);
Integrable low dimensional models for black holes and cosmologies from high dimensional theories, hep-th/0504101 (2005).

V. de Alfaro and A.T. Filippov, Multiexponential models of (1+1)-dimensional dilaton gravity and Toda - Liouville integrable models, Theor. Math. Phys. **162**(1) (2010) 34-56.

Main principles (suggested by Einstein's approach)

1. **Geometry:** **dimensionless** `*action*' constructed of a ***scalar*** *density*; its variations give the geometry and main equations ***without complete specification of the analytic form of the Lagrangian .***
2. **Dynamics:** a concrete Lagrangian constructed of the ***geometric variables*** - homogeneous 4th order function (e.g. , the square root of the determinant of the curvature) produces a physical **effective Lagrangian**.
3. **Duality** between the geometrical and physical variables and Lagrangians.

NB: This looks more artificial than the first two principles and works for rather special models (actually giving *exotic fields*, *tachyons* etc.) (Einstein. did not know this! He was looking for unified theory of EM and Gravity.)

GEOMETRY OF SYMMETRIC CONNECTIONS

$$\gamma^i_{jk} = \Gamma^i_{jk}[g] + a^i_{jk}$$

$$\Gamma^i_{jk}[g] = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l})$$

$$r^i_{jkl} = -\gamma^i_{jk,l} + \gamma^i_{mk}\gamma^m_{jl} + \gamma^i_{jl,k} - \gamma^i_{ml}\gamma^m_{jk}$$

NONSYMMETRIC RICCI CURVATURE

$$r_{jk} = -\gamma^i_{jk,i} + \gamma^i_{mk}\gamma^m_{ji} + \gamma^i_{ji,k} - \gamma^i_{mi}\gamma^m_{jk}$$

Symmetric part of the Ricci curvature

$$s_{ij} \equiv \frac{1}{2}(r_{ij} + r_{j\,i})$$

Anti-symmetric part of the Ricci curvature

$$a_{ij} \equiv \frac{1}{2}(r_{ij} - r_{j\,i}) = \frac{1}{2}(\gamma^m_{jm,i} - \gamma^m_{im,j})$$

$$a_{ij,\,k} + a_{jk,i} + a_{ki,j} \equiv 0$$

VECTON: $a_i \equiv a^m_{im}$

$$a_i \equiv \gamma^m_{mi} - \Gamma^m_{mi} \equiv \gamma_i - \partial_i \ln \sqrt{|g|}$$

$$a_{ij} \equiv -\frac{1}{2}(a_{i,j} - a_{j,i}) \equiv -\frac{1}{2}(\gamma_{i,j} - \gamma_{j,i})$$

EDDINGTON'S SCALAR DENSITY

$$\mathcal{L} \equiv \sqrt{-\det(r_{ij})} \equiv \sqrt{-r}$$

$$s_{ij} = -\nabla^{\gamma}_{m}\gamma^{m}_{ij} + \frac{1}{2}(\nabla^{\gamma}_{i}\gamma_{j} + \nabla^{\gamma}_{j}\gamma_{i}) - \gamma^{m}_{ni}\gamma^{n}_{mj} + \gamma^{n}_{ij}\gamma_{n}$$

Expressing in terms of the `metric' and using notation $\nabla_i \equiv \nabla^{g}_{i}$

$$s_{ij} = R_{ij}[g] - \nabla_m a^{m}_{ij} + \frac{1}{2}(\nabla_i\, a_j + \nabla_j\, a_i) + a^{m}_{ni}a^{n}_{mj} - a^{m}_{ij}a_m$$

$a_{ij} \equiv -\frac{1}{2}(a_{i,j} - a_{j,i})$ depends only on the vecton

‘GEODESICS’ (PATHS)

$$\ddot{x}^{\,i} + \gamma^{i}_{jk}\,\dot{x}^{j}\,\dot{x}^{k} = 0$$

TRANSFORMATIONS **PRESERVING PATHS**

$$\hat{\gamma}^{i}_{jk} = \gamma^{i}_{jk} + \delta^{i}_{j}\,\hat{a}_{k} + \delta^{i}_{k}\,\hat{a}_{j}$$

GEO-RIEMANNIAN CONNECTIONS

$$\hat{\gamma}^{i}_{jk} = \Gamma^{i}{}_{jk}[g] + \delta^{i}_{j}\,\hat{a}_{k} + \delta^{i}_{k}\,\hat{a}_{j}$$

$\alpha\beta$ - CONNECTION

$$\gamma^i_{jk} = \Gamma^i_{jk}[g] + \alpha(\delta^i_j\, \hat{a}_k + \delta^i_k\, \hat{a}_j) - (\alpha - 2\beta) g_{jk}\, \hat{a}^i$$

Weyl: $\beta = 0$

geo-Riemannian: $\alpha = 2\beta$

Einstein $\alpha = -\beta = \frac{1}{6}$

LINEAR TERMS in $s_{ij} - R_{ij}(g)$

$$(\alpha + \beta)(\nabla_i \hat{a}_j + \nabla_j \hat{a}_i) + (\alpha - 2\beta)\, g_{ij} \nabla_m \hat{a}^m$$

QUADRATIC TERMS in $s_{ij} - R_{ij}(g)$

$$\hat{a}_i \hat{a}_j \left[(\alpha - 2\beta)^2 - 3\alpha^2\right] + 2\, g_{ij} \hat{a}^2 (\alpha - 2\beta)(\alpha + \beta)$$

In addition to this dependence on the vecton,
the generalized Einstein equations will depend on it
through dynamics specified by the chosenLagrangian

FROM GEOMETRY TO DYNAMICS

REQUIREMENTS TO LAGRANGIAN DENSITIES

1. IT IS INDEPENDENT OF DIMENSIONAL CONSTANTS.

2. ITS INTEGRAL OVER SPACE-TIME IS DIMENSIONLESS.

3. IT CAN DEPEND ON TENSOR VARIABLES HAVING
a DIRECT GEOMETRIC MEANING and
a NATURAL PHYSICAL INTERPRETATION.

4. THE RESULTING GENERALIZED THEORY MUST AGREE WITH ALL ESTABLISHED EXPERIMENTAL CONSEQUENCES OF EINSTEIN'S THEORY.

r_{ij}, s_{ij}, a_{ij}, and $a_k \equiv a^i_{ik}$ satisfy requirement **3**.

Einstein's choice is $\mathcal{L} = \mathcal{L}(s_{ij}, a_{ij})$

A simple nontrivial choice of a geometric Lagrangian density generalizing the Eddington – Einstein Lagrangian ,

$$\mathcal{L} \equiv \sqrt{-\det(r_{ij})} \equiv \sqrt{-r} \ ,$$

is the following, depending on one dimensionless parameter:

$$\mathcal{L} = \mathcal{L}(s_{ij} + \nu a_{ij}) = \sqrt{-\det(s_{ij} + \nu a_{ij})}$$

$$\det(s_{ij}) < 0$$

When $\nu a_{ij} \rightarrow 0$ it will give Einstein's gravity with the cosmological constant.

Define the following densities of the weight two

$$d_0 \equiv 4!\,\det(s_{ij}) = \epsilon^{ijkl} s_{im} s_{jn} s_{kr} s_{ls} \epsilon^{mnrs} \equiv \epsilon \cdot s \cdot s \cdot s \cdot s \cdot \epsilon \,.$$

$$d_1 \equiv \epsilon \cdot s \cdot s \cdot s \cdot \bar{a} \cdot \epsilon \,, \qquad d_2 \equiv \epsilon \cdot s \cdot s \cdot a \cdot a \cdot \epsilon \,,$$

$$d_4 \equiv \epsilon \cdot a \cdot a \cdot a \cdot a \cdot \epsilon$$

where $\bar{a}$ denotes the matrix $a_i a_j$

$$\det(s_{ij} + \nu a_{ij}) = \frac{1}{4!} \left(d_0 + 6\nu^2 d_2 + \nu^4 d_4 \right)$$

A more general Lagrangian

$$\mathcal{L} \equiv \alpha_0 \sqrt{|\, d_0 \; + \; \alpha_1 d_1 \; + \; \alpha_2 d_2 \; + \; \alpha_4 d_4|}$$

Now we **define** (following Einstein) the metric and field densities by a Legendre-like transformation

$$\frac{\partial \mathbf{L}}{\partial s_{ij}} \equiv \mathbf{g}^{ij} \,, \qquad \frac{\partial \mathbf{L}}{\partial a_{ij}} \equiv \mathbf{f}^{ij}$$

dual to

$$s_{ij} = \frac{\partial \mathbf{L}^*}{\partial \mathbf{g}^{ij}} \,, \qquad a_{ij} = \frac{\partial \mathbf{L}^*}{\partial \mathbf{f}^{ij}}$$

$$2\nabla_i^\gamma \, \mathbf{g}^{kl} \;=\; \delta_i^l \, \nabla_m^\gamma \, (\mathbf{g}^{km} + \mathbf{f}^{km}) + \delta_i^k \, \nabla_m^\gamma \, (\mathbf{g}^{lm} + \mathbf{f}^{lm})$$

$$\nabla_i^\gamma \, \mathbf{f}^{kl} \;=\; \partial_i \, \mathbf{f}^{kl} + \gamma_{im}^k \, \mathbf{f}^{ml} + \gamma_{im}^l \, \mathbf{f}^{km} - \gamma_{im}^m \, \mathbf{f}^{kl}$$

$$\nabla_i^\gamma \, \mathbf{f}^{ki} \;=\; \partial_i \mathbf{f}^{ki} \;\equiv\; \mathbf{a}^k. \qquad \nabla_i^\gamma \, \mathbf{g}^{ik} \;=\; -\frac{D+1}{D-1} \hat{\mathbf{a}}^k$$

The **main** equation $$\nabla_i^\gamma \, \mathbf{g}^{jk} \;=\; -\frac{1}{D-1} (\delta_i^j \hat{\mathbf{a}}^k + \delta_i^k \hat{\mathbf{a}}^j)$$

for any dimension D

Defining the Riemann metric tensor g_{ij} by the equations

$$g^{ij}\sqrt{-g} = \mathbf{g}^{ij}\,, \quad g_{ij}\,g^{jk} = \delta_i^k$$

$$\nabla_i\, g_{jk} = 0, \quad \nabla_i\, g^{jk} = 0 \qquad \hat{a}^k \equiv \hat{\mathbf{a}}^k/\sqrt{-g}$$

we can derive the expression for the connection coefficients

$$\gamma^i_{jk} = \Gamma^i_{jk}[g] + \alpha_D\,[\,\delta^i_j\,\hat{a}_k + \delta^i_k\,\hat{a}_j - (D-1)\,g_{jk}\,\hat{a}^i]$$

$$\alpha_D \equiv [(D-1)(D-2)]^{-1}\,, \qquad \beta_D \equiv -[2(D-1)]^{-1}$$

We thus have derived the connection using a rather general dynamics!

Using a simple dimensional reduction to the dimension 1+1 (similar to spherical or cylindrical reductions in the metric case) we easily derive the important relation between geom. and phys.

$$\mathcal{L} = -\frac{1}{2}\sqrt{|\det(s + \lambda^{-1}\, a)|} = -2\Lambda\sqrt{|\det(\mathbf{g} + \lambda\mathbf{f})|} = \mathcal{L}^*$$

Λ having the dimension L^{-2}

Using the above definitions, → we can then write the **generalized Einstein eqs.**

$$s_{ij} = \frac{\partial \mathbf{L}^*}{\partial \mathbf{g}^{ij}}\,, \qquad a_{ij} = \frac{\partial \mathbf{L}^*}{\partial \mathbf{f}^{ij}}$$

In **dimension D** we can similarly derive the relation

$$\mathcal{L}^* \equiv \sqrt{-\det(s_{ij} + \nu a_{ij})} \sim \sqrt{-g}\;[\det(\delta_i^j + \lambda f_i^j)]^{1/(D-2)}$$

The generalized Einstein–Eddington-Weyl model in dimension D

$$\mathcal{L}_{eff} = \sqrt{-g}\left[-2\Lambda\left[\det(\delta_i^j + \lambda f_i^j)\right]^{1/(D-2)} + R(g) + c_a\, g^{ij} a_i a_j\right]$$

Restoring the dimensions and expanding the root term up to the second order in the vector and scalar fields

$$\mathcal{L}_{eff} \cong \sqrt{-g}\left[R[g] - 2\Lambda - \kappa\left(\frac{1}{2}F_{ij}F^{ij} + \mu^2 A_i A^i + g^{ij}\partial_i\psi\,\partial_j\psi + m^2\psi^2\right)\right]$$

$$A_i \sim a_i,\; F_{ij} \sim f_{ij},\; \kappa \equiv G/c^4$$

NB: $\partial_i\psi$ Is proportional to F_{ij} for i < 4, j=4

Dimensional reductions of

$$\mathcal{L}_{\rm ph} = \sqrt{-g}\left[-2\Lambda\left[\det(\delta_i^j + \lambda f_i^j)\right]^\nu + R(g) + c_a\, g^{ij} a_i a_j\right]$$

Spherical reduction of the theory

$$ds_D^2 = ds_2^2 + ds_{D-2}^2 = g_{ij}\, dx^i\, dx^j \;+\; \varphi^{2\nu}\, d\Omega_{D-2}^2(k)$$

$$\mathcal{L}_D^{(2)} = \sqrt{-g}\left[\varphi R(g) + k_\nu\, \varphi^{1-2\nu} + \frac{1-\nu}{\varphi}(\nabla\varphi)^2 + X(\varphi, \mathbf{f}^2) - m^2\varphi\, \mathbf{a}^2\right]$$

$$X(\varphi, \mathbf{f}^2) \equiv -2\Lambda\varphi\left[1 + \frac{1}{2}\lambda^2\mathbf{f}^2\right]^\nu \qquad \mathbf{f}^2 \equiv f_{ij} f^{ij} \qquad \nu \equiv (D-2)^{-1}$$

Weyl rescaling

$$g_{ij} = \hat{g}_{ij}\, w^{-1}(\varphi), \quad w(\varphi) = \varphi^{1-\nu} \qquad \mathbf{f}^2 = w^2\, \hat{\mathbf{f}}^2, \quad \mathbf{a}^2 = w\, \hat{\mathbf{a}}^2$$

$$\mathcal{L}_{DW}^{(2)} = \sqrt{-g}\left[\varphi R(g) + k_\nu\,\varphi^{-\nu} - 2\Lambda\varphi^{\nu}\left[1 + \frac{1}{2}\lambda^2\varphi^{2(1-\nu)}\mathbf{f}^2\right]^{\nu} - m^2\varphi\,\mathbf{a}^2\right]$$

3-dimensional theory

$$\mathcal{L}_3^{(2)} = \sqrt{-g}\left[\varphi R(g) - 2\Lambda\varphi - \lambda^2\Lambda\varphi\,\mathbf{f}^2 - m^2\varphi\,\mathbf{a}^2\right]$$

Vecton – Scalaron DUALITY

$$ds^2 = -4\,h(u,v)\,du\,dv\,, \qquad \sqrt{-g} = 2h \qquad f_{uv}^n \equiv a_{u,v}^n - a_{v,u}^n$$

L /2*h* = $\varphi R + V(\varphi,\psi) + X(\varphi,\psi;\mathbf{f}_n^2)$ $\qquad -2\mathbf{f}_n^2 = (f_{uv}^n/h)^2$

L' /2*h* = $\varphi R + V(\varphi,\psi) + X_{\text{eff}}(\varphi,\psi;q_n)$ $\qquad$ & $\quad q_n(u,v) \equiv h^{-1}X_n\,f_{uv}^n$

Effective action on \`mass shell'; *f* – from eq, & $\qquad X_n \equiv \dfrac{\partial X}{\partial\,\mathbf{f}_n^2}$

$$X_{\rm eff}(\varphi,\psi;q_n) = X(\varphi,\psi;\bar{\mathbf{f}}_n^2) + \sum q_n(u,v)\sqrt{-2\,\bar{\mathbf{f}}_n^2}$$

where: $2\,\bar{\mathbf{f}}_n^2 = -(q_n/\bar{X}_n)^2$ $\qquad \bar{X}_n \equiv \frac{\partial}{\partial\,\bar{\mathbf{f}}_n^2} X(\varphi,\psi;\bar{\mathbf{f}}_n^2)$

$$\partial_u(h^{-1}X_n f^n_{uv}) = -Z_n a^n_u \;=\; \partial_u\, q_n(u,v)$$

This defines a^n_u in terms of $q_n(u,v)$ and (φ,ψ)

$$X_{\rm eff} = -2\Lambda\sqrt{\varphi}\left[1 + q^2/\lambda^2\Lambda^2\varphi^2\right]^{\frac{1}{2}}$$ for ***D*** = 4

$V = 2k\varphi^{-\frac{1}{2}}\,, \quad \bar{Z} = -1/m^2\varphi$ **N.B**: normally, ***Z*** ~ to dilaton φ

$X_{\rm eff}(\varphi;\, q(u,v)) = -q^2/\lambda^2\Lambda\varphi \quad V = -2\Lambda\varphi$ ***D*** = 3

The result: we can study **DSG** instead of **DVG**

A general theory of **HORIZONS** in DSG

$L'/2h =$ $$\varphi R + U(\varphi, \psi, q) + \bar{Z}(\varphi)(\nabla q)^2$$ (omitting normal scalars)

Consider **STATIC** solutions that normally **have horizons** when there are **no scalars**

All the equations can be derived from the Hamiltonian (constraint)

H = $$\dot{\varphi}\,\dot{h}/h + hU + \bar{Z}\,\dot{q}^2 + Z\,\dot{\psi}^2$$ (**= 0** in the end)

Without the scalars the EXACT solutions is: $h = C_0^2\,[N_0 - N(\varphi)]$

where $N(\varphi) \equiv \int U(\varphi) d\varphi$ $\qquad C_0 \tau = \int d\varphi\,[N_0 - N(\varphi)]^{-1}$

There is always a horizon, i.e. $h \longrightarrow 0 \text{ for } \varphi \longrightarrow \varphi_0$

Horizons are classified into:

:

regular **simple**, regular **degenerate**, **singular**

We find a gen. sol. near horizon as locally convergent power series in: $\tilde{\varphi} \equiv \varphi - \varphi_0$

$$h = \sum h_n \tilde{\varphi}^n, \qquad \chi = \sum \chi_n \tilde{\varphi}^n, \qquad q = \sum q_n \tilde{\varphi}^n, \qquad \chi(\varphi) \equiv \dot{\varphi}$$

$$h_0 = \chi_0 = 0 \qquad q_0 \neq 0 \qquad \tilde{\varphi} \equiv \varphi - \varphi_0$$

The equations for these functions are **not integrable** and we do not know exact solutions of the recurrence relations

For **D = 3** theory one may hope to find some exact solutions. For very similar equations given by ATF in 0812.2616 *S.Vernov* derived some **exact solutions in terms of elliptic functions.** Nontrivial cylindrical sym. solutions are studied by *E.Davydov*

Practically the **same equations** are applicable to studies of the **cosmological models** with vecton. The best chance to **test the theory is in cosmology**

Vecton dark matter can be produced
in ***strong*** *gravitational fields* only.
Quantum gravity is necessary!

Effects of *nonlinear Lagrangians*
must be studied (like in `B-I cosmology')

Inflation and ***dark matter***
are crucial things to study and test
the theory in cosmological models

THE

END