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Group-Theoretical Classification of BPS and Possibly Protected States in D=4 Conformal Supersymmetry

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Plan

Motivations

Representations of $su(2,2/N)$

Decompositions of long superfields

Reduction of supersymmetry in short and semi-short UIRs

Classification of BPS and possibly protected states

Motivations

In the last 25 years, superconformal field theories in various dimensions are attracting more interest, especially in view of their applications in string theory. From these very important is the AdS/CFT correspondence, namely, the remarkable proposal of Maldacena, according to which the large N limit of a conformally invariant theory in d dimensions is governed by supergravity (and string theory) on $d+1$-dimensional AdS space (often called AdS_{d+1}) times a compact manifold. Actually the possible relation of field theory on AdS_{d+1} to field theory on $\mathcal{M}_d$ has been a subject of long interest, cf., e.g., [FF1,NS,DMPPT]. The proposal of Maldacena was elaborated by Gubser-Klebanov-Polyakov and Witten who proposed a precise correspondence between conformal field theory observables and those of supergravity. More recently, there were developments of integrability in the context of the

AdS/CFT correspondence, in which superconformal field theories, especially in 4D, were also playing important role. For this we refer to the review of Beisert [1012.4004 hep-th] and references therein.

Clearly, the classification of the UIRs of the conformal superalgebras is of great importance. For some time such classification was known only for the $D = 4$ superconformal algebras $su(2,2/1)$ [FF] and $su(2,2/N)$ for arbitrary N [DP1,DP2,DP3]). Then, more progress was made with the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1, 2$) in [M] (some results being conjectural), then the $D = 6$ case (for arbitrary N) was finalized in [D1]. Finally, the cases $D = 9, 10, 11$ were treated by finding the UIRs of $osp(1/2n)$, [DZ].

After the list of UIRs is found the next problem to address is to find their characters since these give the spectrum which is important for the applications. This problem is solved in principle, though not all formulae are explicit, for the UIRs of $D = 4$ conformal superalgebras $su(2,2/N)$ in [D2]. From the mathematical point of view this question is clear only for representations with conformal dimension above the unitarity threshold viewed as irreps of the corresponding complex superalgebra $sl(4/N)$ (see Serganova, etc.). But for $su(2,2/N)$ even the UIRs above the unitarity threshold are truncated for small values of spin and isospin. Furthermore, in the applications the most important role is played by the representations with "quantized" conformal dimensions at the unitarity threshold and at discrete points below. In the quantum field or string theory framework some of these correspond to operators with "protected" scaling dimension and therefore imply "non-renormalization theorems" at the quantum level, cf., e.g., [Heslop

& Howe, Ferrara & Sokatchev]. Especially important in this context are the so-called BPS states, cf., [Andrianopoli, Ferrara, Sokatchev & Zupnik, Arutyunov et al, Ryzhov, etc.].

These investigations require deeper knowledge of the structure of the UIRs, in particular, more explicit results on the decompositions of long superfields as they descend to the unitarity threshold . Fortunately, most of the needed information is contained in [DP1,DP2,DP3,DP4,D2],

Preliminaries

Representations of D=4 conformal supersymmetry

The conformal superalgebras in $D = 4$ are $\mathcal{G} = su(2,2/N)$. The even subalgebra of $\mathcal{G}$ is the algebra $\mathcal{G}_0 = su(2,2) \oplus u(1) \oplus su(N)$. We label their physically relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [\, d\,;\, j_1\,,\, j_2\,;\, z\,;\, r_1\,, \ldots, r_{N-1}\,] \qquad (1)$$

where d is the conformal weight, j_1, j_2 are non-negative (half-)integers which are Dynkin labels of the finite-dimensional irreps of the $D = 4$ Lorentz subalgebra $so(3,1)$ of dimension $(2j_1 + 1)(2j_2 + 1)$, z represents the $u(1)$ subalgebra which is central for $\mathcal{G}_0$ (and is central for $\mathcal{G}$ itself when $N = 4$), and $r_1, \ldots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or R) symmetry algebra $su(N)$.

We recall a root system of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$. The positive root system Δ^+ is comprised of α_{ij}, $1 \leq i < j \leq 4 + N$. The even positive root system $\Delta^+_{\bar{0}}$ is comprised of α_{ij}, with $i, j \leq 4$ and $i, j \geq 5$; the odd positive root system $\Delta^+_{\bar{1}}$ is comprised of α_{ij}, with $i \leq 4, j \geq 5$. The generators corresponding to the latter (odd) roots will be denoted as $X^+_{i,4+k}$, where $i = 1, 2, 3, 4$, $k = 1, \ldots, N$. The simple roots are chosen as:

$$\begin{aligned} \gamma_1 \ &= \ \alpha_{12} \ , \ \gamma_2 = \alpha_{34} \ , \ \gamma_3 = \alpha_{25} \ , \ \gamma_4 = \alpha_{4,4+N} \\ &\gamma_k = \alpha_{k,k+1} \ , \quad 5 \leq k \leq 3 + N. \end{aligned} \tag{2}$$

Thus, the Dynkin diagram is:

$$\begin{array}{ccccccccccc} \bigcirc & --- & \bigotimes & --- & \bigcirc & --- \cdots --- & \bigcirc & --- & \bigotimes & --- & \bigcirc \\ 1 & & 3 & & 5 & & 3{+}N & & 4 & & 2 \end{array}$$

This is a non-distinguished simple root system with two odd simple roots [Kac].

Remark: We recall that the group-theoretical approach to $D = 4$ conformal supersymmetry developed in [DP1,DP2,DP3] involves two related constructions - on function spaces and as Verma modules. The first realization employs the explicit construction of induced representations of $\mathcal{G}$ (and of the corresponding supergroup $G = SU(2,2/N)$) in spaces of functions (superfields) over superspace which are called elementary representations (ER). The UIRs of $\mathcal{G}$ are realized as irreducible components of ERs, and then they coincide with the usually used superfields in indexless notation. The Verma module realization is also very useful as it provides simpler and more intuitive picture for the relation between reducible ERs, for the construction of the irreps, in particular, of the UIRs. For the latter the main tool is an adaptation of the Shapovalov form to the Verma modules [DP]. Here we shall need only the second - Verma module - construction. ■

We use lowest weight Verma modules V^Λ over $\mathcal{G}^{I\!\!C}$, where the lowest weight Λ is characterized by its values on the Cartan subalgebra $\mathcal{H}$ and is in 1-to-1 correspondence with the signature χ. If a Verma module V^Λ is irreducible then it gives the lowest weight irrep L_Λ with the same weight. If a Verma module V^Λ is reducible then it contains a maximal invariant submodule I^Λ and the lowest weight irrep L_Λ with the same weight is given by factorization: $L_\Lambda = V^\Lambda / I^\Lambda$.

There are submodules which are generated by the singular vectors related to the even simple roots $\gamma_1, \gamma_2, \gamma_5, \ldots, \gamma_{N+3}$. These generate an even invariant submodule I_c^Λ present in all Verma modules that we consider and which must be factored out. Thus, instead of V^Λ we shall consider the factor-modules:

$$\tilde{V}^\Lambda = V^\Lambda / I_c^\Lambda \qquad (3)$$

The Verma module reducibility conditions for the $4N$ odd positive roots of $\mathcal{G}^{\mathbb{C}}$ were derived in [DP] adapting the results of Kac for the complex case:

$$\begin{aligned} d &= d^1_{Nk} - z\delta_{N4} \\ d^1_{Nk} &\equiv 4 - 2k + 2j_2 + z + 2m_k - 2m/N \end{aligned} \tag{4a}$$

$$\begin{aligned} d &= d^2_{Nk} - z\delta_{N4} \\ d^2_{Nk} &\equiv 2 - 2k - 2j_2 + z + 2m_k - 2m/N \end{aligned} \tag{4b}$$

$$\begin{aligned} d &= d^3_{Nk} + z\delta_{N4} \\ d^3_{Nk} &\equiv 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N \end{aligned} \tag{4c}$$

$$\begin{aligned} d &= d^4_{Nk} + z\delta_{N4} \\ d^4_{Nk} &\equiv 2k - 2N - 2j_1 - z - 2m_k + 2m/N \end{aligned} \tag{4d}$$

where in all four cases of (4) $k = 1, \ldots, N$, $(m_N \equiv 0)$, and

$$m_k \equiv \sum_{i=k}^{N-1} r_i \ , \quad m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} k r_k \tag{5}$$

Note that we shall use also the quantity m^* which is conjugate to m :

$$m^* \equiv \sum_{k=1}^{N-1} k r_{N-k} = \sum_{k=1}^{N-1} (N-k) r_k \ , \qquad (6)$$

$$m + m^* = N m_1 \ . \qquad (7)$$

We need the result of [DP2] (cf. part (i) of the Theorem there) that the following is the complete list of lowest weight (positive energy) UIRs of $su(2,2/N)$:

$$d \geq d_{\max} = \max(d^1_{N1}, d^3_{NN}) \ , \qquad (8a)$$

$$d = d^4_{NN} \geq d^1_{N1} \ , \quad j_1 = 0 \ , \qquad (8b)$$

$$d = d^2_{N1} \geq d^3_{NN} \ , \quad j_2 = 0 \ , \qquad (8c)$$

$$d = d^2_{N1} = d^4_{NN} \ , \quad j_1 = j_2 = 0 \ , \qquad (8d)$$

where $d_{\max}$ is the threshold of the continuous unitary spectrum. Note that in case (d) we have $d = m_1$, $z = 2m/N - m_1$, and that it is trivial for $N = 1$.

Next we note that if $d > d_{\rm max}$ the factorized Verma modules are irreducible and coincide with the UIRs L_Λ. These UIRs are called **long** in the modern literature. Analogously, we shall use for the cases when $d = d_{\rm max}$, i.e., (8a), the terminology of **semi-short** UIRs, while the cases (8b,c,d) are also called **short**.

Next consider in more detail the UIRs at the four distinguished reducibility points determining the UIRs list above: d^1_{N1}, d^2_{N1}, d^3_{NN}, d^4_{NN}. The above reducibilities occur for the following odd roots, resp.:

$$\begin{aligned} \alpha_{3,4+N} &= \gamma_2 + \gamma_4 \ , \quad \alpha_{4,4+N} = \gamma_4 \ , \\ \alpha_{15} &= \gamma_1 + \gamma_3 \ , \quad \alpha_{25} = \gamma_3 \ . \end{aligned} \tag{9}$$

We note a partial ordering of these four points:

$$d^1_{N1} > d^2_{N1} \ , \qquad d^3_{NN} > d^4_{NN} \ . \tag{10}$$

Due to this ordering *at most two* of these four points may coincide.

First we consider the situations in which *no two of the distinguished four points coincide*. There are four such situations:

$$
\begin{aligned}
\mathbf{a}: \quad & d = d_{\rm max} = d^1_{N1} = d^a \equiv \\
& \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN} & (11a) \\
\mathbf{b}: \quad & d \ = \ d^2_{N1} = d^b \equiv \\
& \equiv z - 2j_2 + 2m_1 - 2m/N > d^3_{NN} \ , \\
& \ j_2 = 0 \ ; & (11b) \\
\mathbf{c}: \quad & d \ = \ d_{\rm max} \ = \ d^3_{NN} \ = \ d^c \equiv \\
& \equiv 2 + 2j_1 - z + 2m/N > d^1_{N1} \ ; & (11c) \\
\mathbf{d}: \quad & d \ = \ d^4_{NN} = d^d \equiv \\
& \equiv 2m/N - 2j_1 - z > d^1_{N1} \ , \quad j_1 = 0 & (11d)
\end{aligned}
$$

where for future use we have introduced notations d^a, d^b, d^c, d^d, each definition including also the corresponding inequality.

We shall call these cases **single-reducibility-condition (SRC)** Verma modules or UIRs, depending on the context. In addition, as already stated, we use for the cases when $d =$

$d_{\max}$, i.e., (11a,c), the terminology of semi-short UIRs, while the cases (11b,d), are also called short UIRs.

The factorized Verma modules $\tilde{V}^\Lambda$ with the unitary signatures from (11) have only one invariant odd submodule which has to be factorized in order to obtain the UIRs. These odd embeddings and factorizations are given as follows:

$$\tilde{V}^\Lambda \ \to \ \tilde{V}^{\Lambda+\beta} \ , \qquad L_\Lambda \ = \ \tilde{V}^\Lambda / I^\beta \ , \qquad (12)$$

where we use the convention that arrows point to the oddly embedded module, and we give only the cases for β that we shall use later:

$$\begin{aligned} \beta \quad &= \alpha_{3,4+N}, \quad \text{for } (11a), \quad j_2 > 0, && (13a)\\ &= \alpha_{3,4+N} + \alpha_{4,4+N}, \ (11a), \ j_2 = 0, && (13b)\\ &= \alpha_{15} \ , \quad \text{for } (11c), \ j_1 > 0, && (13c)\\ &= \alpha_{15} + \alpha_{25}, \quad (11c), \ j_1 = 0. && (13d) \end{aligned}$$

We consider now the four situations in which *two* distinguished points coincide:

$$
\begin{aligned}
\mathbf{ac}: \quad d &= d_{\mathsf{max}} = d^1_{N1} = d^3_{NN} = \\
&= d^{ac} \equiv 2 + j_1 + j_2 + m_1 \qquad (14a) \\
\mathbf{ad}: \quad d &= d^1_{N1} = d^4_{NN} = \\
&= d^{ac} \equiv 1 + j_2 + m_1 \ , \quad j_1 = 0 \qquad (14b) \\
\mathbf{bc}: \quad d &= d^2_{N1} = d^3_{NN} = \\
&= d^{bc} \equiv 1 + j_1 + m_1 \ , \quad j_2 = 0 \qquad (14c) \\
\mathbf{bd}: \quad d &= d^2_{N1} = d^4_{NN} = \\
&= d^{bd} \equiv m_1 \ , \quad j_1 = j_2 = 0 \qquad (14d)
\end{aligned}
$$

We shall call these **double-reducibility-condition (DRC)** Verma modules or UIRs. The cases in (14a) are semi-short UIR, while the other cases are short.

The odd embedding diagrams and factorizations for the DRC modules are [DP1]:

$$\begin{array}{ccc} \tilde{V}^{\Lambda+\beta'} & \to & \tilde{V}^{\Lambda+\beta+\beta'} \\ \uparrow & & \uparrow \\ \tilde{V}^{\Lambda} & \to & \tilde{V}^{\Lambda+\beta} \end{array} \tag{15}$$

$$L_\Lambda = \tilde{V}^\Lambda / I^{\beta,\beta'} , \quad I^{\beta,\beta'} = I^\beta \cup I^{\beta'}$$

and we give only the cases for β, β' to be used later:

$$\begin{aligned}
(\beta,\beta') &= (\alpha_{15}, \alpha_{3,4+N}), \\
&\quad \text{for } (14a), \quad j_1 j_2 > 0 \ ; \qquad \text{(16a)} \\
&= (\alpha_{15}, \alpha_{3,4+N} + \alpha_{3,4+N}), \\
&\quad \text{for } (14b), \quad j_1 > 0,\ j_2 = 0 \ ;\text{(16b)} \\
&= (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N}), \\
&\quad \text{for } (14c), \quad j_1 = 0,\ j_2 > 0 \ ;\text{(16c)} \\
&= (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N} + \alpha_{3,4+N}), \\
&\quad \text{for } (14d), \quad j_1 = j_2 = 0 \qquad \text{(16d)}
\end{aligned}$$

Decompositions of long superfields

First we present the results on decompositions of long irreps as they descend to the unitarity threshold [D2].

In the SRC cases we have established that for $d = d_{\rm max}$ there hold the two-term decompositions:

$$\left(\hat{L}_{\rm long}\right)_{|_{d=d_{\rm max}}} = \hat{L}_{\Lambda} \oplus \hat{L}_{\Lambda+\beta}\ , \qquad r_1 + r_{N-1} > 0\ , \tag{17}$$

where Λ is a semi-short SRC designated as type **a** (then $r_1 > 0$) or **c** (then $r_{N-1} > 0$) and there are four possibilities for β depending on the values of j_1, j_2 as given in (13). In cases (13a,c) also the second UIR on the RHS of (17) is semi-short, while in cases (13b,d) the second UIR on the RHS of (17) is short of type **b**, **d**, resp.

In the DRC cases we have established that for $N > 1$ and $d = d_{\rm max} = d^{ac}$ hold the four-term decompositions:

$$\left(\hat{L}_{\rm long}\right)_{|_{d=d^{ac}}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta} \oplus \hat{L}_{\Lambda+\beta'} \oplus \hat{L}_{\Lambda+\beta+\beta'}, \quad r_1 r_{N-1} > 0 \ , \tag{18}$$

where Λ is the semi-short DRC designated as type **ac** and there are four possibilities for β, β' depending on the values of j_1, j_2 as given in (16a,b,c,d). Note that in case (16a) all UIRs in the RHS of (18) are semi-short. In the case (16b) the first two UIRs in the RHS of (18) are semi-short, the last two UIRs are short of type **bc**. In the case (16c) the first two UIRs in the RHS of (18) are semi-short, the last two UIRs are short of type **ad**. In the case (16d) the first UIR in the RHS of (18) is semi-short, the other three UIRs are short of types **bc**, **ad**, **bd**, resp.

Next we note that for $N = 1$ all SRC cases enter some decomposition, while no DRC cases enter any decomposition. For $N > 1$ the situation is more diverse and so we give the list of UIRs that do **not** enter decompositions together with the restrictions on the R-symmetry quantum numbers:

- **SRC cases** $N > 1$:

•a $d = d^a$, $r_1 = 0$.

•b $d = d^b$, $r_1 \leq 2$.

•c $d = d^c$, $r_{N-1} = 0$.

•d $d = d^d$, $r_{N-1} \leq 2$.

- **DRC cases:**

all non-trivial cases for $N = 1$, while for $N > 1$ the list is:

•ac $d = d^{ac}$, $r_1 r_{N-1} = 0$.

•ad $d = d^{ad}$, $r_{N-1} \leq 2$, $r_1 = 0$ for $N > 2$.

•bc $d = d^{bc}$, $r_1 \leq 2$, $r_{N-1} = 0$ for $N > 2$.

•bd $d = d^{bd}$, $r_1, r_{N-1} \leq 2$ for $N > 2$, $1 \leq r_1 \leq 4$ for $N = 2$.

Note that representations and cases are conjugated as follows:

$$j_1 \leftrightarrow j_2$$

$$(r_1, \ldots, r_{N-1}) \leftrightarrow (r_{N-1}, \ldots, r_1)$$

$$\mathbf{a} \leftrightarrow \mathbf{c}$$

$$\mathbf{b} \leftrightarrow \mathbf{d}$$

$$\mathbf{ad} \leftrightarrow \mathbf{bc}$$

Thus, we would omit the conjugated cases.

Reduction of supersymmetry in short and semi-short UIRs

Our first task is to present explicitly the reduction of the supersymmetries in the irreducible UIRs. This means to give explicitly the number κ of odd generators which are eliminated from the corresponding lowest weight module, (or equivalently, the number of superderivatives that annihilate the corresponding superfield).

R-symmetry scalars

We start with the simpler cases of R-symmetry scalars when $r_i = 0$ for all i, which means also that $m_1 = m = m^* = 0$. These cases are valid also for $N = 1$. More explicitly:

• **a** $\quad d = d^a_{|m=0} = 2 + 2j_2 + z \;, \quad j_1$ arbitrary,

$\kappa = N + (1-N)\delta_{j_2,0} \;, \quad$ or casewise :

$\kappa = N, \quad$ if $\; j_2 > 0,$

$\kappa = 1, \quad$ if $\; j_2 = 0$

Here, κ is the number of anti-chiral generators $X^+_{3,4+k}$, $k = 1, \ldots, \kappa$, that are eliminated. Thus, in the cases when $\kappa = N$ the semi-short UIRs may be called semi-chiral since they lack half of the anti-chiral generators.

In the conjugated case **c** when $\kappa = N$ the semi-short UIRs may be called semi–anti-chiral since they lack half of the chiral generators.

• **b** $\quad d = d^b_{|m=0} = z \,, \quad j_1$ arbitrary, $\quad j_2 = 0,$
$\kappa = 2N$ (19)

These short UIRs may be called chiral since they lack all anti-chiral generators $X^+_{3,4+k}\,,$ $X^+_{4,4+k}\,,\ \ k = 1, \ldots, N.$

In the conjugated case **d** the short UIRs may be called anti-chiral since they lack all chiral generators $X^+_{1,4+k}\,,\ X^+_{2,4+k}\,,\ \ k = 1, \ldots, N.$

• **ac** $d = d^{ac}_{|m=0} = 2 + j_1 + j_2 \ , \quad z = j_1 - j_2 \ ,$

$$\kappa = 2N + (1-N)(\delta_{j_1,0} + \delta_{j_2,0}),$$

or casewise :

$$\kappa = 2N, \quad \text{if} \quad j_1, j_2 > 0,$$
$$\kappa = N+1, \quad \text{if} \quad j_1 > 0, \ j_2 = 0,$$
$$\kappa = N+1, \quad \text{if} \quad j_1 = 0, \ j_2 > 0,$$
$$\kappa = 2, \quad \text{if} \quad j_1 = j_2 = 0.$$

Here, κ is the number of mixed elimination: chiral generators $X^+_{1,4+k}$, and anti-chiral generators $X^+_{3,4+k}$. Thus, in the cases when $\kappa = 2N$ the semi-short UIRs may be called semi–chiral–anti-chiral since they lack half of the chiral and half of the anti-chiral generators. (They may be called Grassmann-analytic following [FS].)

• **ad** $d = d^{ad}_{|m=0} = 1 + j_2 = -z \,, \quad j_1 = 0,$
$\kappa = 3N + (1-N)\delta_{j_2,0}, \quad$ or casewise :
$\kappa = 3N, \quad$ if $\quad j_2 > 0,$
$\kappa = 2N+1, \quad$ if $\quad j_2 = 0.$

Here, κ is the number of mixed elimination: chiral generators $X^+_{1,4+k}$, and both types anti-chiral generators $X^+_{3,4+k}$, $X^+_{4,4+k}$. Thus, in the cases when $\kappa = 3N$ the semi-short UIRs may be called chiral - semi–anti-chiral since they lack half of the chiral and all of the anti-chiral generators.

In the conjugated case **bc** in the cases when $\kappa = 3N$ the semi-short UIRs may be called semi–chiral - anti-chiral since they lack all the chiral and half of the anti-chiral generators.

The last two cases **ad,bc** form two of the three series of massless states, holomorphic and antiholomorphic [DP2].

The case **•bd** for R-symmetry scalars is trivial, since also all other quantum numbers are zero ($d = j_1 = j_2 = z = 0$).

R-symmetry non-scalars

Here we need some additional notation.
Let $N > 1$ and let i_0 be an integer such that $0 \leq i_0 \leq N-1$, $r_i = 0$ for $i \leq i_0$, and if $i_0 < N-1$ then $r_{i_0+1} > 0$.
Let now i_0' be an integer such that $0 \leq i_0' \leq N-1$, $r_{N-i} = 0$ for $i \leq i_0'$, and if $i_0' < N-1$ then $r_{N-1-i_0'} > 0$.
(Both definitions are formally valid for $N = 1$ with $i_0 = i_0' = 0$.)

With this notation the cases of R-symmetry scalars occur when $i_0 + i_0' = N-1$, thus, from now on we have the restriction:

$$0 \leq i_0 + i_0' \leq N-2 \qquad (20)$$

Now we can make a list for the values of κ for R-symmetry non-scalars:

- **a** $d = d^a$, j_1, j_2 arbitrary,
  $$\kappa = 1 + i_0(1 - \delta_{j_2,0}) \leq N - 1 \ . \qquad (21)$$

- **b** $d = d^b$, $j_2 = 0$, j_1 arbitrary,
  $$\kappa = 2 + 2i_0 \leq 2N - 2 \ . \qquad (22)$$

We omit the conjugated cases **c,d**.

- **ac** $d = d^{ac}$, $z = j_1 - j_2 + 2m/N - m_1$, j_1, j_2 arbitrary, (23)
  $$\kappa = 2 + i_0(1 - \delta_{j_2,0}) + i'_0(1 - \delta_{j_1,0}) \leq N \ .$$

Here, the eliminated chiral generators are $X^+_{1,4+k}$, $k \leq 1 + i'_0$, and the eliminated anti-chiral generators are $X^+_{3,4+k}$, $k \leq 1 + i_0$.

- **ad** $d \ = \ d^{ad} = 1 + j_2 + m_1 \ , \quad j_1 = 0 \ ,$

$z \ = \ 2m/N - m_1 - 1 - j_2 \,, \quad j_2$ arbitrary,

$\kappa = 3 + i_0(1 - \delta_{j_2,0}) + 2i'_0 \leq 1 + N + i'_0 \leq 2N - 1$

Here, the eliminated chiral generators are $X^+_{1,4+k}$, $k \leq 1 + i'_0$, and the eliminated anti-chiral generators are $X^+_{3,4+k}$, $X^+_{4,4+k}$, $k \leq 1 + i_0$.

We omit the conjugated case **bc**.

• **bd** $\quad d \ = \ d^{bd} = m_1 \ , \quad j_1 = j_2 = 0 \ ,$ (25)

$$z = 2m/N - m_1 \ ,$$
$$\kappa = 4 + 2i_0 + 2i'_0 \leq 2N \ .$$

Here, the eliminated chiral generators are $X^+_{1,4+k}$, $X^+_{2,4+k}$, $k \leq 1 + i'_0$, and the eliminated anti-chiral generators are $X^+_{3,4+k}$, $X^+_{3,4+k}$, $k \leq 1 + i_0$.
Note that the case $\kappa = 2N$ is possible exactly when $i_0 + i'_0 = N - 2$, i.e., when there is only one nonzero r_i, namely, $r_{i_0+1} \neq 0$, $i_0 = 0, 1, \ldots, N-2$:

• **bd** $\quad \kappa = 2N \ : \ d \ = \ m_1 \ = \ r_{i_0+1} \ ,$ (26)

$$j_1 = j_2 = 0 \ , \quad z = r_{i_0+1} \frac{2 + 2i_0 - N}{N} \ .$$

When $d = m_1 = 1$ these $\frac{1}{2}$-eliminated UIRs form the 'mixed' series of massless representations [DP2]. This series is absent for $N = 1$.

Remark: Here we use the Verma (factor-)module realization of the UIRs. We give here a short remark on what happens with the ER realization of the UIRs. As we know, cf. [DP3], the ERs are superfields depending on Minkowski space-time and on $4N$ Grassmann coordinates $\theta^i_a, \quad \bar\theta^k_b,\ a,b = 1,2,\ i,k = 1,\ldots,N$. There is 1-to-1 correspondence in these dependencies and the odd null conditions. Namely, if the condition $X^+_{a,4+k}\,|\Lambda\rangle = 0,\ a = 1,2,$ holds, then the superfields of the corresponding ER do not depend on the variable θ^k_a, while if the condition $X^+_{a,4+k}\,|\Lambda\rangle = 0,\ a = 3,4,$ holds, then the superfields of the corresponding ER do not depend on the variable $\bar\theta^k_{a-2}$. These statements were used in the proof of unitarity for the ERs picture, cf. [DP4], but were not explicated. They were analyzed in detail in papers of Ferrara and Sokatchev, using the Dubna notions of 'harmonic superspace analyticity' and Grassmann analyticity. $\diamondsuit$

In the next Section we shall apply the above classification to the so-called BPS states.

BPS states

PSU(2,2/4)

The most interesting case is when $N = 4$. This is related to super-Yang-Mills and contains the so-called BPS states. They are characterized by the number κ of odd generators which annihilate them - then the corresponding state is called $\frac{\kappa}{4N}$-BPS state. Group-theoretically the case $N = 4$ is special since the $u(1)$ subalgebra carrying the quantum number z becomes central and one can invariantly set $z = 0$.

We give now the explicit list of these states:

•**a** $d = d^1_{41} = 2+2j_2+2m_1-\frac{1}{2}m > d^3_{44}$. The last inequality leads to the restriction:

$$2j_2 + r_1 > 2j_1 + r_3 \ . \tag{27}$$

In the case of R-symmetry scalars, i.e., $m_1 = 0$, follows that $j_2 > j_1$, i.e., $j_2 > 0$, and then we have:

$$\kappa = 4, \qquad m_1 = 0,\ j_2 > 0 \ . \tag{28}$$

In the case of R-symmetry non-scalars, i.e., $m_1 \neq 0$, we have the range: $i_0 + i'_0 \leq 2$, and thus:

$$\kappa = 1 + i_0(1 - \delta_{j_2,0}) \leq 3 \ . \tag{29}$$

•**b** $d = d^2_{41} = \frac{1}{2}m^* > d^3_{44}$, $j_2 = 0$. The last inequality leads to the restriction:

$$r_1 > 2 + 2j_1 + r_3 \ . \tag{30}$$

The latter means that $r_1 > 2$, i.e., $m_1 \neq 0$, $i_0 = 0$, and thus:

$$\kappa = 2 \ . \tag{31}$$

We omit the conjugated cases **c**, **d**.

•**ac** $d = d^{ac} = 2 + j_1 + j_2 + m_1$. From $z = 0$ follows:

$$2j_2 + r_1 = 2j_1 + r_3 \ . \tag{32}$$

In the case of R-symmetry scalars, i.e., $m_1 = 0$, follows that $j_2 = j_1 = j$, and then we have:

$$\kappa \ = \ 8 - 6\delta_{j,0} \ , \quad d = 2 + 2j \ . \tag{33}$$

In the case of R-symmetry non-scalars, i.e., $m_1 \neq 0$, $i_0 + i'_0 \leq 2$, and thus:

$$\kappa = 2 + i_0(1 - \delta_{j_2,0}) + i'_0(1 - \delta_{j_1,0}) \leq 4 \ . \tag{34}$$

•**ad** From $z = 0$ follows: $r_3 = 2 + 2j_2 + r_1 \implies r_3 \geq 2 \implies m_1 \neq 0$, and $i'_0 = 0$, $i_0 \leq 2 \implies$

$$\kappa \ = \ 3 + i_0(1 - \delta_{j_2,0}) \leq 5 \ , \tag{35}$$
$$d \ = \ d^{ad} \ = \ 1 + j_2 + m_1 \ = \ 3 + 3j_2 + 2r_1 + r_2,$$
$$\chi_4 \ = \ \{\, 0 \, ; \, r_1, r_2, 2 + 2j_2 + r_1 \, ; \, 2j_2 \,\} \ .$$

We omit the conjugated case **bc**.

•**bd** From $z = 0$ follows: $r_1 = r_3 = r$, thus, $i_0 = i'_0 = 0, 1$ and then we have:

$$\kappa = 4(1 + i_0) \ , \tag{36}$$
$$d \ = \ d^{bd} \ = \ m_1 \ = \ 2r + r_2 \ \neq \ 0 \ ,$$
$$r, r_2 \in \mathbb{Z}_+ \ ,$$
$$\chi_4 = \{\, 0\,;\, r, r_2, r\,;\, 0\,\} \ .$$

We summarize the results in a Table:

Table 1

$PSU(2,2/4)$ BPS states

	d	j_1, j_2	r_1, r_2, r_3	κ
a	$2+2j_2$	$j_2 > 0$	$m_1 = 0$	4
a	$2+2j_2+$ $2m_1 - m/2$	$2j_2 + r_1 >$ $2j_1 + r_3$	$m_1 \neq 0$	$1+i_0(1-\delta_{j_2,0}$ ≤ 3
b	$m^*/2$	$j_2 = 0$	$r_1 > 2+$ $2j_1 + r_3$	2
ac	$2+2j$	$j = j_1 = j_2$	$m_1 = 0$	$8 - 6\delta_{j,0}$
ac	$2+j_1+$ $j_2 + m_1$	$2j_2 + r_1 =$ $2j_1 + r_3$	$m_1 \neq 0$	$2+i_0(1-\delta_{j_2,0}$ $+i'_0(1-\delta_{j_1,0})$ ≤ 4
ad	$1+j_2$ $+m_1 \geq 3$	$r_3 = 2+$ $2j_2 + r_1,$ $j_1 = 0$	$m_1 \neq 0$	$3+i_0(1-\delta_{j_2,0}$ ≤ 5
bd	m_1	0,0	$m_1 \neq 0$	$4(1+i_0)$ $= 4, 8$

From the above BPS states we list now the most interesting ones in three Tables:

Table 2

$PSU(2,2/4)$, $\frac{1}{2}$-BPS states, $(\kappa = 8)$

	d	j_1, j_2	r_1, r_2, r_3	prot.
ac	$2 + 2j \geq 3$	$j = j_1 = j_2 \geq 1/2$	$m_1 = 0$	
bd	$r_2 \geq 1$	$j_1 = j_2 = 0$	$m_1 = r_2$	

Table 3

$PSU(2,2/4)$, $\frac{1}{4}$-BPS states, $(\kappa = 4)$

	d	j_1, j_2	r_1, r_2, r_3	prot.
a	$2 + 2j_2 \geq 3$	$j_2 \geq 1/2$	$m_1 = 0$	
ac	$2 + j_1 + j_2 + r_{1+i_0} \geq 3$	$j_1 - j_2 = \frac{1}{2} r_{1+i_0}(1 - i_0)$	$r_{1+i_0} \neq 0$, $i_0 \leq 2$	
ad	$\frac{m}{2} \geq \frac{9}{2}$	$j_1 = 0$, $j_2 \geq 1/2$	$r_1 = 0$, $r_3 = 2 + 2j_2$	N
bd	$m_1 \geq 2$	$j_1 = j_2 = 0$	$r_1 = r_3 \geq 1$	N, if $r_1 > 2$

Table 4

$PSU(2,2/4)$, $\frac{1}{8}$-BPS states, $(\kappa = 2)$

	d	j_1, j_2	r_1, r_2, r_3	prot.
a	$2 + 2j_2 + r_2 + \frac{1}{2}r_3$	$2j_2 > 2j_1 + r_3$	$r_1 = 0$, $r_2 > 0$	
b	$m^*/2$	$j_2 = 0$	$r_1 > 2 + 2j_1 + r_3$	N
ac	$2 + m_1 \geq 2$	$j_1 = j_2 = 0$		N, if $r_1 r_3 \neq 0$

SU(2,2/N), N arbitrary

We can set $z = 0$ also for $N \neq 4$ though this does not have the same group-theoretical meaning as for $N = 4$. We summarize the corresponding results in a several Tables:

Table 5

$SU(2,2/1)$

	d	j_1, j_2	κ	prot.
a	$2+2j_2 \geq 3$	$j_2 > j_1$	1	N
ac	$2+2j \geq 2$	$j = j_1 = j_2$	2	

Table 6

$SU(2,2/2)$

	d	j_1, j_2	κ	prot.
a	$2+2j_2+r$	$j_2 > j_1$	$1+i_0 \leq 2$	N, if $r > 0$
ac	$2+2j+r$	$j = j_1 = j_2$	$2+2\delta_{i_0 j,0} \leq 4$	N, if $r > 0$
bd	$r \geq 1$	$j_1 = j_2 = 0$	4	N, if $r > 4$

Table 7

$SU(2,2/N)$, $\frac{1}{2}$-BPS states, $\kappa = 2N$, $N \geq 3$

	d	j_1, j_2	$r_1, \ldots, r_{N-1}$	prot.
ac	$2 + 2j \geq 3$	$j = j_1 = j_2 \geq 1/2$	$m_1 = 0$	
bd N even	$r_{\frac{N}{2}} \geq 1$	$j_1 = j_2 = 0$	$m_1 = r_{\frac{N}{2}}$	

Table 8

$SU(2,2/3)$, $\frac{1}{4}$-BPS states, $\kappa = 3$

	d	j_1, j_2	r_1, r_2	prot.
a	$2 + 2j_2 \geq 3$	$j_2 \geq 1/2$	$m_1 = 0$	
ac	$2 + j_1 + j_2 + r_{1+i_0} \geq 6$	$j_1 - j_2 = \frac{r_1 - r_2}{3} \neq 0$	$m_1 = r_{1+i_0} \neq 0$, $i_0 \leq 1$	
ad	$\frac{2m}{3} \geq 4$	$j_1 = 0$, $j_2 i_0 = 0$	$r_2 = 3 + r_1 + 3j_2$	N, if $r_1 > 0$

Table 9

$SU(2,2/N)$, $\frac{1}{4}$-BPS states, $\kappa = N$, $N > 4$

	d	j_1, j_2	$r_1, \ldots, r_{N-1}$	prot.
a	$2 + 2j_2 \geq 3$	$j_2 \geq 1/2$	$m_1 = 0$	
ac	$2 + j_1 + j_2 + r_{1+i_0} \geq 3$	$j_1 - j_2 = r_{1+i_0}(1 - \frac{2}{N}(1 + i_0))$	$m_1 = r_{1+i_0} \neq 0$, $i_0 \leq N - 2$	
ad N odd	$1 + m_1$	$j_1 = j_2 = 0$	$i'_0 = \frac{N-3}{2}$	N, if $r_1 \neq 0$
ad	$\frac{2m}{N}$	$j_1 = 0$, $j_2 \geq 1/2$	$i_0 + i'_0 \leq N - 3$	N, if $r_1 \neq 0$
bd N even	m_1	$j_1 = j_2 = 0$	$i_0 + i'_0 = \frac{N}{2} - 2$	N, if $r_1, r_{N-1} > 2$

THANK YOU !