Riemanian geometry and Harmonic Superspace : some recent results

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Outline :

- Motion of a particle on a manifold
- Supersymmetric extension and torsion
- N=2 supersymmetry : constraints
- N=2 supersymmetry : superspace action
- N=2 supersymmetry : particular cases from D=2 and D=4
- N=4 supersymmetry : constraints
- N=4 supersymmetry : particular cases from D=2 and D=4
- Harmonic superspace
- Superfield constraints and action
- Components : bridges and metric
- General results : weak HKT geometry
- Beyond weak HKT ?

- Motion of a particle on a manifold

Motion of a particle in a Riemannian manifold M

Coordinates on M : $x^i(t)$

Action : $$S[x] = \int dt g_{ij}(x)\dot{x}^i\dot{x}^j$$

$g_{ij}(x)$ is a metric on M

E. o. M : $$\ddot{x}^i + \gamma^i_{jk}\dot{x}^j\dot{x}^j = 0$$

Christoffel symbols $\gamma^k_{ij}(x)$

$$\gamma^i_{jk} = \tfrac{1}{2}g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_k g_{jk})$$

Supersymmetric extension :

Superspace (t,θ) and superfields $X^i(t,\theta)$

Supersymmetric derivative : $D = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}, \quad D^2 = i\frac{\partial}{\partial t}$

Action : $S[X] = \int dtd\theta(ig_{ij}(X)\dot{X}^i DX^j +$
$+\frac{1}{3!}c_{ijk}(X)DX^i DX^j DX^k)$

where $c_{ijk}(x)$, antisymmetric tensor, will be the torsion

Covariant derivatives $\nabla_i V^j = \frac{\partial}{\partial x^i}V^j + \Gamma^j_{ik}V^k$

Connexion $\Gamma^j_{ik} = \gamma^j_{ik} + \frac{1}{2}g^{jl}c_{ikl}$

Thus, for any geometry (metric and torsion) there is an N=1 supersymmetric extension.

N=2 supersymétrie :

Infinitesimal transformation $\delta X^i = \epsilon J^i_j(X) D X^j$

Supersymmetry algebra : $J^i_j(x)$ is an integrable complex structure

$$J^i_j J^j_k = -\delta^i_k$$

Invariance of the action :

- Metric is Hermitian $g_{ik} J^k_j + J^k_i g_{kj} = 0$

- Symmetrized covariant derivatives of the complex structure vanish

$$D_i J^k_j + D_j J^k_i = 0$$

- Constraint mixing torsion and complex structure

$$\partial_{[i}(J^m_j c_{kl]m}) - 2 J^m_{[i} \partial_{[m} c_{jkl]]} = 0$$

(R. Coles, G. Papadopoulos 1990)

Solution of the constraints :

- Complex coordinates $z^{\alpha}, \bar{z}^{\bar{\alpha}}$

  such that $J_{\alpha}^{\beta} = i\delta_{\alpha}^{\beta} \quad J_{\bar{\alpha}}^{\bar{\beta}} = -i\delta_{\bar{\alpha}}^{\bar{\beta}}$
- Hermitian metric $g_{\alpha\bar{\beta}}$
- Torsion is determined by the metric and a 2-form

$$B_{\alpha\beta}, \quad \bar{B}_{\bar{\alpha}\bar{\beta}}$$

Superspace $(t, \theta, \bar{\theta})$ and supersymmetric derivatives $D, \bar{D}$

Chiral superfields $Z^{\alpha}, \bar{Z}^{\bar{\alpha}} \quad \bar{D}Z^{\alpha} = 0, DZ^{\bar{\alpha}} = 0$

Action : (C. Hull, 1999)

$$S[Z, \bar{Z}] = \int dt d\theta d\bar{\theta} (g_{\alpha\bar{\beta}} DZ^{\alpha} \bar{D}\bar{Z}^{\bar{\beta}} + B_{\alpha\beta} DZ^{\alpha} DZ^{\beta} + \bar{B}_{\bar{\alpha}\bar{\beta}} \bar{D}\bar{Z}^{\bar{\alpha}} \bar{D}\bar{Z}^{\bar{\beta}})$$

The action is constructed from the objects which determine the geometry (prepotentials)

Particular cases :

- N=1, D=4 (or N=(2,2), D=2) : Torsion vanishes, covariant derivatives of the complex structure vanish :

Kähler geometry $$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z, \bar{z})$$ (B. Zumino, 1979)

- N=(2,0), D=2 : Torsion is a closed 3-form, covariant derivatives of the complex structure vanish

Kähler with torsion (KT)

$$g_{\alpha\bar{\beta}} = \partial_{\bar{\beta}} V_\alpha + \partial_\alpha V_{\bar{\beta}}$$

Torsion is also determined in terms of the vector prepotential V_α

(C. Hull, E. Witten, 1985)

N=4 supersymmetry in D=1

Infinitesimal transformation $\delta X^i = \sum_{a=1}^{3} \epsilon^a (J_a)^i_j D X^j$

Supersymmetry algebra : $(J_a)^i_j(x)$ are 3 integrable complex structures which anticommute with each other

$$J_a J_b + J_b J_a = -2\delta_{ab}\mathbf{1}$$

Invariance of the action :

- Metric is hermitian $g_{ik}(J_a)^k_j + (J_a)^k_i g_{kj} = 0$

- Symmetrized covariant derivatives of the complex structures vanish

$$D_i (J_a)^k_j + D_j (J_a)^k_i = 0$$

- Constraint mixing torsion and complex structures

$$\partial_{[i}((J_a)^m_j c_{kl]m}) - 2(J_a)^m_{[i}\partial_{[m} c_{jkl]]} = 0$$

Particular cases :

N=2, D=4 (or D=2, N=(4,4)) : torsion vanishes, complex structures are annihilated by covariant derivatives and form the quaternionic algebra

$$J_a J_b = -\delta_{ab}\mathbf{1} + \epsilon_{abc} J_c$$

➔ Hyper-Kähler (HK) geometry

N=(4,0), D=2 : torsion is a closed 3 form, complex structures are annihilated by covariant derivatives and form the quaternionic algebra

➔ Hyper-Kähler with torsion (HKT) geometry

In both cases, prepotentials are known

(A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev, 1985)

(FD, S. Kalitzin, E. Sokatchev, 1990)

Harmonic superspace :

Apart from ordinary N=4 superspace variables $(t, \theta^i, \bar{\theta}_i),\ i = 1, 2$

add harmonic variables $(u_i^{\pm}),\ i = 1, 2$ such that the 2 by 2 matrix

$$\begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix}$$

is in SU(2). All fields depend on harmonic variables, and have definite charge under the right action of the diagonal U(1) subgroup of SU(2).

In harmonic superspace, one can find a subspace invariant under all supersymmetries

$$(t_A, \theta^+ = \theta^i u_i^+, \bar{\theta}^+ = \bar{\theta}^i u_i^+, u_i^{\pm})$$

(analytic subspace)

To describe HK or HKT geometry, one needs a set of 2n charge 1 analytic superfields (hypermultiplets)

$$q^{+a}(t_A, \theta^+, \bar{\theta}^+, u^\pm), \;\; a = 1 \cdots p$$

HK case : the prepotential is a scalar analytic function

$$L^{+4}(q^{+a}, u^\pm)$$

HKT case : the prepotential has an SP(n) index

$$L^{+3a}(q^{+b}, u^\pm)$$

HK is a special case of HKT, with

$$L^{+3a} = \frac{\partial}{\partial q^{+a}} L^{+4}$$

Harmonic derivatives

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}},\ \ D^{--} = u^{-i}\frac{\partial}{\partial u^{-i}}$$

$$D^{0} = u^{+i}\frac{\partial}{\partial u^{+i}} + u^{-i}\frac{\partial}{\partial u^{-i}}$$

The D^{++} derivative acts inside the analytic subspace.

In two dimensions, the field equations of HKT read

$$D^{++}q^{+a} = L^{+3a}(q^{+b}, u^{\pm})$$

When restricted to one dimension, this equation is a harmonic constraint which restrict the SU(2) content of the superfields. It may be shown that the content of q^{+a} is just that of n (4,4,0) multiplets of D=1 N=4 supersymmetry

Action : $S = \int dt d^4\theta du \mathcal{L}(q^{+a}, q^{-a}, u^{\pm})$ $\qquad q^{-a} = D^{--} q^{+a}$

Coupling to an external field

$$S_{WZ} = \int du dt_A d^2\theta^{(-2)} \mathcal{L}^{+2}(q^{+a}, u^{\pm})$$

+ non-linear constraint $\quad D^{++} q^{+a} = L^{+3a}(q^{+b}, u^{\pm})$

Components $\quad q^{+a} = f^{+a}(t,u) + \theta^{+}\chi^{a}(t,u) + \bar{\theta}^{+}\bar{\chi}^{a}(t,u) + \theta^{+}\bar{\theta}^{+}A^{-a}(t,u)$
are still harmonic functions

$$D^{++} f^{+a} = L^{+3a}(f^{+b}, u^{\pm})$$

One has to separate $\quad f^{+a} = f^{ia} u_i^{+} + v^{+a}(f^{ia}, u^{\pm})$
where v^{+a} may be seen as a bridge between two different sets of coordinates on the manifold. $f^{ai}(t)$ are the coordinate fields

Fermionic harmonic field : $D^{++}\chi^a - \frac{\partial L^{+3a}}{\partial f^{+b}}\chi^b = 0$

Define the frame bridge as a 2n by 2n matrix satisfying

$$D^{++}M_a^{\underline{b}} + \frac{\partial L^{+3c}}{\partial f^{+a}}M_c^{\underline{b}} = 0$$

Then $\chi^{\underline{a}} = M_b^{\underline{a}}\chi^b$ is independent of harmonic variables $D^{++}\chi^{\underline{a}} = 0$

Vielbeins : $\frac{\partial f^{+a}}{\partial f^{ib}}M_a^{\underline{a}} = -e_{ib}^{k\underline{a}}u_k^+$

Symplectic metric

$$G_{\underline{ab}} = \int du (M^{-1})_{\underline{a}}^a (M^{-1})_{\underline{b}}^b (\partial_{+[a}\partial_{-b]}\mathcal{L} + \cdots)$$

Metric

$$g_{ia\,kb} = G_{\underline{cd}}\epsilon_{\underline{lt}}e_{ia}^{\underline{lc}}e_{kb}^{\underline{td}}$$

Complex structures

$$(J_{(\underline{lk})})_{jc}^{ia} = ie_{(\underline{lb}}^{ia}e_{jc}^{\underline{tb}}\epsilon_{\underline{k})\underline{t}}$$

Results : $$S = \int dt \left[\frac{1}{2}\, g_{ia\,kb}\, \dot{f}^{ia} \dot{f}^{kb} - \frac{i}{4}\, G_{[\underline{a}\,\underline{b}]} \left(\nabla \bar{\chi}^{\underline{a}} \chi^{\underline{b}} - \bar{\chi}^{\underline{a}} \nabla \chi^{\underline{b}} \right) - \frac{1}{16} \left(\epsilon^{\underline{i}\,\underline{k}} \nabla_{\underline{i}[\underline{a}} \nabla_{\underline{kb}]}\, G_{[\underline{c}\,\underline{d}]} \right) \bar{\chi}^{\underline{a}} \bar{\chi}^{\underline{b}} \chi^{\underline{c}} \chi^{\underline{d}} \right]$$

- complex structures form a quaternionic algebra
- complex structures are covariantly constant
- torsion is in general not closed

➔ Weak HKT geometry (P. Howe, G. Papadopoulos, 1996)

(same geometrical constraints as D=2 N=(4,0) non-linear sigma models, apart from the fact that the torsion is not closed)

Wess-Zumino action

$$S_{WZ} = \int dt (\mathcal{A}_i a \dot{f}^{ia} - \frac{i}{4} \epsilon^{\underline{ik}} \mathcal{F}_{\underline{ia}\,\underline{kb}} \chi^{\underline{a}} \bar{\chi}^{\underline{b}})$$

where the field strength of the external vector potential satisfies the self-duality condition $\mathcal{F}_{(ia\,k)b} = 0$

Particular cases :

- HKT $\mathcal{L} = q^{+a} D^{--} q^{+a}$, general constraint

  ➔ Torsion is closed

- HK $\mathcal{L} = q^{+a} D^{--} q^{+a}$, $L^{+3a} = \frac{\partial}{\partial q^{+a}} L^{+4}$

If one restricts the constraint $L^{+3a} = \frac{\partial}{\partial q^{+a}} L^{+4}$ but keeps a general lagrangian $\mathcal{L}$, one gets a geometry intimately connected to the hyperkähler geometry encoded in L^{+4}, but which includes torsion. In particular, if the manifold has dimension 4, the HKT metric is conformal to the KT metric, with a conformal factor which is a harmonic function on the KT manifold (C. Callan,L. Harvey, A. Strominger 1991)

Conjecture : in order to describe the general D=1, N=4 geometry, two types of hypermultiplets are needed

Automorphism group of N=4 supersymmetry :

$$SO(4) \sim SU(2) \times SU(2)$$

One of these SU(2)'s act on the harmonic variables $u_i^{\pm}$

One may define two types of hypermultiplets, depending on which the two SU(2) which is harmonized. Very probably, when using the two types together one may describe the general N=4 geometry.

A computation in support of this conjecture was done in N=2 superspace

Starting from chiral superfields $z^\alpha, \ y^a$

Write 2 more supersymmetry transformations as

$$\delta z^\alpha = \epsilon J^\alpha_\beta D z^\beta$$
$$\delta y^a = \bar{\epsilon} \tilde{J}^a_b D y^b$$

where ϵ is a complex Grassmann parameter.

z's and y's are in different representations of N=4 supersymmetry.

We have checked that, generically, the complex structures do not form the quaternionic algebra and only symmetrized covariant derivatives of complex structures vanish

General geometry ?