Interpolating differential reductions of multidimensional dispersionless integrable hierarchies

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General context

- Lax pairs in terms of vector fields (diff. operators of the first order) – Zakharov, Shabat (1979)
- Differential reductions, N-orthogonal coordinate systems – Zakharov (1998). The works of Kyoto school on KP hierarchy reductions (BKP, CKP, etc.)
- Dispersionless limit of integrable systems in (2+1)
- Integrable systems of twistor theory, Plebański heavenly equations and generalizations, hyper-Kähler hierarchies – multidimensional integrable models
- Manakov-Santini hierarchy: generalizes dKP, it is a simplest non-degenerate example of the hierarchy for general vector fields. Dressing method, inverse scattering method for vector fields
- Dunajski interpolating system – describes "a symmetry reduction of the anti-self-dual Einstein equations in (2, 2) signature by a conformal Killing vector whose selfdual derivative is null". On the other hand, it is a simple differential reduction of the Manakov-Santini system

Outline

1. The Manakov-Santini system and Dunajski interpolating equation
2. d2DTL generalization and Dunajski-Tod equation
3. The Manakov-Santini hierarhy
4. A class of differential reductions of the Manakov-Santini hierarchy
5. Reductions in general (N+2)-dimensional case. Systems connected with Dunajski generalization of the second heavenly equation
6. Two-point case. Reductions for dispersionless 2DTL generalization

The Manakov-Santini system

The Manakov-Santini system – two-component integrable generalization of the dKP equation,

$$
\begin{aligned}
u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\
v_{xt} &= v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y.
\end{aligned}
$$

Lax pair

$$
\begin{aligned}
&\partial_y \boldsymbol{\Psi} = ((p - v_x)\partial_x - u_x \partial_p)\boldsymbol{\Psi}, \\
&\partial_t \boldsymbol{\Psi} = ((p^2 - v_x p + u - v_y)\partial_x - (u_x p + u_y)\partial_p)\boldsymbol{\Psi},
\end{aligned}
$$

where p plays a role of a spectral variable. For $v = 0$ reduces to dKP (Khohlov-Zabolotskaya equation)

$$
u_{xt} = u_{yy} + (uu_x)_x,
$$

reduction $u = 0$ gives the equation (Pavlov, Martinez Alonso and Shabat)

$$
v_{xt} = v_{yy} + v_x v_{xy} - v_{xx} v_y.
$$

Dunajski interpolating system

The condition used by Dunajski (JPA 2008) to reduce the Manakov-Santini system to the interpolating system

$$\alpha u = v_x,$$

The reduced MS system can be written as deformed dKP,

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_x &= \alpha u, \end{aligned}$$

it also implies a single equation for v,

$$v_{xt} = v_{yy} + \alpha^{-1} v v_{xx} + v_x v_{xy} - v_{xx} v_y.$$

The limit $\alpha \to 0$ corresponds to dKP, $\alpha \to \infty$ – to equation, introduced by Pavlov, Martinez Alonso and Shabat

Dunajski interpolating system describes "a symmetry reduction of the anti-self-dual Einstein equations in (2, 2) signature by a conformal Killing vector whose selfdual derivative is null".

Elementary description of reductions

MS Lax equation

$$\partial_y \boldsymbol{\Psi} = ((p - v_x)\partial_x - u_x\partial_p)\boldsymbol{\Psi}.$$

Basic solutions Ψ_1, Ψ_2, general solution $F(\Psi_1, \Psi_2)$.
Existence of polynomial solution $p^n + f_{n-2}p^{n-2} + \dots$ for L operator defines Gelfand-Dikii reduction (for MS no stationarity with respect to higher time!), the case $n = 1$ corresponds to Pavlov equation.
Formally adjoint Lax equation $(u\partial \to -\partial u)$

$$\partial_y J = ((p - v_x)\partial_x - u_x\partial_p)J - v_{xx}J,$$

$J = \{\Psi_1, \Psi_2\}$, $\{f, g\} = f_p g_x - f_x g_p$, general solution $JF(\Psi_1, \Psi_2)$.
Remark For divergence-free vector fields Lax equations are self-adjoint.

Interpolating reductions

(L.V. Bogdanov, JPA 43 (2010) 115206)
Adjoint Lax equation in terms of $\ln J$

$$\partial_y \ln J = ((p - v_x)\partial_x - u_x\partial_p) \ln J - v_{xx},$$

nonhomogeneous linear equation, general solution $\ln J + F(\Psi_1, \Psi_2)$.
Interpolating reduction – nonhomogeneous Lax equations possess a polynomial solution $f = -\alpha p^n + f_{n-2}p^{n-2} + \dots$.
$\alpha = 0$ corresponds to dKP (divergence-free vector fields), $\alpha \to \infty$ – to MS Gelfand-Dikii reduction of the order n.
For $n = 1$, substituting $f = -\alpha p$ to adjoint Lax operator, we obtain

$$\alpha u = v_x,$$

corresponding to Dunajski interpolating system.

One-parametric family of Lax pairs

If we have a Lax pair in terms of vector fields, e.g.

$$\partial_y \mathbf{\Psi} = \hat{u}\mathbf{\Psi},$$
$$\partial_t \mathbf{\Psi} = \hat{v}\mathbf{\Psi},$$

in the general (not divergence-free) case we have a one-parametric family of Lax pairs of more general form,

$$\partial_y \mathbf{\Psi} = \hat{U}\mathbf{\Psi} = (\hat{u} - \alpha^{-1}\,\mathrm{div}\,\hat{u})\mathbf{\Psi},$$
$$\partial_t \mathbf{\Psi} = \hat{V}\mathbf{\Psi} = (\hat{v} - \alpha^{-1}\,\mathrm{div}\,\hat{v})\mathbf{\Psi},$$

having the same compatibility condition because

$$[\hat{U}, \hat{V}] = [\hat{u}, \hat{v}] - \alpha^{-1}\,\mathrm{div}[\hat{u}, \hat{v}]$$

(the Lie algebra of extended vector fields stays the same). The reduction means the existence of solution $\ln \Psi = p^n + f_{n-2}p^{n-2} + \ldots$ of Lax equations for some α.

Hamiltonian interpretation of Dunajski interpolating system

Lax equations for the Dunajski interpolating system can be written in Hamiltonian form, but with the modified Poisson bracket $\{-,-\}' = e^{\alpha p}\{-,-\}$ (S.V. Manakov). Indeed, the equation

$$\partial_y \mathbf{\Psi} = ((p - \alpha u)\partial_x - u_x \partial_p)\mathbf{\Psi},$$

can be written in the form

$$\partial_y \mathbf{\Psi} = \{H_1, \mathbf{\Psi}\}' = e^{\alpha p}\{H_1, \mathbf{\Psi}\},$$
$$H_1 = e^{-\alpha p}(u - \alpha^{-1}(p + \alpha^{-1})).$$

The situation with higher reductions is not so transparent, they should probably have some geometric interpretation.

Two-component generalization of d2DTL hierarchy

Two-component generalization of the dispersionless 2DTL equation (L.V. Bogdanov, JPA 43 (2010) 434008)

$$
\begin{aligned}
(\mathrm{e}^{-\phi})_{tt} &= m_t\phi_{xy} - m_x\phi_{ty}, \\
m_{tt}\mathrm{e}^{-\phi} &= m_{ty}m_x - m_{xy}m_t.
\end{aligned}
$$

The Lax pair

$$
\begin{aligned}
\partial_x \mathbf{\Psi} &= \left((\lambda + \frac{m_x}{m_t})\partial_t - \lambda(\phi_t\frac{m_x}{m_t} - \phi_x)\partial_\lambda\right)\mathbf{\Psi}, \\
\partial_y \mathbf{\Psi} &= \left(\frac{1}{\lambda}\frac{\mathrm{e}^{-\phi}}{m_t}\partial_t + \frac{(\mathrm{e}^{-\phi})_t}{m_t}\partial_\lambda\right)\mathbf{\Psi}
\end{aligned}
$$

For $m = t$ the system reduces to the dispersionless 2DTL equation

$$
(\mathrm{e}^{-\phi})_{tt} = \phi_{xy},
$$

The reduction $\phi = 0$ gives an equation (Pavlov; Shabat and Martinez Alonso)

$$
m_{tt} = m_{ty}m_x - m_{xy}m_t.
$$

Interpolating reduction for d2DTL case

Adjoint Lax equations (nonhomogeneous linear equations for the Jacobian) possess a solution $f = -\alpha \ln \lambda$, defining the reduction

$$\mathrm{e}^{\alpha\phi} = m_t.$$

This reduction makes it possible to rewrite the system as one equation for m,

$$m_{tt} = (m_t)^{\frac{1}{\alpha}}(m_{ty} m_x - m_{xy} m_t). \qquad (*)$$

Equation $(*)$ is equivalent to the generalization of a dispersionless $(1+2)$-dimensional Harry Dym equation, Blaszak (2002). It is also connected with an equation describing ASD vacuum metric with conformal symmetry, Dunajski and Tod (1999)

$$(\eta F_y + F_{y\tau})(\eta F_x - F_{x\tau}) - (\eta^2 F - F_{\tau\tau})F_{xy} = 4e^{2\rho\tau},$$

The limit $\alpha \to 0$ gives the d2DTL equation, the limit $\alpha \to \infty$ gives the equation (Pavlov; Shabat and Martinez Alonso)

$$m_{tt} = m_{ty} m_x - m_{xy} m_t.$$

The Manakov-Santini hierarchy

Lax-Sato equations

$$\frac{\partial}{\partial t_n}\begin{pmatrix} L \\ M \end{pmatrix} = \left(\left(\frac{L^n L_p}{\{L,M\}}\right)_+ \partial_x - \left(\frac{L^n L_x}{\{L,M\}}\right)_+ \partial_p\right)\begin{pmatrix} L \\ M \end{pmatrix},$$

where L, M, corresponding to the Lax and Orlov functions of the dispersionless KP hierarchy, are the series

$$L = p + \sum_{n=1}^{\infty} u_n(\mathbf{t}) p^{-n},$$

$$M = M_0 + M_1, \quad M_0 = \sum_{n=0}^{\infty} t_n L^n,$$

$$M_1 = \sum_{n=1}^{\infty} v_n(\mathbf{t}) L^{-n} = \sum_{n=1}^{\infty} \tilde{v}_n(\mathbf{t}) p^{-n},$$

and $x = t_0$, $(\sum_{-\infty}^{\infty} u_n p^n)_+ = \sum_{n=0}^{\infty} u_n p^n$, $\{L, M\} = L_p M_x - L_x M_p$. A more standard choice of times for the dKP hierarchy corresponds to $M_0 = \sum_{n=0}^{\infty}(n+1)t_n L^n$

Lax-Sato equations are equivalent to the generating relation

$$\left(\frac{\mathrm{d}L \wedge \mathrm{d}M}{\{L, M\}}\right)_{-} = 0,$$

Lax-Sato equations for the first two flows of the hierarchy

$$\partial_y \begin{pmatrix} L \\ M \end{pmatrix} = ((p - v_x)\partial_x - u_x\partial_p) \begin{pmatrix} L \\ M \end{pmatrix}$$

$$\partial_t \begin{pmatrix} L \\ M \end{pmatrix} = ((p^2 - v_x p + u - v_y)\partial_x - (u_x p + u_y)\partial_p) \begin{pmatrix} L \\ M \end{pmatrix}$$

where $u = u_1$, $v = v_1$, $x = t_0$, $y = t_1$, $t = t_2$, correspond to the Lax pair of the Manakov-Santini system

A class of differential reductions of the MS hierarchy

The dynamics of the Poisson bracket $J=\{L,M\}$, $J=1+v_x p^{-1}+\dots$ is described by the nonhomogeneous equation

$$\frac{\partial}{\partial t_n}\ln J=(A_n\partial_x-B_n\partial_p)\ln J+\partial_x A_n-\partial_p B_n,$$
$$A_n=\left(\frac{L^n L_p}{J}\right)_+,\quad B_n=\left(\frac{L^n L_x}{J}\right)_+,$$

A_n, B_n are polynomials in p. $\ln J+F(L,M)$ also satisfies these equations. We define a class of reductions of Manakov-Santini hierarchy by the condition

$$(\ln J-\alpha L^k)_-=0,$$

where α is a constant. Then $\ln J-\alpha L^k$ is a polynomial.

Characterization of the reduction

Proposition

The existence of a polynomial solution

$$f = -\alpha p^k + \sum_{0}^{i=k-2} f_i(\mathbf{t})p^i,$$

(where the coefficients f_i don't contain constants, see below) of equations

$$\frac{\partial}{\partial t_n} f = (A_n \partial_x - B_n \partial_p) f + \partial_x A_n - \partial_p B_n,$$

is equivalent to the reduction condition

$$(\ln J - \alpha L^k)_- = 0,$$

General k

$$(\ln J - \alpha L^k)_- = 0 \Rightarrow (\ln J - \alpha L^k) = (\ln J - \alpha L^k)_+ = -\alpha (L^k)_+,$$

$f = -\alpha (L^k)_+$ is a solution of nonhomogeneous equation of the Proposition.

$$J = \exp \alpha (L^k - ({L^k}_+)) = \exp \alpha ({L^k}_-),$$

and Lax-Sato equations of reduced hierarchy read

$$\frac{\partial}{\partial t_n} L = (e^{-\alpha({L^k}_-)} L^n L_p)_+ \partial_x L - (e^{-\alpha({L^k}_-)} L^n L_x)_+ \partial_p L.$$

Generating relation takes the form

$$\left(e^{-\alpha L^k} \mathrm{d}L \wedge \mathrm{d}M \right)_- = 0.$$

For the first flow $n = 1$ we obtain a condition

$$\partial_y (\alpha {L^k}_+) = ((p - v_x)\partial_x - u_x \partial_p)(\alpha {L^k}_+) + v_{xx}.$$

This condition defines a differential reduction of Manakov-Santini system.

The case $k = 0$ (or $\alpha = 0$) corresponds to Hamiltonian vector fields. Indeed, in this case $J = 1$, and from nonhomogeneous equations we have

$$\partial_x A_n - \partial_p B_n = 0.$$

This is the case of the dKP hierarchy.

Proposition

The reduction with general k is 'interpolating' between the dKP hierarchy ($\alpha \to 0$), and the Gelfand-Dikii reduction of the MS hierarchy of the order k, $L_-^k = 0$, for $\alpha \to \infty$.

(directly follows from the definition of the reduction)

$k = 1$. Dunajski interpolating system

In the case $k = 1$

$$(\ln J - \alpha L)_- = 0 \Rightarrow (\ln J - \alpha L) = (\ln J - \alpha L)_+ = -\alpha p,$$
$$J = \exp \alpha(L - p).$$

Lax-Sato equations

$$\frac{\partial}{\partial t_n} L = (e^{\alpha(p-L)} L^n L_p)_+ \partial_x L - (e^{\alpha(p-L)} L^n L_x)_+ \partial_p L.$$

The generating relation for the reduced hierarchy reads

$$\left(e^{\alpha(p-L)} \mathrm{d}L \wedge \mathrm{d}M\right)_- = 0 \Rightarrow \left(e^{-\alpha L} \mathrm{d}L \wedge \mathrm{d}M\right)_- = 0.$$

Differential reduction reads

$$\alpha u = v_x,$$

which is exactly the condition used by Dunajski (JPA 2008) to reduce the Manakov-Santini system to the interpolating system.

The reduced MS system (equivalent to Dunajski interpolating system) can be written as deformed dKP,

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_x &= \alpha u, \end{aligned}$$

it also implies a single equation for v,

$$v_{xt} = v_{yy} + \alpha^{-1} v v_{xx} + v_x v_{xy} - v_{xx} v_y .$$

The limit $\alpha \to 0$ corresponds to dKP, $\alpha \to \infty$ – to equation, introduced by Pavlov.

Differential reductions. Special cases

The case $k = 2$.

$$J = e^{\alpha (L^2 {}_-)}$$

Differential reduction for the MS system

$$2\alpha(u_y + v_x u_x) = v_{xx}$$

The case $k = 3$. Differential reduction

$$3\alpha\left(\partial_y(u_y + u_x v_x) + \partial_x(u_y v_x + u_x v_x^2 + u u_x)\right) = v_{xxx}.$$

A pair of reductions with different k – reduction to (1+1)

Reductions of interpolating system (i.e., the reduction with $k = 1$, together with the reduction of some order $k \neq 1$ with a constant β).

For $k = 2$ we obtain a system

$$u_y + v_x u_x = (2\beta)^{-1} v_{xx},$$
$$v_x = \alpha u,$$

which implies a hydrodynamic type equation (Hopf type equation) for u,

$$u_y + \alpha u u_x = \frac{\alpha}{2\beta} u_x.$$

The system for $k = 3$ read

$$\partial_y(u_y + u_x v_x) + \partial_x(u_y v_x + u_x v_x^2 + u u_x) = 3\beta^{-1} v_{xxx},$$
$$v_x = \alpha u,$$

it implies an equation for u,

$$u_{yy} + \partial_x(2\alpha u_y u + \alpha^2 u_x u^2 + u u_x - \frac{\alpha}{3\beta} u_x) = 0,$$

which can be rewritten as a system of hydrodynamic type for two functions u, w,

$$
\begin{aligned}
&w_y = \left(\frac{\alpha}{3\beta} - \alpha^2 u^2 - u\right)u_x - 2\alpha u w_x, \\
&u_y = w_x.
\end{aligned}
$$

A system of equations of hydrodynamic type corresponding to the reduction of interpolating system of arbitrary order $k > 3$ can be written explicitly.

Two reductions of higher order

A simple example of a system defined by two reductions of higher order (reductions of the order 2 and 3),

$$u_y + v_x u_x = (2\alpha)^{-1} v_{xx},$$
$$\big(\partial_y(u_y + u_x v_x) + \partial_x(u_y v_x + u_x v_x^2 + u u_x)\big) = (3\beta)^{-1} v_{xxx}.$$

A system of hydrodynamic type for the functions u, $w = v_x$,

$$u_y + w u_x = (2\alpha)^{-1} w_x,$$
$$w_y = \frac{2\alpha}{3\beta} w_x - w w_x - 2\alpha u u_x.$$

The characterization of reductions in terms of the dressing data

A dressing scheme for the MS hierarchy

$$L_{\text{in}} = F_1(L_{\text{out}}, M_{\text{out}}),$$
$$M_{\text{in}} = F_2(L_{\text{out}}, M_{\text{out}}),$$

$L_{\text{in}}(p,\mathbf{t})$, $M_{\text{in}}(p,\mathbf{t})$ are analytic inside the unit circle, the functions $L_{\text{out}}(p,\mathbf{t})$, $M_{\text{out}}(p,\mathbf{t})$ are analytic outside the unit circle with a prescribed singulariry defined by the series.
The Riemann problem implies the analyticity of the differential form

$$\Omega_0 = \frac{\mathrm{d}L \wedge \mathrm{d}M}{\{L, M\}}$$

and the generating relation for the hierarchy.

Let $G_1(\lambda,\mu)$, $G_2(\lambda,\mu)$ define an area-preserving diffeomorphism, $\mathbf{G} \in \mathrm{SDiff}(2)$,

$$\left|\frac{D(G_1,G_2)}{D(\lambda,\mu)}\right| = 1.$$

Let us fix a pair of analytic functions $f_1(\lambda,\mu)$, $f_2(\lambda,\mu)$ (the reduction data) and consider a problem

$$f_1(L_{\mathrm{in}}, M_{\mathrm{in}}) = G_1(f_1(L_{\mathrm{out}}, M_{\mathrm{out}}), f_2(L_{\mathrm{out}}, M_{\mathrm{out}})),$$
$$f_2(L_{\mathrm{in}}, M_{\mathrm{in}}) = G_2(f_1(L_{\mathrm{out}}, M_{\mathrm{out}}), f_2(L_{\mathrm{out}}, M_{\mathrm{out}})),$$

which defines a reduction of the MS hierarchy. In terms of the Riemann problem for the MS hierarchy, which can be written in the form

$$(L_{\mathrm{in}}, M_{\mathrm{in}}) = \mathbf{F}(L_{\mathrm{out}}, M_{\mathrm{out}}),$$

the reduction condition for the dressing data reads

$$\mathbf{f} \circ \mathbf{F} \circ \mathbf{f}^{-1} \in \mathrm{SDiff}(2).$$

In terms of equations of the MS hierarchy the reduction is characterized by the condition

$$(\mathrm{d}f_1(L,M)\wedge\mathrm{d}f_2(L,M))_{\text{out}} = (\mathrm{d}f_1(L,M)\wedge\mathrm{d}f_2(L,M))_{\text{in}},$$

thus the differential form

$$\Omega_{\text{red}} = \mathrm{d}f_1(L,M)\wedge\mathrm{d}f_2(L,M)$$

is analytic in the complex plane, and reduced hierarchy is defined by the generating relation

$$(\mathrm{d}f_1(L,M)\wedge\mathrm{d}f_2(L,M))_- = 0.$$

Taking

$$\begin{aligned} f_1(L,M) &= L, \\ f_2(L,M) &= e^{-\alpha L^n} M, \end{aligned}$$

we obtain the generating relation

$$\left(e^{-\alpha L^k}\mathrm{d}L\wedge\mathrm{d}M\right)_- = 0,$$

coinciding with the generating relation for k-reduced MS hierarchy.

Thus we come to the following conclusion:

Proposition

In terms of the dressing data for the Riemann problem, the class of reductions (defined above) is characterized by the condition

$$\mathbf{f} \circ \mathbf{F} \circ \mathbf{f}^{-1} \in SDiff(2),$$

where the components of $\mathbf{f}$ *are defined as*

$$f_1(L, M) = L, \quad f_2(L, M) = e^{-\alpha L^n} M,$$

For the interpolating equation we have $f_1 = L$, $f_2 = e^{-\alpha L} M$, and the Riemann problem can be written in the form

$$\begin{aligned} L_{\text{in}} &= G_1(L_{\text{out}}, e^{-\alpha L_{\text{out}}} M_{\text{out}}), \\ M_{\text{in}} &= e^{\alpha G_1(L_{\text{out}}, e^{-\alpha L_{\text{out}}} M_{\text{out}})} G_2(L_{\text{out}}, e^{-\alpha L_{\text{out}}} M_{\text{out}}), \end{aligned}$$

where $\mathbf{G} \in \text{SDiff}(2)$.

Hamiltonian structure

Lax-Sato equations for the reduction with $k = 1$ (Dunajski interpolating equation) can be written in Hamiltonian form, but with the modified Poisson bracket (S.V. Manakov). Indeed,

$$\{L, M\} = \exp \alpha(L - p) \Rightarrow e^{\alpha p}\{L, e^{-\alpha L} M\} = 1,$$

that indicates that the dynamics is Hamiltonian with the bracket $\{-,-\}' = e^{\alpha p}\{-,-\}$. The first flow of reduced hierarchy

$$\partial_y \boldsymbol{\Psi} = ((p - \alpha u)\partial_x - u_x \partial_p)\boldsymbol{\Psi},$$

can be written in Hamiltonian form

$$\partial_y \boldsymbol{\Psi} = e^{\alpha p}\{H_1, \boldsymbol{\Psi}\},$$
$$H_1 = e^{-\alpha p}(u - \alpha^{-1}(p + \alpha^{-1})).$$

It is possible to prove that all the flows of the reduced hierarchy are Hamiltonian with the bracket $\{-,-\}' = e^{\alpha p}\{-,-\}$, however, we don't have an explicit formula for H_n.
For higher reductions, there is an anti-symmetric invariant, but the corresponding 'bracket' doesn't satisfy the Jacobi identity.

General (N+2)-dimensional hierarchy

Connection of Jacobian with 'local parameter' is a general type of reduction.

Set of functions

$$\Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi^0_n(\mathbf{t}^1, \dots, \mathbf{t}^N)\lambda^{-n},$$

$$\Psi^k = \sum_{n=0}^{\infty} t^k_n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^k_n(\mathbf{t}^1, \dots, \mathbf{t}^N)(\Psi^0)^{-n}.$$

where $1 \leqslant k \leqslant N$, $\mathbf{t}^k = (t^k_0, \dots, t^k_n, \dots)$.

Generating relation

$$(J_0^{-1} \mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \dots \wedge \mathrm{d}\Psi^N)_- = 0,$$

J_0 is a determinant of Jacobian matrix J,

$$J_{ij} = \partial_i \Psi^j, \quad 0 \leqslant i, j \leqslant N, \quad \partial_0 = \frac{\partial}{\partial \lambda}, \ \partial_k = \frac{\partial}{\partial x^k}, \quad 1 \leqslant k \leqslant N,$$

where $x^k = t^k_0$.

Lax-Sato equations

$$\partial_n^k \mathbf{\Psi} = \sum_{i=0}^{N} (J_{ki}^{-1} (\Psi^0)^n)_+ \partial_i \mathbf{\Psi}, \quad 0 \leqslant n \leqslant \infty \,, 1 \leqslant k \leqslant N.$$

First flows of the hierarchy

$$\partial_1^k \mathbf{\Psi} = (\lambda \partial_k - \sum_{p=1}^{N} (\partial_k u_p) \partial_p - (\partial_k u_0) \partial_\lambda) \mathbf{\Psi}, \quad 0 < k \leqslant N,$$

where $u_k = \Psi_1^k$, $0 \leqslant k \leqslant N$. A compatibility condition for any pair of linear equations (e.g., with ∂_1^k and ∂_1^q, $k \neq q$) implies closed nonlinear (N+2)-dimensional system of PDEs for the set of functions u_k, u_0, which can be written in the form

$$\partial_1^k \partial_q \hat{u} - \partial_1^q \partial_k \hat{u} + [\partial_k \hat{u}, \partial_q \hat{u}] = (\partial_k u_0) \partial_q - (\partial_q u_0) \partial_k,$$
$$\partial_1^k \partial_q u_0 - \partial_1^q \partial_k u_0 + (\partial_k \hat{u}) \partial_q u_0 - (\partial_q \hat{u}) \partial_k u_0 = 0,$$

where $\hat{u}$ is a vector field, $\hat{u} = \sum_{p=1}^{N} u_k \partial_k$.

Reductions for (N+2)-dimensional hierarchy

(L.V. Bogdanov, TMPh 167(3): 705–713 (2011))

Nonhomogeneous equations for the Jacobian

$$\partial_n^k \ln J_0 = \sum_{i=0}^{N} (J_{ki}^{-1}(\Psi^0)^n)_+ \partial_i \ln J_0 + \sum_{i=0}^{N} \partial_i (J_{ki}^{-1}(\Psi^0)^n)_+.$$

Solution $(\ln J_0 - \alpha(\Psi^0)^k)$. Reduction

$$(\ln J_0 - \alpha(\Psi^0)^k)_- = 0.$$

In terms of the dressing data reductions belong to the class

$$\mathbf{f} \circ \mathbf{F} \circ \mathbf{f}^{-1} \in \text{SDiff(N+1)}.$$

Generating relation for the reduced hierarchy

$$(\exp(-\alpha(\Psi^0)^k) \mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \cdots \wedge \mathrm{d}\Psi^N)_- = 0.$$

Reductions

$\mathbf{k}=\mathbf{0}$ $J_0=1$, divergence-free vector field in Lax-Sato equations and vector fields $\hat{u}$.
On the other hand, volume-preserving reduction can be obtained from the reduction with arbitrary k in the limit $\alpha\to 0$. Thus the reduction with arbitrary k is an 'interpolating' reduction between the volume-preserving hierarchy and the hierarchy, characterized by the existence of polynomial solution of Lax-Sato equations, $(\Psi^0)^k_-=0$ (Gelfand-Dikii reduction).
$\mathbf{k}=\mathbf{1}$ $J_0=\exp\alpha(\Psi_0-\lambda)$. Reduction implies the existence of the solution $-\alpha\lambda$ of equations for $\ln J_0$ and leads to the condition

$$\operatorname{div}\hat{u}:=\sum_{p=1}^{N}\partial_p u_p=\alpha u_0.$$

The reduced system for $\hat{u}$ is

$$\partial_1^k\partial_q\hat{u}-\partial_1^q\partial_k\hat{u}+[\partial_k\hat{u},\partial_q\hat{u}]=\alpha^{-1}((\partial_k\operatorname{div}\hat{u})\partial_q-(\partial_q\operatorname{div}\hat{u})\partial_k).$$

N=2 (heavenly equation and connected systems)

For $N = 2$ (the setting connected with the heavenly equation) the general system reads, $\hat{u} = u_1\partial_x + u_2\partial_y$, $\phi = u_0$,

$$
\begin{aligned}
&(\partial_{zy} + \partial_{wx})\hat{u} + [\partial_y \hat{u}, \partial_x \hat{u}] = (\partial_y \phi)\partial_x - (\partial_x \phi)\partial_y, \\
&(\partial_{zy} + \partial_{wx} + (\partial_y \hat{u})\partial_x - (\partial_x \hat{u})\partial_y)\phi = 0,
\end{aligned}
$$

Reduction with $k = 0$ (volume-preserving) corresponds to Dunajski generalization of the second heavenly equation,

$$
\begin{aligned}
&\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = \phi, \\
&\phi_{xw} + \phi_{yz} + \Theta_{yy}\phi_{xx} + \Theta_{xx}\phi_{yy} - 2\Theta_{xy}\phi_{xy} = 0.
\end{aligned}
$$

$N = 2,\ k = 1$

The reduction condition for $k = 1$

$$\operatorname{div} \hat{u} := \partial_x u_1 + \partial_y u_2 = \alpha\phi.$$

the reduced system

$$(\partial_{zy} + \partial_{wx})\hat{u} + [\partial_y \hat{u}, \partial_x \hat{u}] = \alpha^{-1}((\partial_y \operatorname{div} \hat{u})\partial_x - (\partial_x \operatorname{div} \hat{u})\partial_y).$$

The limit $\alpha \to 0$ corresponds to the Dunajski system, while the limit $\alpha \to \infty$ corresponds to the hierarchy characterized by the relation $\Psi^0 = \lambda$. For this hierarchy vector fields of Lax-Sato equations do not contain a derivative with respect to a spectral variable, and ϕ is equal to zero,

$$(\partial_{zy} + \partial_{wx})\hat{u} + [\partial_y \hat{u}, \partial_x \hat{u}] = 0, \qquad (1)$$

This hierarchy is a 'precursor' of Plebański second heavenly equation hierarchy corresponding to Hamiltonian vector fiels,

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0.$$

$N = 2$, $k = 2$

Reduction with $k = 2$ is characterized by the relation

$$J_0 = \exp(\alpha(\Psi^0)^2_-).$$

Generating equation for the reduced hierarchy is

$$(\exp(-\alpha(\Psi^0)^2)\mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2)_- = 0.$$

Reduction conditions are

$$\partial_z\phi - (\partial_x\hat{u})\phi - \frac{1}{2\alpha}\partial_x \operatorname{div}\hat{u} = 0,$$
$$\partial_w\phi + (\partial_y\hat{u})\phi + \frac{1}{2\alpha}\partial_y \operatorname{div}\hat{u} = 0.$$

The limit $\alpha \to 0$ corresponds to Dunajski system, and the limit $\alpha \to \infty$ – to the second Gelfand-Dikii reduction $(\Psi^0)^2_- = 0$ for the general system.

$N = 2,\ k = 3$

Reduction with $k = 3$ is characterized by the relation

$$J_0 = \exp(\alpha(\Psi^0)^3_-).$$

Generating equation for the reduced hierarchy is

$$(\exp(-\alpha(\Psi^0)^3)\mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2)_- = 0.$$

Reduction conditions are

$$\partial_z\left((\partial_x \hat{u})\phi\right) = \partial_x\left((\partial_x \hat{u})(\partial_x \hat{u})\phi + \phi(\partial_x \phi) - \frac{1}{3\alpha}\partial_x \operatorname{div}\hat{u}\right),$$
$$\partial_w\left((\partial_y \hat{u})\phi\right) = -\partial_y\left((\partial_y \hat{u})(\partial_y \hat{u})\phi + \phi(\partial_y \phi) - \frac{1}{3\alpha}\partial_y \operatorname{div}\hat{u}\right).$$

The limit $\alpha \to 0$ corresponds to Dunajski system, and the limit $\alpha \to \infty$ – to the third Gelfand-Dikii reduction $(\Psi^0)^3_- = 0$.

Reductions in terms of the dressing data

Riemann-Hilbert problem on the unit circle S in the complex plane of the variable λ,

$$\mathbf{\Psi}_{\text{in}} = \mathbf{F}(\mathbf{\Psi}_{\text{out}}).$$

The reduction condition for the dressing data reads

$$\mathbf{f} \circ \mathbf{F} \circ \mathbf{f}^{-1} \in \text{SDiff(N+1)} \tag{2}$$

(a 'twisted' volume-preservation condition).
The reduced hierarchy is defined by the generating relation

$$(\mathrm{d}f_0(\mathbf{\Psi}) \wedge \cdots \wedge \mathrm{d}f_N(\mathbf{\Psi}))_- = 0$$

For the considered class of reductions

$$f_0(\mathbf{\Psi}) = \Psi^0,$$
$$f_n(\mathbf{\Psi}) = \exp(-\alpha N^{-1}(\Psi^0)^k)\Psi^n, \quad 1 \leqslant n \leqslant N.$$

Two-component generalization of d2DTL hierarchy

Two-component generalization of the dispersionless 2DTL equation

$$(\mathrm{e}^{-\phi})_{tt} = m_t\phi_{xy} - m_x\phi_{ty},$$
$$m_{tt}\mathrm{e}^{-\phi} = m_{ty}m_x - m_{xy}m_t.$$

The Lax pair is

$$\partial_x\mathbf{\Psi} = \left((\lambda + \frac{m_x}{m_t})\partial_t - \lambda(\phi_t\frac{m_x}{m_t} - \phi_x)\partial_\lambda\right)\mathbf{\Psi},$$
$$\partial_y\mathbf{\Psi} = \left(\frac{1}{\lambda}\frac{\mathrm{e}^{-\phi}}{m_t}\partial_t + \frac{(\mathrm{e}^{-\phi})_t}{m_t}\partial_\lambda\right)\mathbf{\Psi}$$

For $m = t$ the system reduces to the dispersionless 2DTL equation

$$(\mathrm{e}^{-\phi})_{tt} = \phi_{xy},$$

Respectively, the reduction $\phi = 0$ gives an equation (Pavlov; Shabat and Martinez Alonso)

$$m_{tt} = m_{ty}m_x - m_{xy}m_t.$$

The hierarchy

$$\Lambda^{\text{out}} = \ln\lambda + \sum_{k=1}^{\infty} l_k^{+}\lambda^{-k}, \quad \Lambda^{\text{in}} = \ln\lambda + \phi + \sum_{k=1}^{\infty} l_k^{-}\lambda^{k},$$

$$M^{\text{out}} = M_0^{\text{out}} + \sum_{k=1}^{\infty} m_k^{+}\mathrm{e}^{-k\Lambda^{+}}, \quad M^{\text{in}} = M_0^{\text{in}} + m_0 + \sum_{k=1}^{\infty} m_k^{-}\mathrm{e}^{k\Lambda^{-}},$$

$$M_0 = t + x\mathrm{e}^{\Lambda} + y\mathrm{e}^{-\Lambda} + \sum_{k=1}^{\infty} x_k\mathrm{e}^{(k+1)\Lambda} + \sum_{k=1}^{\infty} y_k\mathrm{e}^{-(k+1)\Lambda},$$

where λ is a spectral variable.
The generating relation

$$((J_0)^{-1}\mathrm{d}\Lambda\wedge\mathrm{d}M)^{\text{out}} = ((J_0)^{-1}\mathrm{d}\Lambda\wedge\mathrm{d}M)^{\text{in}},$$

$J_0 = \{\Lambda, M\}$, the Poisson bracket is $\{f, g\} = \lambda(f_\lambda g_t - f_t g_\lambda)$,
$J_0^{\text{out}} = 1 + O(\lambda^{-1})$, $J_0^{\text{in}} = 1 + \partial_t m_0 + O(\lambda)$.

Lax-Sato equations

$$\left(\frac{\partial_n^+}{n+1} - \left(\frac{\lambda(\mathrm{e}^{(n+1)\Lambda})_\lambda}{\{\Lambda, M\}}\right)_+^{\text{out}} \partial_t + \left(\frac{(\mathrm{e}^{(n+1)\Lambda})_t}{\{\Lambda, M\}}\right)_+^{\text{out}} \lambda\partial_\lambda\right)\begin{pmatrix}\Lambda \\ M\end{pmatrix} = 0,$$
$$\left(\frac{\partial_n^-}{n+1} + \left(\frac{\lambda(\mathrm{e}^{-(n+1)\Lambda})_\lambda}{\{\Lambda, M\}}\right)_-^{\text{in}} \partial_t - \left(\frac{(\mathrm{e}^{-(n+1)\Lambda})_t}{\{\Lambda, M\}}\right)_-^{\text{in}} \lambda\partial_\lambda\right)\begin{pmatrix}\Lambda \\ M\end{pmatrix} = 0.$$

Nonlinear Riemann-Hilbert problem on the unit circle S in the complex plane of the variable λ,

$$\Lambda^{\text{out}} = F_1(\Lambda^{\text{in}}, M^{\text{in}}),$$
$$M^{\text{out}} = F_2(\Lambda^{\text{in}}, M^{\text{in}}),$$

Differential reduction

Generating relation

$$(\exp(-\alpha\Lambda)\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{out}} = (\exp(-\alpha\Lambda)\mathrm{d}\Lambda \wedge \mathrm{d}M)^{\text{in}}.$$

Implies that

$$J_0 = \{\Lambda, M\} = \lambda^{-\alpha}\exp(\alpha\Lambda),$$

and Lax-Sato equations for Λ split out from equations for M.
Nonhomogeneous linear equations for the Jacobian possess a solution

$$f = -\alpha \ln \lambda.$$

In terms of the Riemann-Hilbert dressing

$$\mathbf{f} \circ \mathbf{F} \circ \mathbf{f}^{-1} \in \mathrm{SDiff}(2),$$

where $f_1(\Lambda, M) = \Lambda,\ f_2(\Lambda, M) = \exp(-\alpha\Lambda)M$.

In terms of the two-component system we get a reduction

$$\mathrm{e}^{\alpha\phi} = m_t.$$

This reduction makes it possible to rewrite the system as one equation for m,

$$m_{tt} = (m_t)^{\frac{1}{\alpha}}(m_{ty}m_x - m_{xy}m_t), \qquad (*)$$

or in the form of deformed d2DTL equation,

$$(\mathrm{e}^{-\phi})_{tt} = m_t\phi_{xy} - m_x\phi_{ty},$$
$$m_t = \mathrm{e}^{\alpha\phi}.$$

Equation $(*)$ is equivalent to the generalization of a dispersionless $(1+2)$-dimensional Harry Dym equation, Blaszak (2002). It is also connected with an equation describing ASD vacuum metric with conformal symmetry, Dunajski and Tod (1999), see below.
The limit $\alpha \to 0$ gives the d2DTL equation, the limit $\alpha \to \infty$ gives the equation (Pavlov; Shabat and Martinez Alonso)

$$m_{tt} = m_{ty}m_x - m_{xy}m_t.$$

Dunajski-Tod equation

$$(\eta F_y + F_{y\tau})(\eta F_x - F_{x\tau}) - (\eta^2 F - F_{\tau\tau})F_{xy} = 4e^{2\rho\tau},$$

Locally describes general ASD vacuum metric with conformal symmetry, Dunajski and Tod (1999). It can be obtained from eqn. $(*)$, using a Legendre transformation.

Exterior differential form of equation $(*)$

$$\beta^{-1}\mathrm{d}m_t^{\beta} \wedge \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}m_y \wedge \mathrm{d}m \wedge \mathrm{d}y,$$

where $\beta = 1 - \alpha^{-1}$.

Legendre type transform (new independent variable τ, new dependent variable M)

$$m_t = e^{\tau}, \quad M = m - te^{\tau}.$$

Differential of M

$$\mathrm{d}M = M_x \mathrm{d}x + M_y \mathrm{d}y - te^{\tau}\mathrm{d}\tau.$$

Then

$$\beta^{-1}\mathrm{d}e^{\beta\tau} \wedge \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}M_y \wedge \mathrm{d}M \wedge \mathrm{d}y - \mathrm{d}M_y \wedge \mathrm{d}M_\tau \wedge \mathrm{d}y,$$

Transformed equation $(*)$

$$e^{\beta\tau} = (M_{y\tau}M_x - M_{yx}M_\tau) - (M_{y\tau}M_{x\tau} - M_{yx}M_{\tau\tau}) \qquad (3)$$

Scaling the time $\tau \to 2\tau$, in terms of the function $F = e^{-\tau}M$ we get

$$(F_y + F_{y\tau})(F_x - F_{x\tau}) - (F - F_{\tau\tau})F_{xy} = 4e^{-2\alpha^{-1}\tau}.$$

Considering the scaling $x \to \eta^{-1}x$, $y \to \eta^{-1}y$, $\tau \to \eta\tau$, we obtain Dunajski-Tod equation

$$(\eta F_y + F_{y\tau})(\eta F_x - F_{x\tau}) - (\eta^2 F - F_{\tau\tau})F_{xy} = 4e^{2\rho\tau}, \qquad (4)$$

where $\rho = -\alpha^{-1}\eta$.

Hamiltonian structure

(In collaboration with S.V.Manakov)

The Lax-Sato equations are Hamiltonian with the bracket

$$\{f,g\}' = \lambda^{\alpha}\{f,g\} = \lambda^{\alpha+1}(f_\lambda g_t - f_t g_\lambda).$$

$$\{\Lambda, M\} = \lambda^{-\alpha}\exp(\alpha\Lambda) \Rightarrow \{\Lambda, \exp(-\alpha\Lambda)M\}' = 1.$$

The Lax pair

$$\partial_x \mathbf{\Psi} = \left((\lambda + \frac{m_x}{m_t})\partial_t - \lambda(\phi_t\frac{m_x}{m_t} - \phi_x)\partial_\lambda\right)\mathbf{\Psi},$$

$$\partial_y \mathbf{\Psi} = \left(\frac{1}{\lambda}\frac{\mathrm{e}^{-\phi}}{m_t}\partial_t + \frac{(\mathrm{e}^{-\phi})_t}{m_t}\partial_\lambda\right)\mathbf{\Psi}$$

with the reduction $m_t = \mathrm{e}^{\alpha\phi}$ can be written in Hamiltonian form

$$\partial_x \mathbf{\Psi} = \{H_x, \mathbf{\Psi}\}', \quad H_x = (1-\alpha)^{-1}\lambda^{1-\alpha} - \alpha^{-1}\lambda^{-\alpha}\frac{m_x}{m_t},$$

$$\partial_y \mathbf{\Psi} = \{H_y, \mathbf{\Psi}\}', \quad H_y = -\frac{1}{\alpha+1}\lambda^{-\alpha-1}m_t^{-\frac{1}{\alpha}-1}.$$

THANK YOU!