Dynamics of localized states in extended supersymmetric quantum mechanics with multi–well potentials

V.P. Berezovoj, M.I. Konchatnij
e-mail: berezovoj@kipt.kharkov.ua

Institute of Theoretical Physics, NSC KIPT

The aim of the present paper is to describe the dynamics of localized states in multi-well potentials in the non-perturbative approach. We use multi-well potentials obtained in the framework of $N = 4$ $SUSY\ QM$ [1] and describe the dynamics of wave packets with corresponding propagators, calculated using approach of [2, 3, 4].

[1] V.P.Berezovoj, G.I.Ivashkevych and M.I.Konchatnij, Multi-well potentials in quantum mechanics and stochastic processes, SIGMA **6** (2010), 098, 18 pages.

[2] H.R.Jauslin, Exact propagator and eigenfunctions for multistable models with arbitrarily prescribed N lowest eigenvalues, J.Phys.A:Math.Gen.,21 (1988) 2337-2350.

[3] B.F.Samsonov and A.M.Pupasov, Exact propagators for complex SUSY partners of real potentials, Phys.Lett. A356 (2005) 210-214.

[4] A.M.Pupasov, B.F.Samsonov and U.Gunther, Exact propagators for SUSY partners, J.Phys. A: Math.Theor.,40(2007)10557-10587.

$N = 4$ *SUSY QM* and multi-well potentials

- To construct multi-well potentials in $N = 4$ *SUSY QM* [1], the states with energy ε below the ground state E_0 of the initial Hamiltonian H_0 is added (assuming that H_0 has only discrete spectrum).
- The super–Hamiltonian of $N = 4$ *SUSY QM* consists of three non–trivial Hamiltonians [1] $H_+^- = H_-^+ = H_0 - \varepsilon$, H_-^- and H_+^+. Spectra of the latter two have the additional state below the ground state of the initial Hamiltonian in case of exact $N = 4$ *SUSY*, while other states coincide with the those of $H_0 - \varepsilon$.

H_-^- and its wave functions have the following relations to $H_+^- = H_0 - \varepsilon$ and its initial wave functions:

$$H_-^- = H_+^- - \frac{d^2}{dx^2}\ln(\varphi_1(x,\varepsilon) + \varphi_2(x,\varepsilon),$$

$$\psi_-^-(x,E) = \frac{1}{\sqrt{2(E_i - \varepsilon)}}\frac{W\{\psi_+^-(x,E_i),\varphi(x,\varepsilon,1)\}}{\varphi(x,\varepsilon,1)},$$

$$\psi_-^-(x,E=0) = \frac{N^{-1}}{\varphi(x,\varepsilon,1)},\ \varphi(x,\varepsilon,1) = \varphi_1(x,\varepsilon) + \varphi_2(x,\varepsilon)$$

$$N^{-2} = \frac{2W\{\varphi_1,\varphi_2\}}{\Delta(+\infty,\varepsilon) - \Delta(-\infty,\varepsilon)},\ \Delta(x,\varepsilon) = \frac{\varphi_1(x,\varepsilon) - \varphi_2(x,\varepsilon)}{\varphi_1(x,\varepsilon) + \varphi_2(x,\varepsilon)}$$

- $\varphi_i(x,\varepsilon),\ i = 1,2$ are two linear independent non–negative solutions to the auxiliary equation $H_0\varphi(x) = \varepsilon\varphi(x)$. Their asymptotics: $\varphi_1(x) \to +\infty$ ($\varphi_2(x) \to 0$) under $x \to -\infty$, and $\varphi_1(x) \to 0$ ($\varphi_2(x) \to +\infty$) under $x \to +\infty$
- $\psi_+^-(x,E_i)$ are normalized wave functions of H_0
- $W\{\varphi_1,\varphi_2\}$ is Wronskian.

Using the form-invariance of H_+^+ and H_-^- [1] similar expressions for H_+^+, ψ_+^+ could be obtained:

$$H_+^+ = H_+^- - \frac{d^2}{dx^2} \ln \left(\varphi_1(x,\varepsilon) + \Lambda(\varepsilon,\lambda)\ \varphi_2(x,\varepsilon)\right),$$

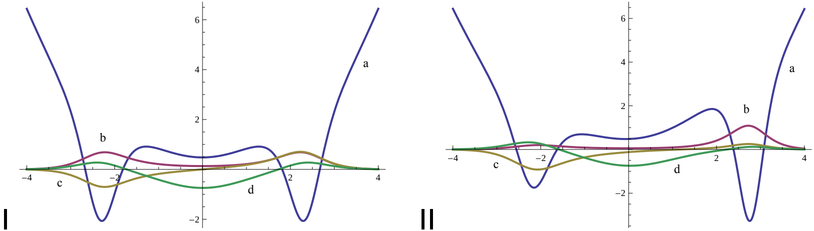
$$\psi_+^+(x, E=0) = \frac{N_\Lambda^{-1}}{(\varphi_1(x,\varepsilon) + \Lambda(\varepsilon,\lambda)\ \varphi_2(x,\varepsilon))}$$

$$\psi_+^+(x, E_i) = \frac{1}{\sqrt{2(E_i - \varepsilon)}} \frac{W\{\psi_+^-(x, E_i), \varphi_1(x,\varepsilon) + \Lambda(\varepsilon,\lambda)\varphi_2(x,\varepsilon)\}}{(\varphi_1(x,\varepsilon) + \Lambda(\varepsilon,\lambda)\varphi_2(x,\varepsilon))},$$

$$\Lambda(\varepsilon,\lambda) = \frac{\Delta(\infty,\varepsilon) - \lambda - (\lambda+1)\ \Delta(-\infty,\varepsilon)}{\Delta(\infty,\varepsilon) + \lambda - (\lambda+1)\ \Delta(-\infty,\varepsilon)},$$

where parameter λ is restricted to $\lambda > -1$ and normalization constant is $N_\Lambda^{-2} = (1+\lambda)\ N^{-2}$.

When choosing Hamiltonian of harmonic oscillator (HO) as initial, potentials U_-^- and U_+^+ and corresponding wave functions are:



Potential (I) $U_-^-(\xi)$ ($\omega = 1$, $\nu = -0.02$) (a), wave functions of ground (b), first excited (c) and second excited states (d). (II) $U_+^+(\xi, \lambda + 1)$ ($\omega = 1$, $\nu = -0.02$, $\lambda = -0.95$) and corresponding wave functions.

Time evolution of localized states in $N = 4$ $SUSY$ QM

The time evolution of the Gaussian wave packet

$$\Phi(x) = \left(\frac{\omega e^{2R}}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\left(x - x_0\right)^2 e^{2R}\right)$$

where R is a squeezing parameter, initially localized in $x = x_0$ is determined by

$$\Phi\left(x, t\right) = \int_{-\infty}^{+\infty} K\left(x, t; x_0, 0\right)\ \Phi(x_0)\ dx_0,$$

$$K\left(x, t; x_0, 0\right) = \sum_{n=0}^{\infty} \psi_n(x)\ \psi_n^*(x_0)\ e^{-iE_n t},$$

where $K\left(x, t; x_0, 0\right)$ is a propagator, which is sufficient to describe the dynamics of localized states in potentials of arbitrary complexity.

In $N=4$ $SUSY\ QM$ three $K_{\sigma_1}^{\sigma_2}(x,t;x_0,0)$ $(\sigma_i=\pm)$ exist. For example, $K_{+}^{+}(x,t;x_0,0)$ is

$$K_{+}^{+}(x,t;y,0)=\frac{1}{2}L_x L_y\int_{-\infty}^{+\infty}dz K_{+}^{-}(x,t;z,0)\ G_{+}^{-}(z,y,\varepsilon)+$$

$$+\frac{N_{\Lambda}^{-2}\ e^{-i\varepsilon t}}{\varphi(x,\varepsilon,\Lambda)\ \varphi(y,\varepsilon,\Lambda)},$$

Here $L_x=\left(\frac{d}{dx}-\frac{\varphi'(x,\varepsilon,\Lambda)}{\varphi(x,\varepsilon,\Lambda)}\right)$ and $G_{+}^{-}(z,y,\varepsilon)$ is the Green function of the Schrödinger equation with energy ε:

$$G_{+}^{-}(x,y,\varepsilon)=-\frac{2}{W\{f_l,f_r\}}\left(f_l(x,\varepsilon)\ f_r(y,\varepsilon)\theta(y-x)+\right.$$

$$\left.+f_l(y,\varepsilon)\ f_r(x,\varepsilon)\theta(x-y)\right).$$

According to the notation used: $f_l(x,\varepsilon) = \varphi_2(x,\varepsilon)$, $f_r(x,\varepsilon) = \varphi_1(x,\varepsilon)$. Acting by the operator L_y and simplifying, the expression for propagator becomes

$$K_+^+(x,t;y,0) = -\frac{1}{\varphi(y,\varepsilon,\Lambda)} L_x \left[\Lambda(\varepsilon,\lambda) \int_{-\infty}^{y} dz\, K_+^-(x,t;z,0)\ \varphi_2(z,\varepsilon) - \right.$$

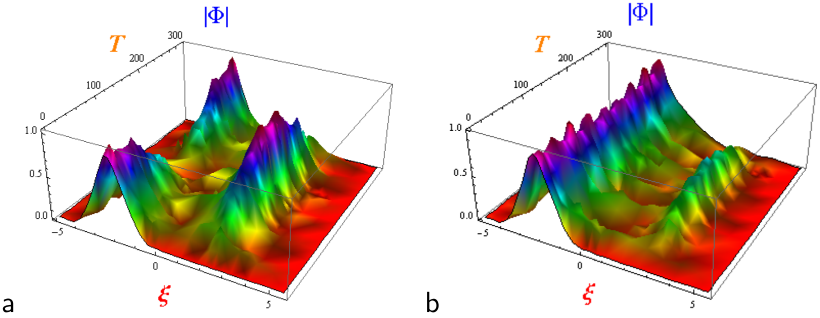
$$\left. - \int_y^{\infty} dz\, K_+^-(x,t;z,0)\ \varphi_1(z,\varepsilon)\right] + \frac{N_\Lambda^{-2}\ e^{-i\varepsilon t}}{\varphi(x,\varepsilon,\Lambda)\varphi(y,\varepsilon,\Lambda)}.$$

For Hamiltonian of HO as initial:

$$K_+^-(x,t;\ y,0) = \left(\frac{\omega\, e^{-i\pi(\frac{1}{2}+n)}}{2\pi\ \sin\omega\tau}\right)^{1/2} \exp\left\{\frac{i\omega}{2\ \sin\omega\ t}\left[\left(x^2+y^2\right)\ \cos\omega\ t - 2xy\right]\right\}$$

Dynamics of localized states in multi–well potentials

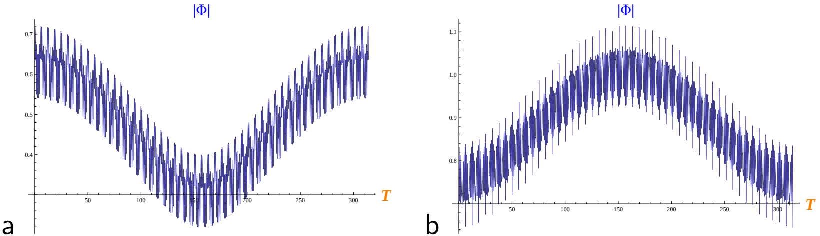
Small squeezed states.



Time dependence of $|\Phi(\xi, T)|$ initially located in left minimum with $R = 0.35$ (a) ($\xi_l = -2.29$) in $U_-^-(\xi)$, (b) ($\xi_l = -2.153$) in $U_+^+(\xi, \lambda)$, $\lambda = -0.95$.

In case of symmetric potential the evolution of wave packet have "Josephson"–type oscillations. But for U_+^+ the phenomenon of partial trapping of the wave packet is observed: in case of asymmetric potential, the portion of wave packet which tunnel from initial well is very small.
This phenomenon is more obvious when the initial WP is uniformly distributed among local minima $(\xi_L\,,\ \xi_R)$ of $U_+^+(\xi,\lambda)$, e.g.

$$\Phi(\xi,0)=\left(\frac{\sigma^{-2}}{4\pi}\right)^{1/4}\ \frac{e^{-\frac{(\xi-\xi_L)^2}{2\sigma^2}}+e^{-\frac{(\xi-\xi_R)^2}{2\sigma^2}}}{\sqrt{1+e^{-\frac{1}{4\sigma^2}(\xi_L-\xi_R)^2}}},\quad \sigma^2=e^{-2R}$$



Time dependence of $|\Phi(\xi,T)|$ in (a) left ($\xi_l=-2.153$) and (b) ($\xi_r=2.755$) wells. $R=0.35$.

It's important to note that in the left well wave packet oscillates and completely restores after time $T_{rev} = \frac{2\pi}{(E_1 - E_0)} \approx 300$. At the same time, the portion of the wave packet in the right well increases at $0 < T < 150$ while complete variation of $|\Phi(\xi, T)|$ is caused by the tunneling from left well. Moreover, when time is close to $T \sim 150$ the strong squeezing of the packet occurs and while time T increasing $|\Phi(\xi, T)|$ decreases and reaches its minimum. This means that some sort of partial "confining" of the portion of WP inside right well occurs. Thus, in one (left) minimum tunneling dynamics has oscillatory nature while in another partial "trapping" of part of WP occurs.

Conclusions

- We propose a non-perturbative approach to the description of the temporal dynamics of localized states. This approach is based on exactly solvable quantum mechanical models with multi-well potentials and their propagators.
- We also consider the properties of the tunneling of wave packets, taking into account all states of Hamiltonians with symmetric and asymmetric potentials, as well as their dependence on the degree of localization and deformations of potentials.
- The study the dynamics of initially localized states shows, that applicability of the two-state approximation for the description of tunneling is considerably limited.