Degression of Any Symmetric Spin in anti-de Sitter Space

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Joint Institute for Nuclear Research, Dubna, July 19, 2011

based on A. A. and M.A. Vasiliev (to appear soon)

Introduction

A conventional way to consider a higher-dimensional theory in lower dimension is by the Kaluza-Klein mechanism going from R^{d+k} to $R^d \times (S^1)^k$ as well as its further generalizations to less trivial compact manifolds. It may be however interesting analyze similar procedure starting from AdS space.
The procedure is divided in the steps

- Foliation of AdS geometry
- Splitting of the world indices
- Splitting of the fiber indices
- Choose the convenient variables
- Fourier expansion

Foliation of AdS geometry

The description of AdS_{d+1} geometry

$$\widehat{\mathbf{d}}\widehat{\mathbf{\Omega}}^{AB} + \widehat{\mathbf{\Omega}}^{A}{}_{C} \wedge \widehat{\mathbf{\Omega}}^{CB} = 0\,, \quad \widehat{\mathbf{\Omega}}^{AB} = -\widehat{\mathbf{\Omega}}^{BA}\,, \; A, B, \ldots = 0, \ldots, d+1\,.$$

The covariant definition of the frame field and Lorentz connection

$$V^A V_A = 1\,, \quad \widehat{\mathbf{H}}^A = \widehat{\mathfrak{D}} V^A\,, \quad \widehat{\mathbf{\Omega}}_L^{AB} = \widehat{\mathbf{\Omega}}^{AB} + V^A \widehat{\mathbf{H}}^B - V^B \widehat{\mathbf{H}}^A\,.$$

The covariant splitting of fiber indices

$$U^A U_A = -1\,, \quad V^A U_A = 0\,.$$

The description of AdS_d geometry

$$\mathbf{d}\boldsymbol{\omega}^{AB} + \boldsymbol{\omega}^{A}{}_{C} \wedge \boldsymbol{\omega}^{CB} = 0\,.$$

$$DU^B = 0\,, \quad \mathbf{h}^A = DV^A\,, \quad \boldsymbol{\omega}_L^{AB} = \boldsymbol{\omega}^{AB} + V^A \mathbf{h}^B - V^B \mathbf{h}^A\,.$$

The splitting of the world indices

$$\widehat{\mathbf{d}} = \mathbf{d} + d\varphi \partial_\varphi\,, \quad \widehat{\mathbf{\Omega}}^{AB} = \mathbf{\Omega}^{AB} + d\varphi \mathbf{\Psi}^{AB}\,,$$

$$\mathbf{\Omega}^{AB} = \boldsymbol{\omega}^{AB} + (1 - \cos^{-1}(\varphi)) \left(V^A \mathbf{h}^B - V^B \mathbf{h}^A \right) + \tan(\varphi) \left(U^A \mathbf{h}^B - U^B \mathbf{h}^A \right)\,,$$

$$\mathbf{\Psi}^{AB} = \cos^{-1}(\varphi) \left(U^A V^B - U^B V^A \right)\,, \quad \varphi \in [0, 2\pi)\,.$$

Symmetric Bosonic Massless Field in AdS_{d+1}

$\widehat{W}^{A(s-1),B(s-1)}$ 1-form carrying the representation of $o(d,2)$

$$\widehat{R}^{A(s-1),B(s-1)} = \widehat{\mathfrak{D}}\widehat{W}^{A(s-1),B(s-1)}\,,$$

$$\delta\widehat{W}^{A(s-1),B(s-1)} = \widehat{\mathfrak{D}}\Lambda^{A(s-1),B(s-1)}\,.$$

On-Mass-Shell theorem

$$\widehat{R}^{A(s-1),B(s-1)} = s^2\widehat{\mathbf{H}}_C \wedge \widehat{\mathbf{H}}_D B^{A(s-1)C,B(s-1)D}\,,$$

$$\eta_{A(2)}B^{A(s),B(s)} = 0, \quad V_B B^{A(s),B(s)} = 0.$$

Weyl module

$$\widehat{\mathfrak{D}}^L B^{A(i+s),B(s)} = (i+s+1)\widehat{\mathbf{H}}_C\left[B^{A(i+s)C,B(s)} + \frac{s}{i+2}B^{A(i+s)B,B(s-1)C}\right]$$
$$+\frac{(i+1)(d+i+2s-3)}{d+2i+2s-1}\left[\widehat{\mathbf{H}}^A B^{A(i+s-1),B(s)} - \frac{i+s-1}{d+2i+2s-3}\eta_L^{A(2)}\widehat{\mathbf{H}}_C B^{A(i+s-2)C,B(s)}\right.$$
$$-\frac{s}{d+i+2s-3}\eta_L^{AB}\widehat{\mathbf{H}}_C B^{A(i+s-1),B(s-1)C}$$
$$\left.+\frac{s(i+s-1)}{(d+2i+2s-3)(d+i+2s-3)}\eta_L^{A(2)}\widehat{\mathbf{H}}_C B^{A(i+s-2)B,B(s-1)C}\right], \quad i = 0,1,\dots$$

Cell Operator Algebra

Consider the generating function $T^{(k,l)} = T^{A(k),B(l)} y_{A_1} \dots y_{A_k} p_{B_1} \dots p_{B_l}$,
$k, l = 0, 1, \dots$, where $T^{A(k),B(l)}$ irrep of $o(d,2)$ and

$$X^A \widehat{\Sigma}^+_{1\,A} T^{(k,l)} \in T^{(k+1,l)}, \quad X^A \widehat{\Sigma}^-_{1\,A} T^{(k,l)} \in T^{(k-1,l)},$$

$$X^A \widehat{\Sigma}^+_{2\,A} T^{(k,l)} \in T^{(k,l+1)}, \quad X^A \widehat{\Sigma}^-_{2\,A} T^{(k,l)} \in T^{(k,l-1)}.$$

$$\widehat{\Sigma}^-_{1\,A} = \frac{\partial}{\partial y^A} + \frac{1}{\hat{x}+2} p^B \frac{\partial}{\partial y^B} \frac{\partial}{\partial p^A}\,, \quad \widehat{\Sigma}^-_{2\,A} = \frac{\partial}{\partial p^A}\,,$$

$\hat{x} = \hat{y} - \hat{p}$, $\hat{y} = y^A \frac{\partial}{\partial y^A}$, $\hat{p} = p^A \frac{\partial}{\partial p^A}$. The explicit form of $\widehat{\Sigma}^+_{1\,A}$ and $\widehat{\Sigma}^+_{2\,A}$ is more complicated.

The formulation of symmetric massless field in AdS_{d+1}

$$\widehat{R}^{(s-1,s-1)} = \widehat{\mathfrak{D}} \widehat{W}^{(s-1,s-1)}\,, \quad \delta \widehat{W}^{(s-1,s-1)} = \widehat{\mathfrak{D}} \Lambda^{(s-1,s-1)}\,,$$

$$\widehat{R}^{(s-1,s-1)} = \widehat{\mathbf{H}}^B \wedge \widehat{\mathbf{H}}^A \widehat{\Sigma}^{-L}_{1\,B} \widehat{\Sigma}^{-L}_{2\,A} B^{(s,s)}\,,$$

$$\widehat{\mathfrak{D}}^L B^{(s+i,s)} = \widehat{\mathbf{H}}^A \widehat{\Sigma}^{-L}_{1\,A} B^{(i+s+1,s)} + \frac{(\hat{x}+1)(\hat{z}-3)}{\hat{z}+\hat{x}-1} \widehat{\mathbf{H}}^A \widehat{\Sigma}^{+L}_{1\,A} B^{(i+s-1,s)}\,,$$

$i = 0, 1, \dots, \quad \hat{z} = d + \hat{y} + \hat{p}\,.$

Splitting space indices

We single out the components of forms along the (d+1)th direction

$$\widehat{R}^{(s-1,s-1)} = R^{(s-1,s-1)} - d\varphi \wedge T^{(s-1,s-1)} ,$$

$$\widehat{W}^{(s-1,s-1)} = W^{(s-1,s-1)} + d\varphi C^{(s-1,s-1)} .$$

$$R^{(s-1,s-1)} = \mathfrak{D} W^{(s-1,s-1)}, \quad T^{(s-1,s-1)} = \mathfrak{D} C^{(s-1,s-1)} - \widehat{\mathfrak{D}}_{\underline{\varphi}} W^{(s-1,s-1)}.$$

The relation of the AdS_{d+1} covariant derivative to that of AdS_d

$$\widehat{\mathfrak{D}} X^A = (\mathfrak{D} + d\varphi \widehat{\mathfrak{D}}_{\underline{\varphi}}) M^A{}_B \widetilde{X}^B = M^A{}_B (D + d\varphi \partial_\varphi) \widetilde{X}^B ,$$

$$R^{A(s-1),B(s-1)} = M^{A_1}{}_{A'_1} \dots M^{B_{s-1}}{}_{B'_{s-1}} \widetilde{R}^{A'(s-1),B'(s-1)}$$

and the same formulas for T, W, C, B.
On mass shell theorem

$$\widetilde{R}^{(s-1,s-1)} = D\widetilde{W}^{(s-1,s-1)} = \cos^{-2}(\varphi) \mathbf{h}^A \wedge \mathbf{h}^B \widehat{\Sigma}^-_{1\,A} \widehat{\Sigma}^-_{2\,B} \widetilde{B}^{(s,s)} ,$$

$$\widetilde{T}^{(s-1,s-1)} = D\widetilde{C}^{(s-1,s-1)} - \partial_\varphi \widetilde{W}^{(s-1,s-1)}$$

$$= \cos^{-2}(\varphi) \mathbf{h}^A \left[\widehat{\Sigma}^-_{1\,A} \widehat{\Sigma}^-_2 (\widetilde{U}) - \widehat{\Sigma}^-_1 (\widetilde{U}) \widehat{\Sigma}^-_{2\,A} \right] \widetilde{B}^{(s,s)} ,$$

where $\widetilde{U}^A = U_B M^{BA} = \cos^{-1}(\varphi) U^A + \tan(\varphi) V^A$.

Splitting fiber indices

The branching rule for $o(d,2) \downarrow o(d-1,2)$ in the case of two-row rectangular Young tableaux

$$\begin{array}{|c|}\hline \text{s-1} \\ \hline \text{s-1} \\ \hline \end{array} \Rightarrow \bigoplus_{t=0}^{s-1} \begin{array}{|c|}\hline \text{s-1} \\ \hline \text{t} \\ \hline \end{array}$$

The vector U^A allows to make the branching in the covariant way

$$\widetilde{R}^{(s-1,s-1)} = \sum_{t=1}^{s} (\widehat{\Sigma}_2^+(U))^{s-t} \widetilde{r}^{(s-1,t-1)} \,,$$

$$\widetilde{W}^{(s-1,s-1)} = \sum_{t=1}^{s} (\widehat{\Sigma}_2^+(U))^{s-t} \widetilde{w}^{(s-1,t-1)} \,,$$

$$\widetilde{T}^{(s-1,s-1)} = \sum_{t=1}^{s} (\widehat{\Sigma}_2^+(U))^{s-t} \widetilde{t}^{(s-1,t-1)} \,,$$

$$\widetilde{C}^{(s-1,s-1)} = \sum_{t=1}^{s} (\widehat{\Sigma}_2^+(U))^{s-t} \widetilde{c}^{(s-1,t-1)} \,.$$

The decomposition of Weyl tensor

The decomposition of Weyl tensor in accordance with the branching rule

$$\widetilde{B}^{(s,s)} = \sum_{t=0}^{s} (\widehat{\Sigma}_2^+(U))^{s-t} \widetilde{b}^{(s,t)} \,.$$

$\widetilde{B}^{(s,s)}$ carries $o(d,1)$ irrep. $\Rightarrow \widehat{\Sigma}^-_{2\,\widetilde{V}} \widetilde{B}^{(s,s)} = 0$. It gives for $\widetilde{b}^{(s,t)}$

$$\begin{aligned} &\Sigma^-_{1\,V} \widetilde{b}^{(s,t)} = 0\,, \\ &\quad (s-t)(s-t+1)\Sigma_2^+(V)\widetilde{b}^{(s,t-1)} - \sin(\varphi)(s-t)(\hat{z}-\hat{x}+s-t-3)\widetilde{b}^{(s,t)} \\ &+ \frac{(\hat{z}-\hat{x}+s-t-3)(\hat{z}-\hat{x}+s-t-2)}{(\hat{z}-\hat{x}-1)} \Sigma_2^-(V)\widetilde{b}^{(s,t+1)} = 0,\ t = 0, \ldots, s-1 \end{aligned}$$

$\widetilde{b}^{(s,t)}$ are not irreps of $o(d-1,1)$. We introduce $b^{(s,t)}$ with $\Sigma^-_{1\,V} b^{(s,t)} = \Sigma^-_{2\,V} b^{(s,t)} = 0$ and

$$\widetilde{b}^{(s,t)} = \sum_{k=0}^{t} (\Sigma_2^+(V))^{t-k} \gamma_k^t(\hat{x},\hat{z}) b^{(s,k)} \,,$$

where $\gamma_k^t(\hat{x},\hat{z}) = N_k^t(\hat{x},\hat{z}) C_{t-k}^{\frac{\hat{z}-\hat{x}-3}{2}}(\sin(\varphi))$.

The choose of convenient variables

The equations in AdS_d take the form of mixture for different spins

$$\widetilde{r}^{(s-1,t-1)} = \mathbf{h}^A \wedge \mathbf{h}^B \left\{ \sum_{k=0}^{t} \cos^{-2}(\varphi) \left(\Sigma_2^+(V)\right)^{t-k} (\dots)\gamma_k^t(\hat{x}, \hat{z}+2)\Sigma^-_{1\,A}\Sigma^-_{2\,B} - \sum_{k=0}^{t-2} \left(\Sigma_2^+(V)\right)^{t-k-2} (\dots)\gamma_{k+2}^{t-2}(\hat{x}, \hat{z})\Sigma^+_{2\,A}\Sigma^-_{1\,B} \right\} b^{(s,k)} ,$$

$$\widetilde{t}^{(s-1,t)} = -\cos^{-1}(\varphi) \sum_{k=0}^{t} (\Sigma_2^+(V))^{t-k}(\dots)\gamma_k^t(\hat{x}, \hat{z})\mathbf{h}^A\Sigma^-_{1\,A} b^{(s,k)} .$$

We require that in new variables the equations take the form

$$r^{(s-1,k-1)} = \frac{\cos^{-2}(\varphi)}{\hat{z} - \hat{x} + 2s - 2k - 2}\mathbf{h}^A \wedge \mathbf{h}^B \Sigma^-_{1\,A}\Sigma^-_{2\,B} b^{(s,k)} ,$$

$$t^{(s-1,k)} = -\frac{\cos^{-1}(\varphi)(\hat{x}+2)(\hat{z}-\hat{x}-1)}{(\hat{x}+1)(\hat{z}-\hat{x}-2)(\hat{z}-\hat{x}+2s-2k-4)}\mathbf{h}^A\Sigma^-_{1\,A} b^{(s,k)} .$$

There is the reversible transformation $(r, t) \Leftrightarrow (\widetilde{r}, \widetilde{t})$.

Fourier expansion

$$w^{(s-1,k-1)} = \sum_{n=0}^{\infty} w_n^{(s-1,k-1)} P_n^{(k)}(\varphi), \quad c^{(s-1,k-1)} = \sum_{n=0}^{\infty} c_n^{(s-1,k-1)} \cos(\varphi) P_n^{(k-1)}(\varphi),$$

$$r^{(s-1,k-1)} = \sum_{n=0}^{\infty} r_n^{(s-1,k-1)} P_n^{(k)}(\varphi), \quad t^{(s-1,k-1)} = \sum_{n=0}^{\infty} t_n^{(s-1,k-1)} \cos(\varphi) P_n^{(k-1)}(\varphi),$$

$$b^{(s,k)} = \sum_{n=0}^{\infty} b_n^{(s,k)} \cos^2(\varphi) P_n^{(k)}(\varphi) \,.$$

The conditions for $P_n^{(k)}(\varphi)$ follow form the curvatures

$$r^{(s-1,k-1)} = \ldots + (\ldots) \cos(\varphi) \left[\partial_\varphi + (d+2k-6) \tan(\varphi)\right] \mathbf{h}^A \mathbf{\Sigma}_2^+{}_A \wedge w^{(s-1,k-2)} \,,$$

$$t^{(s-1,k)} = \ldots + (\ldots) \partial_\varphi w^{(s-1,k)} \,.$$

The equations

$$\partial_\varphi^2 P_n^{(k)} + (d+2k-5) \tan(\varphi) \partial_\varphi P_n^{(k)} + (\Delta_n^{(k)})^2 P_n^{(k)} = 0 \,, \ k = 0, 1, \ldots, s.$$

The condition of normalizability of $P_n^{(k)}$ gives the spectrum

$$(\Delta_n^{(k)})^2 = (n+1)(n+d+2k-4) \,.$$

On mass shell theorem and Weyl module

On mass shell theorem

$$r_n^{(s-1,k-1)} = \frac{1}{d+2s-4}\mathbf{h}^A \wedge \mathbf{h}^B \Sigma^-_{1\ A}\Sigma^-_{2\ B} b_n^{(s,k)} \ ,$$

$$t_n^{(s-1,k)} = -\frac{(s-k+1)(d+2k-1)}{(s-k)(d+2k-2)(d+2s-4)}\mathbf{h}^A \Sigma^-_{1\ A} b_n^{(s,k)} \ .$$

Weyl module we deduce from the above equations

$$D^L b_n^{(s+i,k)} = \mathbf{h}^A \Sigma^-_{1\ A} b_n^{(s+i+1,k)} + \alpha^i_n(\hat{x},\hat{z})\mathbf{h}^A \Sigma^{+L}_{1\ A} b_n^{(s+i-1,k)}$$
$$+ \beta^i_n(\hat{x},\hat{z})\mathbf{h}^A \Sigma^-_{2\ A} b_{n-1}^{(s+i,k+1)} + \gamma^i_n(\hat{x},\hat{z})\mathbf{h}^A \Sigma^{+L}_{2\ A} b_{n+1}^{(s+i,k-1)} \ ,$$

for $k = 0, 1, \dots, s$ and $i = 0, 1, \dots$

The curvature

$$r_n^{(s-1,k-1)} = Dw_n^{(s-1,k-1)} + \frac{(s-k+1)(d+2k-3)\Delta_n^{(k)}}{(d+2k-6)(d+s+k-2)}\mathbf{h}^A\Sigma^+_{2\,A} \wedge w_{n+1}^{(s-1,k-2)}\,,$$

$$t_n^{(s-1,k)} = Dc_n^{(s-1,k)} + \frac{(s-k+1)(d+2k-1)\Delta_n^{(k)}}{(d+2k-2)(d+s+k-3)}\mathbf{h}^A\Sigma^+_{2\,A}c_{n+1}^{(s-1,k-1)}$$
$$+\frac{(d+s+k-2)\Delta_{n-1}^{(k+1)}}{(s-k)(d+2k-2)}w_{n-1}^{(s-1,k)} - \frac{d+2k-1}{d+2k-2}\Sigma^+_2(V)w_n^{(s-1,k-1)}$$
$$-\frac{(s-k+1)(d+2k-3)(d+2k-1)\Delta_n^{(k)}}{(d+2k-6)(d+2k-4)(d+2k-2)(d+s+k-3)}(\Sigma^+_2(V))^2 w_{n+1}^{(s-1,k-2)}.$$

They are invariant under gauge transformations

$$\delta w_n^{(s-1,k-1)} = D\xi_n^{(s-1,k-1)} + \frac{(s-k+1)(d+2k-3)\Delta_n^{(k)}}{(d+2k-6)(d+s+k-3)}\mathbf{h}^A\Sigma^+_{2\,A}\xi_{n+1}^{(s-1,k-2)}\,,$$

$$\delta c_n^{(s-1,k)} = -\frac{(d+s+k-2)\Delta_{n-1}^{(k+1)}}{(s-k)(d+2k-2)}\xi_{n-1}^{(s-1,k)} + \frac{d+2k-1}{d+2k-2}\Sigma^+_2(V)\xi_n^{(s-1,k-1)}$$
$$+\frac{(s-k+1)(d+2k-3)(d+2k-1)\Delta_n^{(k)}}{(d+2k-6)(d+2k-4)(d+2k-2)(d+s+k-3)}(\Sigma^+_2(V))^2\xi_{n+1}^{(s-1,k-2)}.$$