Modulational instability for extended NLS equations

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• The modulational instability (MI) is one of the most frequent instability in nature. It was studied in hydrodynamics, electrodynamics and nonlinear optics.

• In a very simple way the MI is the interaction of a strong carrier wave of frequency ω and small sidebands of frequencies $\omega_{1,2} = \omega \pm \Omega$ with the fulfillement of resonance condition on the corresponding wave vectors $k_1 + k_2 = 2k$ also.

• All these approaches can be termed as deterministic, as they generally study the evolution of small perturbations of the amplitude of the carrier wave.

Review paper V.E. Zakharov, L.A. Ostrovsky, Physica D ***238****, 540-548 (2009)*

- Statistical description of MI (SAMI), an alternative and complementary approach, provide a bridge between deterministic and random schools.
- In this approach the attention was concentrated on the wave-wave energy transfer due to weak nonlinear coupling.
- Very important in hydrodynamics (stability of surface wave trains in deep oceans) and in incoherent light propagation in nonlinear media.
- One of the methods used in SAMI was based on the Wigner-Moyal transform method.

Consider the extended derivative NLS equation

$$i\partial_t \Psi + \alpha \partial_x^2 \Psi + \beta |\Psi|^2 \Psi + i\gamma |\Psi|^2 \partial_x \Psi = 0 \qquad (1)$$

A 2-points correlation function

$$W(1,2) = < \Psi(x_1)\Psi^*(x_2) > = < \Psi(1)\Psi^*(2) >$$

Wigner function

$$\rho(x,k,t) = \frac{1}{2\pi} \int e^{-ik\xi} < \Psi(x + \frac{\xi}{2})\, \Psi(x - \frac{\xi}{2}) >$$

$$x = \frac{1}{2}(x_1 + x_2), \qquad \xi = x_1 - x_2$$

The Wigner function ρ is a real function.

$$n(x,t) =< |\Psi(x)|^2 >, \qquad n(x,t) = \int \rho(x,k,t)dk$$

$$n(x,t) = \begin{cases} \text{pulse intensity (optics)} \\ \text{fluid density (hydrodynamics)} \end{cases}$$

Introduce the quantities

$$q(x,t) =< (\partial_x \Psi(x))\Psi^*(x) >, \qquad q^*(x,t) =< \Psi(x)(\partial_x \Psi^*(x)) >$$

$$q + q^* = \partial_x n(x,t).$$

Another relation between q and q^* comes from the conservation low for $n(x,t)$; it writes

$$i\partial_t n(x,t) + \alpha\partial_x(q(x,t) - q^*(x,t)) + 2i\gamma n(x,t)\partial_x n(x,t) = 0,$$

where we used a Gaussian approximation

$$< |\Psi(x,t)|^4 > \simeq 2(n(x,t))^2.$$

Kinetic equation (Alber - 1978)

- write equation (1) for $x = x_1$ and multiply by $\Psi^*(2)$;
- write the complex conjugate of equation (1) for $x = x_2$ and multiply it by $\Psi(1)$;
- add the equations and take an ansamble average;
- use a Gaussian decoupling for averages of four Ψ functions.

Decoupling examples:

$$< |\Psi(1)|^2 \Psi(1) \Psi^*(2) > \simeq 2n(1) W(1,2)$$

$$< |\Psi(1)|^2 \left(\partial_x \Psi(x) \right) \Psi^*(2) > \simeq n(1) \partial_{x_1} W(1,2) + q(1) W(1,2).$$

The kinetic equation satisfied by $W(1,2)$ will be

$$i\partial_t W(1,2) + \alpha(\partial_{x_1}^2 - \partial_{x_2}^2)W(1,2) + 2\beta(n(1) - n(2))W(1,2) +$$
$$i\gamma(n(1)\partial_{x_1} + n(2)\partial_{x_2})W(1,2) + i\gamma(q(1) + q^*(2))W(1,2) = 0.$$

Wigner transform

In performing the Fourier transform with respect to the relative coordinate $\xi = x_1 - x_2$, we use

$$\xi^j e^{-ik\xi} = (i)^j \partial_k^j e^{-ikx}.$$

Exemple:

$$\mathrm{FT}\left((n(1) - n(2))W(1,2)\right) =$$
$$2\sum_{j=0} \frac{1}{2^{2j+1}(2j+1)!}\partial_x^{2j+1} n(x) \mathrm{F}T\left(\xi^{2j+1} W(1,2)\right) =$$
$$2i\sum_{j=0} \frac{(-)^j}{2^{2j+1}(2j+1)!}\partial_x^{2j+1} n(x)\partial_k^{2j+1}\rho(x,k) =$$
$$2i\, n(x) \sin\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)\rho(x,k,t),$$

where the sin(...) is defined by its Fourier transform and the arrows are indicating the direction in which the derivatives are acting.
The Fourier transform of the kinetic equation becomes

$$\partial_t \rho(x,k,t) + 2\alpha k \partial_x \rho(x,k,t) + 4\beta n(x,t) \sin\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)\rho(x,k,t) +$$

$$\gamma(\partial_x n(x,t)) \cos\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)\rho(x,k,t) - 2\gamma n(x,t) \sin\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)\rho(x,k,t) +$$

$$\gamma n(x,t) \cos\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)\rho(x,k,t) +$$

$$i\gamma(q(x) - q^*(x)) \sin\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)\rho(x,k,t) = 0$$

This result improves a similar result existing in literature (Marklund , Shukla, Bingham, Mendonca, Phys. Rev. 2006), where an incomplete decoupling procedure was used (the terms with q are missing).

The system equilibrium state $W_0(|\xi|)$: $\begin{cases} \text{homogeneous} \\ \text{isotropic} \end{cases}$

The Fourier transform $FT\{W_0(|\xi|)\} = f(k) \rightarrow$ even function

First order perturbation

$$\rho(x,k,t) = f(k) + \epsilon\rho_1(x,k,t), \qquad n(x,t) = n_0 + \epsilon n_1(x,t)$$

$$n_0 = \int f(k)dk; \qquad n_1(x,t) = \int \rho_1(x,k,t)dk.$$

Taking into account the continuity equation, denoting $h(k) = kf(k)$ the kinetic equation becomes

$$\partial_t\partial_x\rho_1(x,k,t) + (2\alpha k + \gamma n_0)\partial_x^2\rho_1(x,k,t) + 4\beta(\partial_x n_1)\sin\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)f(k) +$$

$$\gamma(\partial_x^2 n_1)\cos\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)f(k) - 2\gamma(\partial_x n_1)\sin\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)h(k) +$$

$$\frac{\gamma}{\alpha}(\partial_t n_1 + 2\gamma n_0\partial_x n_1)\sin\left(\frac{1}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_k}\right)f(k) = 0$$

Plane wave solutions

$$\rho_1(x,k,t) = g(k)e^{i(Qx-\Omega t)}$$

$$n_1(x,t) = Ge^{i(Qx-\Omega t)}$$

$$G = \int g(k)dk$$

$$\partial_x \to iQ, \qquad \partial_t \to -i\Omega$$

$$\sin\left(i\frac{Q}{2}\partial_k\right) \to i\sinh\left(\frac{Q}{2}\right)$$

$$\cos\left(i\frac{Q}{2}\partial_k\right) \to \cosh\left(\frac{Q}{2}\right)$$

$$2\sinh\left(\frac{Q}{2}\partial_k\right)f(k) = f\left(k+\frac{Q}{2}\right) - f\left(k-\frac{Q}{2}\right)$$

$$2\cosh\left(\frac{Q}{2}\partial_k\right)f(k) = f\left(k+\frac{Q}{2}\right) + f\left(k-\frac{Q}{2}\right)$$

Denoting

$$\omega = \frac{\Omega}{2\alpha Q} - \frac{\gamma}{2\alpha} n_0$$

$$I = \frac{1}{Q} \int_{-\infty}^{+\infty} \frac{f\left(k + \frac{Q}{2}\right) - f\left(k - \frac{Q}{2}\right)}{\omega - k} dk,$$

$$J = \frac{1}{Q} \int_{-\infty}^{+\infty} \frac{h\left(k + \frac{Q}{2}\right) - h\left(k - \frac{Q}{2}\right)}{\omega - k} dk,$$

$$K = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{f\left(k + \frac{Q}{2}\right) + f\left(k - \frac{Q}{2}\right)}{\omega - k} dk,$$

one obtains the implicit dispersion relation

$$1 - \frac{\beta}{\alpha} I + \frac{\gamma}{2\alpha} J - \frac{\gamma}{2\alpha} K + \frac{\gamma}{2\alpha} (\omega - \frac{\gamma}{2\alpha} n_0) I = 0$$

$\gamma = 0 \rightarrow$ NLS and $\beta = 0 \rightarrow$ dNLS-2.

$$\underline{f(k) = n_0\delta(k)}$$

$$I = -\frac{n_0}{\omega^2 - \frac{Q^2}{4}}$$

$$J = 0$$

$$K = \frac{n_0\omega}{\omega^2 - \frac{Q^2}{4}}$$

$$\left(\omega - \frac{\gamma}{2\alpha}n_0\right)^2 + \frac{\beta}{\alpha}n_0 - \frac{Q^2}{4} = 0$$

$$\gamma = 0, \qquad \omega = \omega_i, \qquad \omega_i = \frac{\Omega_i}{2\alpha Q}, \qquad \Omega_i = 2|\alpha|Q\sqrt{\frac{\beta}{\alpha}n_0 - \frac{Q^2}{4}}$$

$$\gamma \neq 0, \quad \omega = \omega_r + i\omega_i, \quad \omega_i = \sqrt{\frac{\beta}{\alpha}n_0 - \frac{Q^2}{4}}, \quad \Omega_i = 2|\alpha|Q\sqrt{\frac{\beta}{\alpha}n_0 - \frac{Q^2}{4}}$$

The same as for the NLS case.

Lorentzian case

$$f(k) = n_0 \frac{p}{\pi} \frac{1}{k^2 + p^2}$$

$Im\, \omega > 0$

$\frac{1}{\omega - k \pm \frac{Q}{2}}$ has a pole in the upper complex k semiplane.

$$I = -n_0 \frac{1}{(\omega + ip)^2 - \frac{Q^2}{4}},$$

$$J = n_0 \frac{ip}{(\omega + ip)^2 - \frac{Q^2}{4}},$$

$$K = n_0 \frac{\omega + ip}{(\omega + ip)^2 - \frac{Q^2}{4}}.$$

Denoting $\Delta = \left(\frac{\beta}{\alpha} n_0 - \frac{Q^2}{4}\right)^2 + \left(\frac{\gamma}{\alpha} n_0 p\right)^2 > 0,$

$$\omega_i = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{\beta}{\alpha} n_0 - \frac{Q^2}{4}\right)^2 + \sqrt{\Delta}} - p \geq 0$$

$$\sqrt{\Delta} \geq 2p^2 - \left(\frac{\beta}{\alpha} n_0 - \frac{Q^2}{4}\right)$$

Case a

$$\frac{\beta}{\alpha} n_0 - \frac{Q^2}{4} \geq 2p^2, \qquad \frac{Q^2}{4} \leq \frac{\beta}{\alpha} n_0 - 2p^2.$$

As $\Delta > 0$ and the r.h.s. is negative, the inequality is satisfied.

Case b

$$\frac{\beta}{\alpha} n_0 - \frac{Q^2}{4} \leq 2p^2, \qquad \frac{Q^2}{4} \geq \frac{\beta}{\alpha} n_0 - 2p^2.$$

$$\frac{Q^2}{4} \leq \frac{\beta}{\alpha} n_0 + \left(\frac{\gamma}{2\alpha} n_0\right)^2 - p^2$$

The modulational instability is restricted to long wave length region.

Conclusions

- MI from a statistical point of view was discussed for extended NLS equations (containing nonlinear derivative terms).
- The kinetic equation for the two-point correlation function $W(1,2)$ was obtained using a complete Gaussian decoupling procedure for averaged values of products of four Ψ field variable. Then the equation satisfied by Wigner's function $\rho(k,x,t)$ is containing not only the density $n(x,t)$, but also the quantities $q(x,t), \quad q^*(x,t)$ related to the derivatives of $n(x,t)$.
- In the linear approximation and for plane wave solutions an implicit dispersion relation is obtained. This is solved for a δ-function and a Lorentzian form of the equilibrium $f(k)$.
- For a δ-function the deterministic result is recovered.
- For a Lorentzian form the instability region is reducing when the parameter (p) of the distribution becomes greater and greater.

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