# The impact of isospin breaking effects on the scattering lengths extracted from the $K_{e4}$ decay

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**Abstract**

The final state interaction of pions in $K_{e4}$ decay allows one to obtain the value of the isospin and angular momentum zero $\pi\pi$ scattering length. We have shown that the electromagnetic interaction of pions and isospin symmetry breaking effects caused by different masses of neutral and charged pions, have an essential impact on the procedure of scattering length extraction from $K_{e4}$ decay.

## 1 Introduction

For many years the decay

$$K^{\pm} \rightarrow \pi^{+}\pi^{-}e^{\pm}\nu \tag{1}$$

was considered as the cleanest method to determine the isospin and angular momentum zero scattering length $a_0$ [1]. At present the value of $a_0$ is predicted by Chiral Perturbation Theory (ChPT) with high precision [2] ($\sim 2\%$) and its measurement with relevant accuracy can provide useful constraints on the ChPT Lagrangian. The appearance of new precise experimental data [3, 4] requires approaches taking into account the effects, which

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have been neglected up to now in extracting the scattering length from experimental data on $K_{e4}$ decays.
The common way to get the scattering length $a_0$ from the decay probability is based on the classical works [5, 6]. The transition amplitude for decay (1) can be written as the product of the lepton and hadronic currents:

$$A = \frac{G_F \sin\theta_c}{\sqrt{2}} \langle \pi^+\pi^- | J^{\mu}_{had} | K^+ \rangle \langle e^+ \nu_e | J^{lep}_{\mu} | 0 \rangle. \quad (2)$$

The leptonic part of this matrix element is known exactly, while the hadronic part can be described by three form factors $F, G, H$ [6]. Making the partial-wave expansion of the hadronic current with respect to the angular momentum of the dipion system and taking only $S$ and $P$ waves,[1] the hadronic form factors can be written in the following form:

$$\begin{aligned} F &= f_s e^{i\delta_0^0(s)} + f_p e^{i\delta_1^1(s)} \cos\theta_\pi, \\ G &= g_p e^{i\delta_1^1(s)}, \quad H = h_p e^{i\delta_1^1(s)}. \end{aligned} \quad (3)$$

Here $s = M^2_{\pi\pi}$ is the square of dipion invariant mass; $\theta_\pi$ is the polar angle of the pion in the dipion rest frame measured with respect to the flight direction of dipion in the $K$ meson rest frame. The coefficients $f_s, f_p, g_p, h_p$ can be parameterized as functions of pion momenta $q$ in the dipion rest system and of the invariant mass of lepton pair $s_{e\nu}$ in the known way [8]. The phases $\delta_l^I$ relevant to isospin $I$ and orbital momenta $l$ of the dipion system due to Fermi—Watson theorem [9] coincide with the corresponding phase shifts in elastic $\pi\pi$ scattering. From the other hand the phases can be related to the scattering lengths by the set of Roy equations [10].
Recently, the experiment NA48/2 at CERN [11] has observed the anomaly (*cusp*) at the two charged pions production threshold in the neutral pions mass distribution from the decays $K^\pm \to \pi^\pm\pi^0\pi^0$. As N. Cabibbo pointed [12], this is a result of isospin breaking in the final state due to the difference of masses of neutral and charged pions in the reaction [2] $\pi^+\pi^- \to \pi^0\pi^0$.
The final state interaction of pions in $K_{e4}$ decay is usually considered using the Fermi—Watson theorem [9] valid only in the isospin symmetry limit i.e. at $m_c = m_0$.
As was independently shown in [15, 16, 17, 18] using different approaches to the problem of $K_{e4}$ decay, the distinction in masses of neutral and charged

[1] As was shown in [7], the contribution of higher waves are small and can be safely neglected.

[2] The possibility of cusp in $\pi^0\pi^0$ scattering due to different pion masses in charge exchange reaction $\pi^+\pi^- \to \pi^0\pi^0$ was firstly predicted in [14].

pions leads to breaking of this theorem[3] and results in the corrections, which are not small even far from the production threshold [4].
In the present paper we consider all isospin symmetry breaking effects including the electromagnetic interaction in the dipion system and calculate their impact on the value of scattering length $a_0$ extracted from $K_{e4}$ decay rates.

## 2 Isospin symmetry breaking due to pions mass difference

The phase shift $\delta_0^0$ relevant to scattering length $a_0$, has an impact only on hadronic form factor $F$, whereas the form factors G and H depend only on $P$-wave phase shift $\delta_1^1$. The $P$-wave neutral pions production in inelastic process $\pi^+\pi^- \to \pi^0\pi^0$ is forbidden due to identity of neutral pions. Thus inelastic transitions can change only the first term in the form factor $F$, relevant to production of $S$-wave pions in the state with isospin $I = 0$.
It can be shown that in one loop approximation of nonperturbative effective field theory (see e.g. [20]), the decay amplitude M relevant to dipion in the state with $I = l = 0$

$$M = M_1(1 + ik_2a_{+-}) + ik_1a_xM_2. \tag{4}$$

Here $M_1, M_2$ are the so called "unperturbed" amplitudes [12] corresponding to the decays with charged and neutral dipions in the final state; $k_1 = \sqrt{M_{\pi\pi}^2 - 4m_0^2}/2$, and $k_2 = \sqrt{M_{\pi\pi}^2 - 4m_c^2}/2$ are the relative momenta in the $\pi^0\pi^0$ and $\pi^+\pi^-$ systems with the same invariant mass $M_{\pi\pi}$; $a_{+-}$ and $a_x$ are the $S$-waves amplitudes of the elastic scattering $\pi^+\pi^- \to \pi^+\pi^-$ and charge exchange reaction $\pi^0\pi^0 \to \pi^+\pi^-$.
As discussed in [13, 20], these amplitudes are related with scattering lengths $a_0, a_2$ through the following relations[4]:

$$\begin{aligned} a_{+-} &= \frac{2a_0 + a_2}{3}(1 + \epsilon), \\ a_x &= \frac{\sqrt{2}(a_0 - a_2)}{3}\left(1 + \frac{\epsilon}{3}\right), \quad \epsilon = \frac{m_0^2 - m_c^2}{m_c^2}. \end{aligned} \tag{5}$$

A simple relation between the "unperturbed" amplitudes $M_1 = \sqrt{2}M_2$, follows from the rule $\Delta I = 1/2$ for semi-leptonic decays. Thus, in the isospin

[3]The breakdown of Fermi—Watson theorem in photoproduction has been discussed in [19].
[4]Our definition of amplitudes coincide with the one adopted in [23], and differs from the accepted definition in [13, 20].

symmetry limit ($m_0 = m_c$) [5] :

$$M = M_1(1 + ika_0) = M_1\sqrt{1 + k^2 a_0^2}\,\exp(i\delta_0^0). \tag{6}$$

This equation is nothing else than the Fermi—Watson theorem for the $\pi\pi$ interaction in the final states.
The considered picture can be generalized to higher orders [21]. Summing all subsequent loops of $\pi\pi$ scattering we obtain the following:

$$\begin{aligned} M &= \frac{M_1(1 - ik_1a_{00}) + ik_1a_xM_2}{D}, \\ D &= (1 - ik_1a_{00})(1 - ik_2a_{+-}) + k_1k_2a_x^2, \end{aligned} \tag{7}$$

where the $\pi^0\pi^0$ elastic amplitude $a_{00} = (a_0 + 2a_2)(1 - \epsilon)/3$.
Using the relation between "unperturbed" amplitudes $M_1 = \sqrt{2}M_2$, it is convenient to rewrite this equation in the following form:

$$\begin{aligned} M &= \frac{M_1\sqrt{1 + k_1^2\left(a_{00} - a_x/\sqrt{2}\right)^2}}{|D|}\exp(i\delta_0^0), \\ \delta_0^0 &= \arctan\left\{\frac{k_1a_{00} + k_2a_{+-}}{1 + k_1k_2(a_x^2 - a_{00}a_{+-})}\right\} \\ &- \arctan\{k_1(a_{00} - a_x/\sqrt{2})\}. \end{aligned} \tag{8}$$

Thus, unlike to isospin symmetry limit the decay amplitude depends on $a_2$.
The expression (8) is the generalization of Fermi—Watson theorem for the case of the isospin symmetry breaking in the strong phase relevant to the $S$-wave $\pi\pi$ scattering.
Another effect which can be important in the procedure of the scattering lengths extraction from the experimental data on $K_{e4}$ decay, is the Coulomb interaction among the charged pions. The widely spread wisdom is that in order to take the electromagnetic effects into account it is sufficient to multiply the square of matrix element (2) by Gamov factor

$$G = \frac{2\pi w}{1 - \exp(-2\pi w)}, \quad w = \frac{\alpha}{v}. \tag{9}$$

Here $v$ is the relative velocity in the dipion system and $\alpha = e^2/(4\pi)$ is fine structure constant.
Later on we show that besides this multiplier the electromagnetic interaction between pions also change the expression (8) for the strong phase results in essential influence on the value of scattering length $a_0$ extracted from experimental data.

[5] In this limit the scattering lengths $a_I$ corresponding to $\pi\pi$ states with isospin $I = 0, 2$ are connected with elements of K-matrix by the relation $\exp(2i\delta_I) = (1+ika_I)/(1-ika_I)$.

# 3 Electromagnetic interaction in $\pi\pi$ system

In order to take into account the electromagnetic interactions between pions, we take an advantage of the trick successfully used in [21]. To switch on the electromagnetic interaction, one has to replace the charged pion momenta $k_1$ by a logarithmic derivative of the pion wave function in the Coulomb potential at the boundary of the strong field $r_0$:

$$ik_1 \to \tau = \left.\frac{d\log[G_0(kr) + iF_0(kr)]}{dr}\right|_{r=r_0}. \tag{10}$$

Here $F_0, G_0$ are the regular and irregular solutions of the Coulomb problem. In the region $kr_0 \ll 1$, where the electromagnetic effects are significant, this expression can be simplified:

$$\begin{aligned} \tau &= ik - \alpha m\left[\log(-2ikr_0) + 2\gamma + \psi(1 - i\xi)\right], \\ Re\,\tau &= -\alpha m\left[\log(2kr_0) + 2\gamma + Re\,\psi(1 - i\xi)\right], \\ Im\,\tau &= kG, \quad \xi = \alpha m/(2k), \end{aligned} \tag{11}$$

where Euler constant $\gamma = 0.5772$ and digamma function $\psi(\xi) = d\log\Gamma(\xi)/d\xi$. Substituting these expressions in (8) one can express the modified phase for $\pi^+\pi^-$ state ($I = l = 0$) $\tilde{\delta}_0^0$ through the standard phases [1] $\delta_0^0, \delta_0^2$ relevant to exact isospin symmetry limit.

Dividing the modified $S$-wave phase as a sum of strong $\delta_{st}^0$ and Coulomb $\delta_c^0$ terms, we obtain:

$$\begin{aligned} \tilde{\delta}_0^0 &= \delta_{st}^0 + \delta_c^0, \\ \delta_{st}^0 &= \arctan(A\tan\delta_0^0 + B\tan\delta_0^2), \\ A &= [2G(1+\epsilon) + \lambda(1+\epsilon/3)]/3, \quad B = [G(1+\epsilon) - \lambda(1+\epsilon/3)]/3, \\ \delta_c^0 &= Arg\{\Gamma(1 - i\alpha/\beta)\}, \quad \beta = \sqrt{1-4v}/(1-2v), \\ \lambda &= \sqrt{(1-4u_0)/(1-4u_c)}, \quad u_c = m_c^2/s, \quad u_0 = m_0^2/s. \end{aligned} \tag{12}$$

Let us note that,whereas the Coulomb phase $\delta_c$ has a common textbook form [22], the strong phase $\delta_{st}^0$ is essentially modified by electromagnetic effects as well as by isospin symmetry breaking effects provided by pions mass difference.

Using the same approach one can show that the modified $P$-wave phase becomes as follows:

$$\begin{aligned} \tilde{\delta}_1^1 &= \delta_{st}^1 + \delta_c^1, \\ \tilde{\delta}_{st}^1 &= \arctan\left\{(1 + \alpha^2/\beta^2)G\tan\delta_1^1\right\}, \quad \delta_c^1 = Arg\{\Gamma(2 - i\alpha/\beta)\}. \end{aligned} \tag{13}$$

The difference of $S$ and $P$-wave Coulomb phases has a simple form:

$$\delta_c^0 - \delta_c^1 = \arctan(\alpha/\beta). \quad (14)$$

In the limit of exact isospin symmetry ($m_c = m_0$; $\alpha = 0$) the above expressions turn to well known one.
Setting in accordance with ChPT $a_0 = 0.225$, $a_2 = -0.037$ and using the relevant phases $\delta_0^0$, $\delta_1^1$ from Appendix D of [1], we calculated the modified phases differences $\delta = \tilde{\delta_0^0} - \tilde{\delta_1^1}$ as a function of the invariant mass $M_{\pi\pi}$.
Fig.1 shows these dependencies in the two limiting cases. The dashed line corresponds to exact isospin symmetry limit ($m_0 = m_c$, $\alpha = 0$). To get $\delta = \delta_0^0 - \delta_1^1$ we use the phases values from Appendix D of work [1]. The solid line is the result of all the isospin breaking effects, calculated by obtained above expressions. The experimental data are from [4].
This figure demonstrates agreement between experimental data and the predictions of ChPT, when isospin symmetry breaking corrections are taken into account.

In table 1 we cite $\delta$ as a function of dipion invariant mass $M_{\pi\pi}$ in respect to different isospin breaking corrections. This allows one to estimate separately the contribution of considered above effects.

Table 1: The impact of considered corrections on phase difference $\delta = \tilde{\delta_0^0} - \delta_1^1$: 1) standard case [1] with $a_0 = 0.225$, $a_2 = -0.037$; 2) the case with charge exchange process $\lambda = \sqrt{(1-4u_0)/(1-4u_c)}$; 3) the impact of parameter $\epsilon$ (expression (5)); 4) the case with electromagnetic effects in the strong phases; 5) the case with the Coulomb phases difference (14)

| $M_{\pi\pi}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **0.285** | **0.048** | **0.059** | **0.061** | **0.063** | **0.082** |
| **0.300** | **0.096** | **0.103** | **0.108** | **0.110** | **0.122** |
| **0.315** | **0.134** | **0.140** | **0.147** | **0.149** | **0.159** |
| **0.330** | **0.170** | **0.175** | **0.184** | **0.186** | **0.195** |
| **0.345** | **0.205** | **0.210** | **0.220** | **0.223** | **0.231** |
| **0.360** | **0.239** | **0.244** | **0.256** | **0.259** | **0.267** |
| **0.375** | **0.274** | **0279** | **0.292** | **0.296** | **0.304** |
| **0.390** | **0.309** | **0.314** | **0.328** | **0.333** | **0.340** |

# 4 Conclusions

All the isospin symmetry breaking corrections considered above increase the phase difference $\delta$. Their contribution is the largest near the threshold, but even far from it they are essential.

The $K_{e4}$ decay amplitude if to take the isospin symmetry breaking effects into account depends on scattering length $a_2$ unlike the common approach.

Our results are in accordance with approach developed in [15, 17, 18], thus allowing to extract the values of scattering lengths from $K_{e4}$ decays with higher accuracy than in standard approximation.

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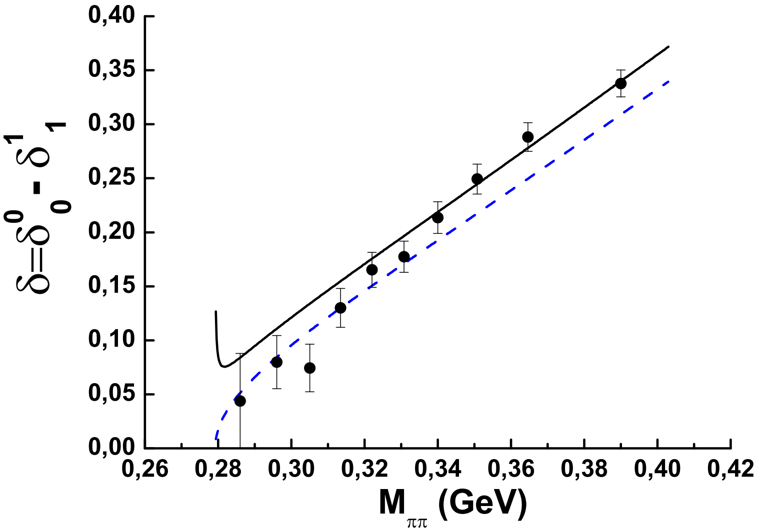


Figure 1: The dependence of $\delta = \tilde{\delta}_0^0 - \tilde{\delta}_1^1$ on dipion invariant mass $M_{\pi\pi}$ in the exact isospin symmetry case (dashed line) and with all isospin symmetry breaking corrections taken into account (solid line).