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Inönü–Wigner Contractions and Separation of Variables on the Spaces with Constant Curvature

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Abstract

In this article we review some recent results on the application of Lie algebra and Lie group contractions to special function theory and to the separation of variables in Laplace–Beltrami equations on homogeneous spaces. The concept of analytic contractions is defined. The contraction parameters are introduced into the the coordinates on the homogeneous spaces and thus also into the differential operators realizing the Lie algebras, into the Laplace–Beltrami operators, into the basis functions of representations and all other objects figuring in the representation theory of the groups involved.

1. Introduction

Lie algebra contractions were introduced into physics by Inönü and Wigner [2] in 1953 as a mathematical expression of a philosophical idea, namely the *correspondence principle*. This principle tells us that whenever a new physical theory surplants an old one, there should exist a well defined limit in which the results of the old theory are recovered. A typical example of such a limiting procedure is the relation between relativistic and nonrelativistic theories where the limit $c \to \infty$ for the velocity of the light takes the Poincaré group into the Galilei one. Similarly, de Sitter space with its $SO(3,2)$ or $SO(4,1)$ isometry group is contracted to a flat Minkowski space with its Poincaré isometry group $P(3,1)$, in the limit $R \to \infty$ for the radius of the universe.

The Inönü–Wigner contractions can be viewed as singular changes of basis in a given Lie algebra L [2]. Indeed, consider a basis $\{X_1, X_2, \dots$

$\ldots, X_n\}$ of L. Introduce a new basis

$$\widetilde{X}_i = U_{ik}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p) X_k, \qquad U_{ik}(1, 1, \ldots, 1) = \delta_{ik}, \tag{1.1}$$

where the matrix $U(\varepsilon)$ realizing the transformation (1.1) depends on some parameters ε_i and is nonsingular for $\varepsilon_i \neq 0$, $|\varepsilon_i| < \infty$. For $\varepsilon \to 0$ (i.e. some, or all of the ε_i vanishing) the matrix $U(\varepsilon)$ is singular. In this limit the commutation relations of L change (continuously) into those of a different, nonisomorphic, Lie algebra $\widetilde{L}$.

Let us consider a simple *example*, namely the rotation algebra $L = o(3)$ with the basis X_i, $i = 1, 2, 3$ chosen such that the following commutation relation are valid

$$[X_i, X_j] = \varepsilon_{ijk} X_k. \tag{1.2}$$

Introduce a new basis

$$Y_i = U(\varepsilon)_i^j X_i \tag{1.3}$$

where

$$U(\varepsilon) = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The commutation relations (1.2) are transformed into

$$[Y_1, Y_2] = \varepsilon^2 Y_3, \quad [Y_2, Y_3] = Y_1, \quad [Y_3, Y_1] = Y_2$$

and in the limit $\varepsilon \to 0$ we obtain the Euclidean Lie algebra $\widetilde{L} = e(2)$:

$$[Y_1, Y_2] = 0, \quad [Y_2, Y_3] = Y_1, \quad [Y_3, Y_1] = Y_2.$$

It is well known that practically all properties of large classes of special functions can be obtained from the representation theory of Lie groups, making use of the fact that the special functions occur as basis functions of irreducible representations, as matrix elements of transformation matrices, as Clebsch–Gordon coefficients, or in some other guise. One very fruitful application of Lie theory, in this context, is the algebraic approach to the separation of variables in partial differential equations. In this approach separable coordinate systems (for Laplace-Beltrami, Schrödinger and other invariant partial differential equations) are characterized by complete sets of commuting second order operators.

By «separable coordinates» we mean curvilinear coordinates $(\xi_1, \xi_2, \ldots, \xi_n)$ on a hyperboloid H_n or sphere S_n such that the Laplace–Beltrami equation

$$\Delta_{LB} \Psi = E \Psi, \qquad \Delta_{LB} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \sqrt{g}\, g^{ij} \frac{\partial}{\partial \xi^j}, \tag{1.4}$$

allows the «multiplicative» separation of variables

$$\Psi_{\lambda_1, \lambda_2, \ldots, \lambda_n}(\xi_1, \xi_2, \ldots\ldots, \xi_n) = \prod_{i=1}^{n} \Phi_i(\xi_i; \lambda_1, \lambda_2, \ldots, \lambda_n). \tag{1.5}$$

The metric is

$$ds^2 = g_{ij} d\xi^i d\xi^j, \qquad g = |\det g_{ij}|, \qquad g_{ij} g^{jk} = \delta_i^k, \tag{1.6}$$

In eq. (1.5) each function Φ_i depends only on one variable ξ_i, but can depend on all the separation constants λ_j.

The separated solutions of eq. (1.4) are simultaneous eigenfunctions of a complete set of commuting operators $\{Y_1, Y_2, ..., Y_n\}$ (including the Laplace–Beltrami operator), where n is the dimension of the space. We thus have

$$Y_a \Psi = -\lambda_a \Psi, \qquad a = 1, ..., n. \tag{1.7}$$

The operators Y_a are second order operators in the enveloping algebra of L, where L is the Lie algebra of the isometry group G of the corresponding space. Let $\{X_1, ..., X_n\}$ be a basis of L and

$$Y_a = A^a_{ik} X_i X_k, \qquad [Y_a, Y_b] = 0, \qquad A^a_{ik} = A^a_{ki}; \quad a = 1, 2, ...n. \tag{1.8}$$

The commuting sets of operators $\{Y_a\}$ can be classified into conjugacy classes under the action of the isometry group G.

The classification of the sets $\{Y_a\}$ provides a classification of inequivalent coordinate systems. Particularly simple coordinate systems are obtained if all operators Y_a in a given set are either squares of elements in the Lie algebra L

$$Y_j = \left\{ \sum_{k=1}^{N} a_{jk} X_k \right\}^2 \tag{1.9}$$

or Casimir operators of subalgebras L. Such coordinate systems have been called «**subgroup type coordinates**» [10].

Subgroup type coordinates on homogeneous spaces associated with rotation groups $O(n)$ and unitary groups $SU(n)$ where studied by Vilenkin, Kuznetsov, Smorodinsky and others [14, 15]. They introduced a graphical method, the «method of trees» to describe subgroup coordinates on spheres $S_n \sim O(n+1)/O(n)$ and complex «spheres» $C_n \sim SU(n)/U(n-1)$. The method of trees was extended from S_n spheres to Euclidean spaces E_n and hyperboloids $L_n \sim O(n,1)/O(n)$ or $L_n \sim O(n,1)/O(n-1,1)$ in two recent articles [6, 12].

Some new aspects of the theory of Lie group and Lie algebra contractions have recently been presented in the series of papers [3]-[8], namely: *the relation between separable coordinate systems in curved and flat spaces, related by the contraction of their isometry groups*. The approach makes use of specific realizations of Inönü–Wigner contractions. The articles [3]-[5] were devoted to two simple homogeneous spaces:the two-dimensional sphere $S_2 \sim O(3)/O(2)$ and the two-dimensional hyperboloid $H_2 \sim O(2,1)/O(2)$. The new aspect introduced was a contraction procedure, called **analytic contractions**. The contractions are analytic because the contraction parameter R -the radius of sphere, or pseudosphere, appears in the operators of the Lie algebra, in the eigenvalues and eigenfunctions, not only in the structure constants. For two-dimensional spaces all types of coordinates where considered. For example, contractions of $O(3)$ to $E(2)$ relate elliptic coordinates on S_2 to elliptic and parabolic coordinates on E_2. They also relate spherical coordinates on S_2 to polar and Cartesian coordinates on E_2. Similarly, all 9 coordinate systems on the H_2 hyperboloid can be contracted to at least one of the four systems on E_2, or one of the 10 separable systems on $E_{1,1}$ [4, 5, 11]. Using this method, it is possible to observe the contraction limit $R \to \infty$ at all levels. The level of the Lie algebra as realized by vector fields. That Laplace–Beltrami operators in the four homogeneous spaces (sphere or hyperboloid on one hand and Euclidean or pseudo-Euclidean space on the other). The second order operators in the enveloping algebras, characterizeng separable systems. The separable coordinate systems themselves, the separated (ordinary) differential equations, the separated eigenfunctions of the invariant operators and the interbases expansions.

In paper [6] the dimension of the space was arbitrary, but only the simplest types of coordinates were considered, namely subgroup ones. Furthemore, we introduce a *graphical method* for connecting subgroup-type coordinates on the sphere $S_n \sim O(n+1)/O(n)$ (characterized by tree diagrams) and on the Euclidean space E_n (characterized by cluster diagrams) and give the rules relating the contraction limit $R \to \infty$ of the coordinates, eigenvalues and basis functions. The analytic contractions from the rotation group $O(n+1)$ to the Euclidean group $E(n)$ are used to obtain asymptotic relations for matrix elements between the eigenfunctions of the Laplace–Beltrami operator corresponding to separation of variables in the subgroup–type coordinates on S_n [7, 8, 13]. The contraction for non subgroup coordinates have been described in [9].

2. Subgroup Coordinates on S_n and the Method of Trees

Let us consider the n - dimensional sphere S_n:

$$u_0^2 + \sum_{\nu=1}^{n} u_\nu^2 = R^2, \quad R^2 > 0, \tag{2.1}$$

where u_i are Cartesian coordinates in the Euclidean ambient space E_{n+1}. Its isometry group is $O(n+1)$. We choose a standard basis L_{ik} for the Lie algebra $o(n+1)$:

$$L_{ik} = u_i\partial_k - u_k\partial_i, \tag{2.2}$$

$$[L_{ij}, L_{rs}] = \delta_{jr}L_{is} + \delta_{is}L_{jr} - \delta_{js}L_{ir} - \delta_{ir}L_{js}, \tag{2.3}$$

$$i, j, r, s = 0, 1, 2, ..., n.$$

The metric tensor in this case has the form: $g_{ij} = diag(1, 1, ..., 1)$ and the Laplace-Beltrami operator on S_n is:

$$\Delta(S_n) = \frac{1}{2R^2} \sum_{i,k=0}^{n} L_{ik}^2. \tag{2.4}$$

We are dealing with the Laplace–Beltrami equation (1.4) on the sphere S_n and use a graphical method, the «method of trees», for characterizing different types of subgroup coordinates, or hyperspherical coordinates on S_n, complete sets of commuting operators and their eigenvalues and separated solutions. These methods are best presented in the original article [15] and in the book [14].

Let us briefly describe some basic facts concerning the method of trees [15]. Each end point $u_i, i = 0, 1, 2, ..., n$ on the tree corresponds to a Cartesian coordinate in the ambient space E_{n+1}. At each branching point, we introduce an angle θ_j. To express a cartesian coordinate in terms of hyperspherical ones we move along the tree from the ground upwards to a specific coordinate u_i. At each branching point, we write $\cos\theta_j$, if we go to the left, and $\sin\theta_j$, if we go to the right. For example, to the tree on Fig.1 there correspond the following hyperspherical coordinates:

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2, & u_1 &= R\cos\theta_1\sin\theta_2\cos\theta_3, \\ u_2 &= R\cos\theta_1\sin\theta_2\sin\theta_3, & u_3 &= R\sin\theta_1\cos\theta_4\cos\theta_5, \\ u_4 &= R\sin\theta_1\cos\theta_4\sin\theta_5 & u_5 &= R\sin\theta_1\sin\theta_4, \end{aligned}$$

To each branching point on the tree diagram we also associate non-negative quantum numbers l_j. This will determine the eigenvalue λ_j of the $O(k)$ Laplace–Beltrami operators according to the formula

$$Y_j\Psi = R^2\Delta_{LB}\Psi = -\lambda_j\Psi, \qquad \lambda_j = l_j(l_j + k - 2), \tag{2.5}$$

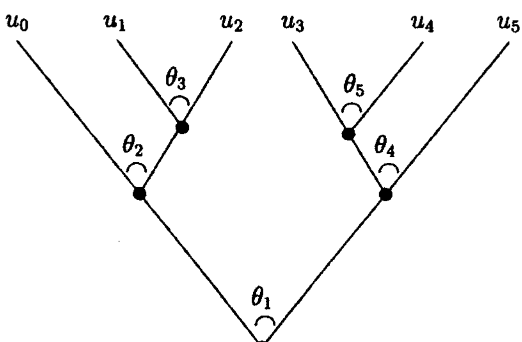

Fig. 1. Example of tree for hyperspherical coordinates on the sphere S_5

where k is the dimension of the ambient space above the corresponding vertex on the tree. Only for $k = 2$ we have $l_j = 0, \pm 1, \pm 2, \ldots$..

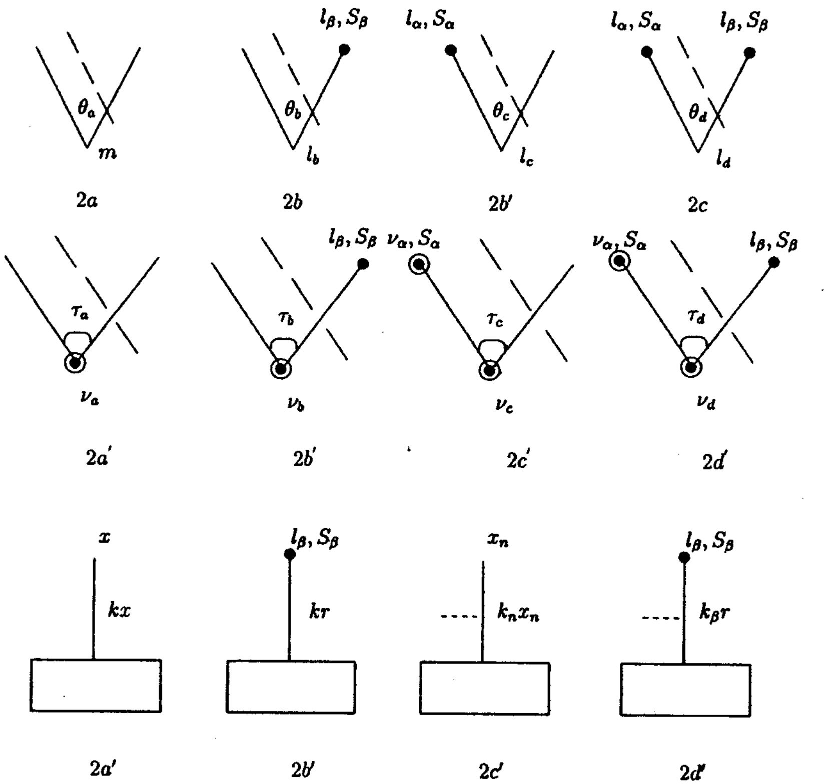

Fig. 2. Elementary cells for S_n (diagrams $2a, \ldots, 2d$) and H_n (diagrams $2a', \ldots, 2d'$), and their contractions to E_n ones (diagrams $2a'', \ldots, 2d''$). Full circles correspond to closed ends. There are S_α further vertices above the vertex alpha. The broken lines are explaned in the text

To specify the separated wave function

$$\Psi = \Pi_{j=1}^{n} \Phi_j(\theta_j) \tag{2.6}$$

on S_n, we follow Refs. [15] and introduce four types of vertices, or «cells» on a tree, as illustrated in Fig 2. The first row, diagrams 2(a)-(d) contains elementary S_n cells (the second and third rows 2(a'-d') and 2(a"- d") will be discussed below). A full circle on diagrams 2(a)-(d) denote a «closed» end, i.e., one that leads to further branches, an open end leads directly to a coordinate in ambient space. The numbers m, ℓ, ℓ_β, ℓ_α are all integers and are related to the separation constant corresponding to each vertex,

S_α = number of vertices above vertex ℓ_α, S_β = number of vertices above vertex ℓ_β.

Each vertex and each angle θ_j provides a «building block» $\Psi_j(\theta_j)$ for the wave function $\Psi(\theta_1, ..., \theta_n)$. Specifically, we have

Cell of type 2a:

$$\Psi_m(\theta_a) = \frac{1}{\sqrt{2\pi}} e^{im\theta_a}; \quad m = 0, \pm 1, \pm 2, \ldots; \quad 0 \leqslant \theta_a < 2\pi. \tag{2.7}$$

Cell of type 2b:

$$\Psi^c_{n,l_\beta}(\theta_b) = N^{c,c}_n (\sin\theta_b)^{l_\beta} P^{(c,c)}_n (\cos\theta_b) \tag{2.8}$$

$$n = l - l_\beta, \; c = l_\beta + \frac{S_\beta}{2}, \; n = 0, 1, 2, \ldots; \quad 0 \leqslant \theta_b \leqslant \pi,$$

where $P^{(a,b)}_n(x)$ are the Jacobi polynomials [1].

Cell of type 2c:

$$\Psi^a_{n,l_\alpha}(\theta_{b'}) = N^{a,a}_n (\cos\theta_{b'})^{l_\alpha} P^{(a,a)}_n (\sin\theta_{b'}) \tag{2.9}$$

$$n = l - l_\alpha, \; a = l_\alpha + \frac{S_\alpha}{2}, \; n = 0, 1, 2, \ldots; \quad -\pi/2 \leqslant \theta_{b'} \leqslant \pi/2.$$

Cell of type 2d:

$$\Psi^{b,a}_{n,l_\beta,l_\alpha}(\theta_c) = 2^{(b+a)/2+1} N^{b,a}_n (\sin\theta_c)^{l_\beta} (\cos\theta_c)^{l_\alpha} P^{(b,a)}_n (\cos 2\theta_c), \tag{2.10}$$

$$n = \frac{l - l_\alpha - l_\beta}{2}, \quad b = l_\beta + \frac{S_\beta}{2}, \quad a = l_\alpha + \frac{S_\alpha}{2},$$
$$n = 0, 1, 2, \ldots; \quad 0 \leqslant \theta_c \leqslant \pi/2.$$

The normalization constants are

$$N^{a,b}_n = \left\{ \frac{(2n + a + b + 1)\Gamma(n + a + b + 1) n!}{2^{a+b+1}\Gamma(n + a + 1)\Gamma(n + b + 1)} \right\}^{1/2}.$$

3. Subgroup Coordinates on E_n and Cluster Diagrams

Let us now consider the Euclidean Lie algebra $e(n)$ with a basis

$$L_{ik} = x_i \partial_k - x_k \partial_i, \quad p_i = \partial_{x_i}, \quad i, k = 1, 2, ..., n.$$

The commutation relations are as in (2.3), together with

$$[p_j, L_{ik}] = \delta_{ji} p_k - \delta_{jk} p_i, \quad [p_i, p_k] = 0.$$

The Casimir operator of $e(n)$ is

$$\Delta_n = p_1^2 + p_2^2 + \ldots.. + p_n^2. \tag{3.1}$$

As in the case of the $O(n)$ group it is useful to introduce diagrams for subgroup type coordinate systems on the Euclidean space E_n. They are called «cluster diagrams» [6]. They consist of one or more trees of the $O(k)$ type with a tree «trunk» added, and possibly of individual isolated trunks.

An isolated trunk corresponds to a Cartesian coordinate. A trunk with further branches above it corresponds to a radial coordinate r satisfying $0 \leqslant r < \infty$. The tree above the trunk is treated exactly as in the case of hyperspherical coordinates on S_n spheres.

As an example let us consider the diagram on Fig.3. The coordinates in E_7 are:

$$\begin{aligned} x_1 &= z, & x_4 &= r_2 \cos\theta_2, \\ x_2 &= r_1 \cos\theta_1, & x_5 &= r_2 \sin\theta_2 \cos\theta_3, \\ x_3 &= r_1 \sin\theta_1, & x_6 &= r_2 \sin\theta_2 \sin\theta_3 \cos\theta_4, \\ & & x_7 &= r_2 \sin\theta_2 \sin\theta_3 \sin\theta_4. \end{aligned}$$

The prescriptions for writing the complete sets of commuting operators, eigenvalues and eigenfunctions are now quite simple.

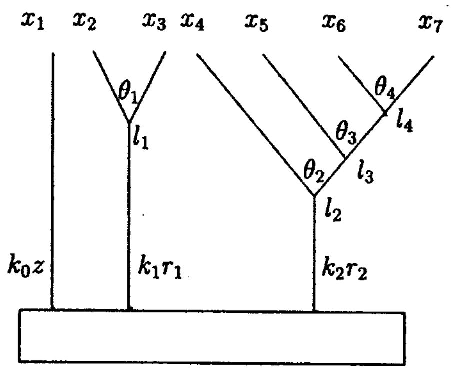

Fig. 3. Example of E_7 cluster diagram

To each tree trunk we associate an M- dimensional Laplace operator where M is the number of end points (Cartesian coordinates) above the trunk. We also associate a number $k \in \mathcal{R} > 0$ with each trunk. The corresponding radial eigenfunction [normalized to the delta function: $\delta(k' - k)$] is

$$\Psi_{kl}(r) = \sqrt{\frac{k}{r^{M-2}}}\, J_{l+\frac{M-2}{2}}(kr), \qquad M \geqslant 2, \tag{3.2}$$

$$\Psi_k(z) = \frac{1}{\sqrt{2\pi}} e^{ikz}, \qquad M = 1. \tag{3.3}$$

The angular part of the eigenfunctions is written following the rules for S_n spheres, as are the invariant operators and their eigenvalues.

4. Pseudospherical Coordinates on H_n

Let us consider the upper sheet of the two-sheeted hyperboloid H_n

$$u_0^2 - \sum_{\nu=1}^{n} u_\nu^2 = R^2, \quad R^2 > 0, \tag{4.1}$$

where u_μ, $\mu = 0, 1, \ldots n$ are Cartesian coordinates in the ambient Minkowski space $M_{n,1}$. The isometry group is SO(n,1), the proper Lorentz group. Its Lie algebra o(n,1) is realized by vector fields with a standard basis $M_{\mu\nu}$, namely

$$M_{ik} = u_i\partial_k - u_k\partial_i, \qquad M_{0k} = u_0\partial_k + u_k\partial_0, \qquad i,k = 1,2,\ldots n \tag{4.2}$$

with the commutation relations

$$[M_{\mu\nu}, M_{\alpha\beta}] = G_{\nu\alpha}M_{\mu\beta} - G_{\nu\beta}M_{\mu\alpha} - G_{\mu\alpha}M_{\nu\beta} + G_{\mu\beta}M_{\nu\alpha}, \quad \alpha,\beta,\mu,\nu = 0,1,2,\ldots n \tag{4.3}$$

where the metric tensor is $G_{\mu\nu} = \mathrm{diag}(1,-1,-1,\ldots,-1)$, $(\mu,\nu = 0,1,2,\ldots n)$.

The Laplace-Beltrami operator and the second order Casimir operator of o(n,1) are related by the formula

$$\Delta(H_n) = Q(n,1), \qquad Q(n,1) = \sum_{i=1}^{n} M_{0i}^2 - \sum_{1\leqslant i<k}^{n} M_{ik}^2. \tag{4.4}$$

In previous sections we have presented the separated solutions of the Laplace-Beltrami equation for all subgroup type coordinates on S_n and E_n. Here we descibe the pseudospherical coordinates and wave functions on hyperboloid H_n. We use the modified, or pseudosherical tree formalism [12]. We will always chose the u_0 coordinate on the left side of the tree. We introduce two types of nodes: trigonometric and hyperbolic ones. For each trigonometric node we introduce an angle θ and for each hyperbolic one a «hyperbolic angle» $\tau \in (-\infty,\infty)$. At trigonomatric nodes we write $\cos\theta$ when going to the left and $\sin\theta$ when going right. Similarly for hyperbolic nodes we write $\operatorname{ch}\mu$ and $\operatorname{sh}\mu$. For example the tree on Fig 4. corresponds to the pseudospherical system of coordinates

$$\begin{aligned} u_0 &= R\operatorname{ch}\tau_1\operatorname{ch}\tau_2\operatorname{ch}\tau_3, & u_1 &= R\operatorname{ch}\tau_1\operatorname{ch}\tau_1\operatorname{sh}\tau_3,\\ u_2 &= R\operatorname{ch}\tau_1\operatorname{sh}\tau_2\cos\theta_1, & u_3 &= R\operatorname{ch}\tau_1\operatorname{sh}\tau_2\sin\theta_1\\ u_4 &= R\operatorname{sh}\tau_1\cos\theta_2 & u_5 &= R\operatorname{sh}\tau_1\sin\theta_2\cos\theta_3\\ u_6 &= R\operatorname{sh}\tau_1\sin\theta_2\sin\theta_3. \end{aligned}$$

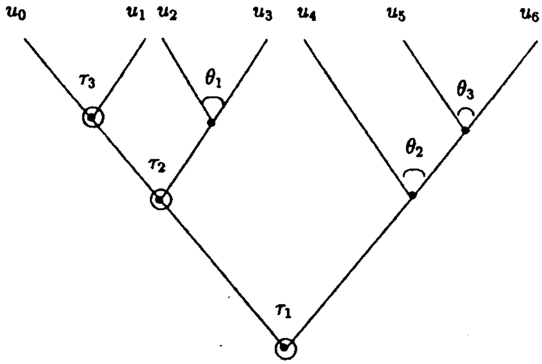

Fig. 4. Example of pseudospherical coordinate diagram for the H_7 hyperboloid

On a «hyperbolic tree» we meet four types of trigonometric vertexes or «cells» and four types of hyperbolic ones [see Fig 2(a,...d) and 2(a'...d')]. The following numbers are associated with each cell: $(\ell, \ell_\beta, \ell_\alpha)$ are related to the separation constant corresponding to each trigonometric node, ν, ν_α to hyperbolic ones and S_α, S_β - numbers of vertexes above vertex $\ell_\alpha(\nu_\alpha)$ or ℓ_β. The numbers m, ℓ, ℓ_β and ℓ_α are integers while ν, ν_α are complex and correspond to the principal series of unitary representations of $O(n,1)$ with

$$\nu_\alpha = -\frac{n'-1}{2} + ip.$$

Here p is real, n' is the number of end points u_i, $i = 0, 1, ..n'-1$ connected with vertex τ_α. We will request that the function Ψ be normalized to the delta function $\delta(p'-p)$ with respect to the invariant measure on H_n.

Let us construct the wave functions corresponding to cells of type 2(a',b',c',d').

Cell of type 2a':

This cell corresponds the wave function $(\tau \equiv \tau_{a'},\ \nu_{a'} = ip)$

$$\Psi_p(\tau) = \frac{e^{ip\tau}}{\sqrt{2\pi}} \tag{4.5}$$

Cell of type 2b':

For this case we have $(\nu \equiv \nu_{b'} = -\frac{S_\beta+1}{2} + ip,\ \tau \equiv \tau_{b'})$

$$\Psi_p^{\ell_\beta}(\tau) = \frac{\left|\Gamma(\ell_\beta + \frac{S_\beta+1}{2} + ip)\right|}{\sqrt{2}\,|\Gamma(ip)|} (\operatorname{sh}\tau)^{-\frac{S_\beta}{2}}\, \mathcal{P}^{-\ell_\beta-\frac{S_\beta}{2}}_{-\frac{1}{2}+ip}(\operatorname{ch}\tau) \tag{4.6}$$

where the $\mathcal{P}_\nu^\mu(z)$ is an associated Legendre function. It can be written in terms of the hypergeometric function

$$\mathcal{P}_\nu^\mu(z) = \frac{2^\mu}{\Gamma(1-\mu)}(z^2-1)^{-\frac{\mu}{2}}\,{}_2F_1\left(1-\mu+\nu, -\mu-\nu; 1-\mu; \frac{1-z}{2}\right) \tag{4.7}$$

Cell of type 2c':

Taking into account that $\nu \equiv \nu_{c'} = -\dfrac{S_\alpha+1}{2} + ip$, $\nu_\alpha = -\dfrac{S_\alpha}{2} + ip$ $(\tau \equiv \tau_{c'})$ we have

$$\Psi_{pp_\alpha}(\tau) = \frac{1}{\sqrt{2\pi}}\,|\Gamma(1-ip)|\,(\mathrm{ch}\,\tau)^{-\frac{S_\alpha+1}{2}}\,\mathcal{P}^{ip}_{-ip_\alpha-1/2}(\mathrm{th}\,\tau) \tag{4.8}$$

Cell of type 2d':

The corresponding wave function $(\tau \equiv \tau_d,\ \nu_d \equiv \nu)$ is

$$\Psi^{\ell_\beta}_{pp_\alpha}(\tau) = \frac{\left|\Gamma(\frac{\ell_\beta+ip+ip_\alpha+1}{2}+\frac{S_\beta}{4})\Gamma(\frac{\ell_\beta+ip-ip_\alpha+1}{2}+\frac{S_\beta}{4})\right|}{2\sqrt{\pi}\,\Gamma(\ell_\beta+\frac{S_\beta}{2}+1)\,|\Gamma(ip)|}$$

$$\times\ (\mathrm{ch}\,\tau)^{\nu_\alpha}(\mathrm{sh}\,\tau)^{\ell_\beta}\,\mathcal{P}^{(\ell_\beta+\frac{S_\beta}{2},\ \nu_\alpha+\frac{S_\alpha}{2})}_{\frac{\nu-\nu_\alpha-\ell_\beta}{2}}(\mathrm{ch}\,2\tau), \tag{4.9}$$

where $\nu = -\dfrac{S_\alpha+S_\beta+2}{2} + ip$, $\nu_\alpha = -\dfrac{S_\alpha}{2} + ip_\alpha$ and $\mathcal{P}^{(\alpha,\beta)}_\mu(z)$ is the Jacobi function defined by the formula

$$\mathcal{P}^{(\alpha,\beta)}_\mu(z) = \left(\frac{z+1}{2}\right)^\mu {}_2F_1\left(-\mu, -\mu-\beta; \alpha+1; \frac{z-1}{z+1}\right). \tag{4.10}$$

5. Contractions of the Lie Algebras

5.1. The $o(n+1) \to e(n)$ contraction. We shall use $\varepsilon = R^{-1}$ as the contraction parameter. To realize the contraction explicitly, let us introduce Beltrami coordinates on the sphere [6], putting

$$x_\mu = R\frac{u_\mu}{u_0} = u_\mu\left(1 - \frac{1}{R^2}\sum_{\nu=1}^n u_\nu^2\right)^{-1/2}, \quad \mu = 1,2,3,\dots,n. \tag{5.1}$$

The $O(n+1)$ generators then can be expressed as:

$$-\frac{L_{0\nu}}{R} \equiv \pi_\nu = p_\nu + \frac{x_\nu}{R^2}\sum_{\nu=1}^{n}(x_\mu p_\mu), \tag{5.2}$$

$$-L_{\mu\nu} \equiv (x_\mu p_\nu - x_\nu p_\mu) = x_\mu \pi_\nu - x_\nu \pi_\mu, \quad \mu,\nu = 1,2,...,n, \tag{5.3}$$

where $p_\nu = \partial/\partial x_\nu$. The commutation relations in the new variables are:

$$[L_{\mu\nu}, L_{\lambda\sigma}] = \delta_{\mu\lambda}L_{\nu\sigma} + \delta_{\nu\sigma}L_{\mu\lambda} - \delta_{\mu\sigma}L_{\nu\lambda} - \delta_{\nu\lambda}L_{\mu\sigma}, \tag{5.4}$$

$$[L_{\mu\nu}, \pi_\lambda] = \delta_{\mu\lambda}\pi_\nu - \delta_{\nu\lambda}\pi_\mu, \quad [\pi_\mu, \pi_\nu] = \frac{L_{\mu\nu}}{R^2}, \tag{5.5}$$

so that for $R \to \infty$ the $o(n+1)$ algebra contracts to the Euclidean $e(n)$ one. The momenta π_ν contract to the Euclidean $e(n)$ ones: $\pi_\nu \to p_\nu$. The $o(n+1)$ Laplace-Beltrami operator (2.4) contracts to the $e(n)$ one:

$$\Delta(S_n) = \sum_{\nu=1}^{n}\pi_\nu^2 + \sum_{\mu,\nu=1}^{n}\frac{L_{\mu\nu}^2}{2R^2} \to \Delta(E_n) = p_1^2 + p_2^2 + ... + p_n^2. \tag{5.6}$$

5.2. The $o(n,1) \to e(n)$ contraction. To realize the contraction from o(n,1) to e(n), as in the case of the sphere S_n we introduce Beltrami coordinates on the hyperboloid H_n putting

$$x_\mu = R\frac{u_\mu}{u_0} = u_\mu\left(1 + \frac{1}{R^2}\sum_{\nu=1}^{n}u_\nu^2\right)^{-1/2}, \quad \mu = 1,2,3,...,n. \tag{5.7}$$

The O(n,1) generators (4.2) can be written as:

$$-\frac{M_{0\mu}}{R} \equiv \pi_\mu = p_\mu - \frac{1}{R^2}x_\mu\sum_{\nu=1}^{n}(x_\mu p_\mu), \tag{5.8}$$

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu = x_\mu\pi_\nu - x_\nu\pi_\mu. \quad \mu = 1,2,3,...,n. \tag{5.9}$$

The commutators of the o(n,1) algebra take the form:

$$[\pi_\mu, \pi_\nu] = \frac{M_{\mu\nu}}{R^2}, \quad [\pi_\delta, M_{\mu\nu}] = \delta_{\delta\mu}\pi_\nu - \delta_{\delta\nu}\pi_\mu, \tag{5.10}$$

In the limit $R \to \infty$ the o(n,1) algebra contracts to e(n) and the o(n,1) Laplace-Beltrami operator (4.4) contracts to the e(n) one:

$$\Delta_{LB} = \sum_{\nu=1}^{n}\pi_\nu^2 - \sum_{\mu,\nu=1}^{n}\frac{M_{\mu\nu}}{R^2} \to \Delta = (p_1^2 + p_2^2 + ... + p_n^2). \tag{5.11}$$

6. Contractions of Coordinate Systems and Basis Functions. The Graphical Method

We shall now describe a graphical method for connecting the subgroup type coordinates on S_n, or H_n with those on E_n and present the rules relating the coordinates, eigenvalues and basis functions. The relations are asymptotic ones for the radius of the sphere or hypersphere $R \to \infty$ and one or more of the coordinates θ_i or τ_i satisfying $\theta_i \to 0$ and $\tau_i \to 0$.

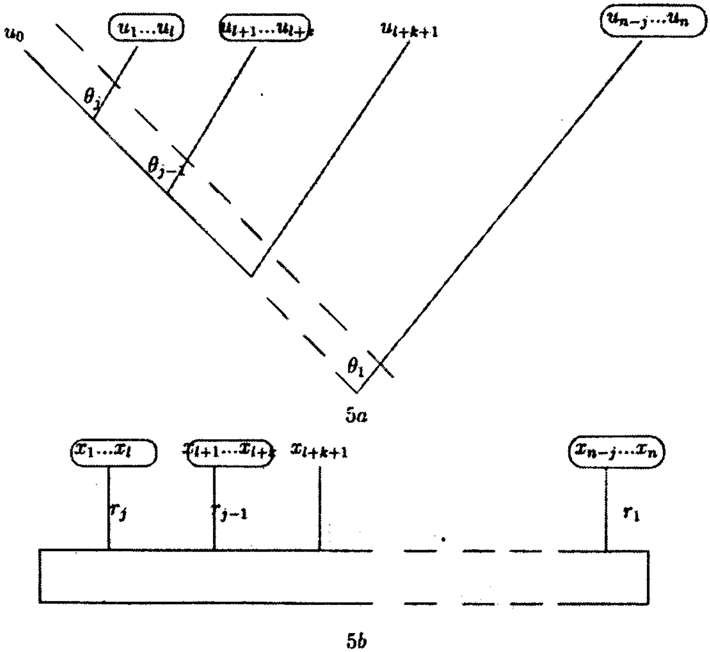

Fig. 5. Contractions of the tree diagrams into cluster ones for $S_n(H_n) \to E_n$ hyperboloid

A general S_n (or H_n) tree diagram can be represented by Fig 5(a) (for H_n the trigonometric angles θ_i must be changed to hyperbolic ones τ_i). Graphically the contraction $R \to \infty$ corresponds to the fact that we cut off the ground to u_0 branch by the dashed line in Fig.5(a). The dashed line then becomes the ground for the corresponding E_n cluster diagram of Fig.5(b). The ambient space coordinates $(u_0, u_1, \dots u_n)$ for S_n or H_n are replaced by the Cartesian coordinates $(x_1, x_2 \dots x_n)$. The angles $(\theta_1, \theta_2, \dots \theta_j)$ or $(\tau_1, \tau_2, \dots \tau_j)$ that lead to branches cut off by the dotted line satisfy $\theta_j \to 0$ or $\tau_j \to 0$ in the contraction and are replaced by radial coordinates r_j, or Cartesian coordinates x_m (if the surviving branch leads directly to a single coordinate on S_n (H_n) and E_n). We have

$$R \to \infty, \qquad \theta_i \to 0, \qquad R\,\mathrm{tg}\,\theta_i \sim R\sin\theta_i \sim R\theta_i \to r_i \tag{6.1}$$

and

$$R \to \infty, \qquad \tau_i \to 0, \qquad R\,\mathrm{th}\,\tau_i \sim R\,\mathrm{sh}\,\tau_i \sim R\tau_i \to r_i. \tag{6.2}$$

The individual trees in an E_n cluster correspond to an $O(k)$ subgroup of $O(n)$ that survives the contraction.

When we cut off the branches of a tree as in Fig 5(a), the cutting line intersects an elementary cell (see Fig.2) at each branch. Each elementary $O(n+1)$ or $O(n,1)$ cell then goes into an elementary trunk for $E(n)$, as indicated by the lower row of diagrams in Fig.2.

Let us now discuss the four cases in Fig. 2. The limiting procedure is always the same, namely for S_n

$$\theta_j \sim \frac{r_j}{R}, \quad l_j \sim kR, \quad R \to \infty, \quad j = a, b, c, d \tag{6.3}$$

and for H_n

$$\tau_j \sim \frac{r_j}{R}, \quad \nu_j \sim kR, \quad R \to \infty, \quad j = a', b', c', d' \tag{6.4}$$

where r_j is the radius of the sphere (pseudosphere) that survives the contraction. So, for $j = a, c$ we have $r_j = x$, a Cartesian coordinate.

Let us now run through the individual cells in Fig.2.

6.1 Contractions of Functions Corresponding to Elementary Cells for S_n

1. Cell 2a to 2a"

Using the eqs. (2.7) and (6.1) we have ($R \to \infty$, $m \sim kR$, $\theta_a \sim x/R$)

$$\lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} e^{im\theta_a} = \frac{1}{\sqrt{2\pi}} e^{ikx} \tag{6.5}$$

2. Cell 2b to 2b"

For the second cell using the eq. (2.8). We have ($\ell \sim kR$, $\theta_b \sim r/R$)

$$\lim_{R\to\infty} \frac{1}{\sqrt{R^{S_\beta+1}}} N_{l-\beta}^{\beta+\frac{S_\beta}{2}} (\sin\theta)^\beta P_{l-\beta}^{(\beta+\frac{S_\beta}{2},\beta+\frac{S_\beta}{2})}(\cos\theta) = \\ = \sqrt{\frac{k}{r^{S_\beta}}}\, J_{\beta+\frac{S_\beta}{2}}(kr). \tag{6.6}$$

3. Cell 2c to 2c"

The contribution of this cell to the $O(n+1)$ separated basis function is given in eq. (2.9). The limit is ($\ell \sim kR$, $\ell_a \sim pR$, $\theta_c \sim x_n/R$)

$$\lim_{R\to\infty} N_{l-\alpha}^{\alpha+\frac{S_\alpha}{2}} (\cos\theta)^\alpha P_{l-\alpha}^{(\alpha+\frac{S_\alpha}{2},\alpha+\frac{S_\alpha}{2})}(\sin\theta) = \sqrt{\frac{2k}{k_n}} \left\{ \begin{array}{c} \cos k_n x_n \\ -i\sin k_n x_n \end{array} \right\} \tag{6.7}$$

where $k^2 = p^2 + k_n^2$.

4. Cell 2d to 2d"

The corresponding basis function given in eq (2.10). Taking the limit (6.1), $\ell \sim kR$, $\ell_a \sim k_a R$ and $\theta_d \sim r/R$) we get

$$\lim_{R\to\infty} \frac{2^{\frac{1}{2}(\alpha+\frac{S_\alpha}{2}+\beta+\frac{S_\beta}{2})+1}}{\sqrt{R^{S_\beta+1}}} N^{\beta+\frac{S_\beta}{2},\alpha+\frac{S_\alpha}{2}}_{\frac{l-\alpha-\beta}{2}} (\sin\theta)^\beta (\cos\theta)^\alpha \times$$
$$\times P^{(\beta+\frac{S_\beta}{2},\alpha+\frac{S_\alpha}{2})}_{\frac{l-\alpha-\beta}{2}}(\cos 2\theta) = \sqrt{\frac{2k}{(k_\beta r)^{S_\beta}}}\, J_{\beta+\frac{S_\beta}{2}}(k_\beta r), \quad (6.8)$$

where $k^2 = k_\alpha^2 + k_\beta^2$.

These contractions for basis functions of the elementary cells 2(a,...d) determine the general contractions for hypergeometrical functions corresponding to any tree for the sphere S_n.

6.2. Contractions of Functions Corresponding to Elementary Cells for H_n

1. Cell 2a' to 2a"

Using the eq. (4.5) and (6.4) we have ($R \to \infty$, $\nu_a = ip_a \sim ikR$, $\tau_a \sim x/R$)

$$\lim_{R\to\infty} \frac{e^{ip_a\tau_a}}{\sqrt{2\pi}} = \frac{e^{ikx}}{\sqrt{2\pi}}. \quad (6.9)$$

2. Cell 2b' to 2b"

The contribution to the separated $O(n,1)$ basis function is given in eq. (4.6). In the contraction limit $R \to \infty$ we put: $p \sim kR$, $\tau \sim r/R$. Using the asymptotic formula [1]

$$\lim_{|y|\to\infty} |\Gamma(x+iy)| \exp(\frac{\pi}{2}|y|)\, |y|^{\frac{1}{2}-x} = \sqrt{2\pi} \quad (6.10)$$

and rewriting the Legendre function in terms of the hypergeometric function as in eq. (4.7), we obtain

$$\lim_{R\to\infty} {}_2F_1\left(\ell_\beta + \frac{S_\beta+1}{2} + ip, \ell_\beta + \frac{S_\beta+1}{2} - ip; \ell_\beta + \frac{S_\beta}{2} + 1; -\mathrm{sh}^2\frac{\tau}{2}\right)$$
$$= \Gamma(1+\ell_\beta+\frac{S_\beta}{2}) \left(\frac{2}{kr}\right)^{\ell_\beta+\frac{S_\beta}{2}} J_{\ell_\beta+\frac{S_\beta}{2}}(kr).$$

So, finally we have

$$\lim_{R\to\infty} \frac{\left|\Gamma(\ell_\beta + \frac{S_\beta+1}{2} + ip)\right|}{\sqrt{2R^{S_\beta+1}}\, |\Gamma(ip)|} (\mathrm{sh}\,\tau)^{-\frac{S_\beta}{2}} P^{-\ell_\beta-\frac{S_\beta}{2}}_{-\frac{1}{2}+ip}(\mathrm{ch}\,\tau) = \sqrt{\frac{k}{r^{S_\beta}}}\, J_{\ell_\beta+\frac{S_\beta}{2}}(kr).$$

3. Cell 2c' to 2c"

The relevant basis function is given in eq. (4.8). To perform the contraction we write the Legendre function in terms of the hypergeometric function

$$P^{ip}_{-ip\alpha-1/2}(\operatorname{th}\tau) = \frac{\sqrt{\pi}\,2^{ip}(\operatorname{ch}\tau)^{-ip}}{\Gamma\left(\frac{3}{4}-a\right)\Gamma\left(\frac{3}{4}-b\right)}\left\{{}_2F_1\left(\frac{1}{4}+a,\frac{1}{4}+b;\frac{1}{2};\operatorname{th}^2\tau\right)+\right.$$
$$\left. 2\operatorname{th}\tau\frac{\Gamma\left(\frac{3}{4}-a\right)\Gamma\left(\frac{3}{4}-b\right)}{\Gamma\left(\frac{1}{4}-a\right)\Gamma\left(\frac{1}{4}-b\right)}{}_2F_1\left(\frac{3}{4}-a,\frac{3}{4}-b;\frac{3}{2};\operatorname{th}^2\tau\right)\right\},$$

where $a = i(p-p_\alpha)/2$, $b = i(p+p_\alpha)/2$. In the contraction limit $R \to \infty$ we put: $p \sim kR$, $p_\alpha \sim k_\alpha R$ and $\tau \sim \frac{x_n}{R}$, where x_n is a Cartesian coordinate. We use the asymptotic formulae:

$$\lim_{R\to\infty} {}_2F_1\left(\frac{1}{4}+a,\frac{1}{4}+b;\frac{1}{2};\operatorname{th}^2\tau_1\right) = \cos k_n x_n,$$

$$\lim_{R\to\infty} {}_2F_1\left(\frac{3}{4}-a,\frac{3}{4}-b;\frac{3}{2};\operatorname{th}^2\tau_1\right) = \frac{1}{k_n x_n}\sin k_n x_n,$$

where $k_\alpha^2 + k_n^2 = k^2$. After using asymptotic formula

$$\lim_{z\to\infty}\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta},$$

we finally obtain

$$\lim_{R\to\infty}\frac{1}{\sqrt{2\pi}}|\Gamma(1-ip)|\,(\operatorname{ch}\tau)^{-\frac{S_\alpha+1}{2}}\;P^{ip}_{-ip-1/2}(\operatorname{th}\tau) = \sqrt{\frac{k}{2\pi k_n}}\;e^{ik_n x_n}$$

4. Cell 2d' to 2d"

The corresponding basis function is given in eq (4.9) ($\nu = -\frac{S_\alpha+S_\beta+2}{2} + {} + ip$, $\nu_\alpha = -\frac{S_\alpha}{2} + ip_\alpha$). To take the limit (6.1), we put $p_\alpha \sim k_\alpha R$, $p \sim kR$ and $\theta \sim r/R$). We use the equation expressing Jacobi functions in terms of hypergeometric functions

$$\lim_{R\to\infty} = \frac{1}{\Gamma(\ell_\beta+\frac{S_\beta}{2}+1)}\,\mathcal{P}^{(\ell_\beta+\frac{S_\beta}{2},\,\nu_\alpha+\frac{S_\alpha}{2})}_{\frac{\nu-\nu_\alpha-\ell_\beta}{2}}(\operatorname{ch}2\tau),$$

$$= \frac{(\operatorname{ch}\tau)^{\nu-\nu_\alpha-\ell_\beta}}{\Gamma(\ell_\beta+\frac{S_\beta}{2}+1)}\,{}_2F_1\left(-\frac{\nu-\nu_\alpha-\ell_\beta}{2},-\frac{\nu+\nu_\alpha-\ell_\beta}{2}+\frac{S_\alpha}{2};\right.$$
$$\left.\ell_\beta+\frac{S_\beta}{2}+1;\operatorname{th}^2\tau\right)$$

where $k_\alpha^2 + k_\beta^2 = k^2$. The final result is

$$\lim_{R\to\infty} \frac{\left|\Gamma(\frac{\ell_\beta+ip+ip_\alpha+1}{2} + \frac{S_\beta}{4})\Gamma(\frac{\ell_\beta+ip-ip_\alpha+1}{2} + \frac{S_\beta}{4})\right|}{2\sqrt{\pi R^{S_\beta+1}}\,\Gamma(\ell_\beta + \frac{S_\beta}{2} + 1)\,|\Gamma(ip)|} (\mathrm{ch}\,\tau)^{\nu_\alpha} (\mathrm{sh}\,\tau)^{\ell_\beta} \times$$

$$\mathcal{P}_{\frac{\nu-\nu_\alpha-\ell_\beta}{2}}^{(\ell_\beta+\frac{S_\beta}{2},\ \nu_\alpha+\frac{S_\alpha}{2})}(\mathrm{ch}\,2\tau) = \sqrt{\frac{2k}{r^{S_\beta}}}\ J_{\ell_\beta+\frac{S_\beta}{2}}(k_\beta r),$$

7. Conclusions

The main conclusion from this article, or rather from the research program that it summarizes, is that the use of analytical contractions makes is possible to apply Lie group theory to a new area of special function theory: asymptotic relations between special functions occuring in the representation theory of different Lie groups. This should be specially fruitful for for less well studied functions than those that occur when subgroup type coordinates are used on spheres and hyperboloid. So far this has been studied only for O(3) and O(2,1) [3] and this has provided asymptotic relations between Lamé and Mathieu functions. Work in this direction is in progress.

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