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VERY HIGH MULTIPLICITY PHYSICS

ON THERMALIZATION OF INELASTIC PROCESSES

© 2004 J. Manjavidze[1)], A. N. Sissakian[2)]

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The paper contains the discussion of the criteria of applicability of statistical methods in the multiple production processes. The main attention is devoted to the thermalization phenomena, while the energy is uniformly distributed over the dynamical degrees of freedom and the energy correlators are relaxed. It is argued that this condition must be satisfied in the deep asymptotics over multiplicity and the "very high multiplicity" (VHM) domain is defined as the region where this thermalization condition is satisfied but for moderate multiplicities. The model independent classification of the multiplicity asymptotics and their physical content is offered. It is shown explicitly that existing multiple production models are not able to predict the range of the VHM domain.

1. INTRODUCTION

It is accepted now that the main road of the particle physics development is the Standard Model. However, it is obvious to expect the existence of other ways, less important at first sight but permitting to observe new interesting phenomena. The multiple production phenomenon can be one of them.

But the multiple production phenomena seem to be "uninteresting" because of a very large number of involved degrees of freedom. This is definitely so and it must be mentioned also that to all appearances a gap between the strict theory based on the non-Abelian gauge symmetry and the obvious hadron multiple production phenomenology never would be surmounted for this reason. We will return to this question in Section 3.

Then the attempts to find the kinematical condition(s), where the multiple production process becomes "describable," seem crucial. The most popular condition is based on the asymptotic freedom. It assumes hardness of the interaction. This allows to investigate only the "local" properties of the hadron.

We are discussing another possibility. In this connection let us remember that the statistical physics deals very well with the enormous number of degrees of freedom (particles). It is natural to engage this rich experience to describe the multiple production phenomenon.

The main attention will be concentrated on the *equilibrium*, since, presumably, only it can be described *completely*. The appearance of such a state in the hadron inelastic collisions will be considered as a phenomenon which can be examined experimentally and may be predicted theoretically.

The multiple production may be considered as the process of colliding particles kinetic energy dissipation into the mass of produced particles. To use this interpretation, one must consider the final-state particles as the "probes", through which the measurement of interacting fields state is performed [1]. Then, one can consider the multiplicity as a measure of entropy S. One *may* expect, therefore, that in the very high multiplicity (VHM) domain the entropy exceeds its maximum. For this reason we will define here the VHM final state through the equilibrium condition.

Using the thermodynamical terminology, we investigate in this case the production and properties of the comparatively "cold" final state of *interacting* fields. One may expect that in this condition the system becomes "calm." This is one more argument why we expect the equilibrium in the VHM domain.

In the conclusion, we consider the VHM processes as the only ones, the complete theory of which can be constructed. Discussing the thermalization phenomenon, we actually try for the condition, in the frame of which this theory would work.

The phenomenology and an idea of a rough (statistical) description of the VHM processes were formulated in our first publications [2]. Later we accumulated our main ideas on the VHM theory in the review paper [3]. The definite connection with the idea of N.N. Bogolyubov concerning transition to the equilibrium was described in [4].

The preferable processes at $n \sim \bar{n}$ are saturated by excitation of the nonperturbative degrees of freedom. These soft processes are described by the creation of quarks and gluons from the vacuum: the kinetic

[1)]Institute of Physics, Georgia Academy of Sciences, Tbilisi, and Joint Institute for Nuclear Research, Dubna, Russia; E-mail: joseph@nusun.jinr.ru

[2)]Joint Institute for Nuclear Research, Dubna, Russia; E-mail: sisakian@jinr.ru

motion of partons leads to increasing, because of confinement phenomenon, polarization of the vacuum and in result to its instability concerning quarks creation [5]. In other words, there is a long-range correlation among hadrons constituents at $n \sim \bar{n}$.

The most popular field-theoretical description of statistical systems at a finite temperature is based on the formal analogy between imaginary time and inverse temperature β ($\beta = 1/T$) [6]. This approach is fruitful, if we did not want to clear up the dynamical aspects [7, 8]. The further attempts led to the real-time finite-temperature field theory [9–13].

2. CLASSIFICATION OF ASYMPTOTICS OVER MULTIPLICITY

We will use the following quantitative definition of discussed high energy VHM hadron reactions. Let $\varepsilon_{\max}$ be the energy of the fastest particle in the given frame and let E be the total incident energy in the same frame. Then the difference $(E - \varepsilon_{\max})$ is the energy spent on the production of less energetic particles. It is useful to consider the inelasticity coefficient

$$\kappa = 1 - \frac{\varepsilon_{\max}}{E} \leq 1. \tag{2.1}$$

It defines the *portion* of spent energy. Therefore, we wish to consider processes with

$$1 - \kappa \ll 1, \tag{2.2}$$

and the produced particles would have the comparatively small energies. Using the energy conservation law, the produced hadrons multiplicity n is defined by inequality:

$$n(1 - \kappa) > 1. \tag{2.3}$$

So, (2.2) means roughly the VHM region.

Following the natural at finite CM energies, $\sqrt{s}$, condition:

$$n \ll n_{\max} = \sqrt{s}/m_h, \quad m_h \simeq 0.2\,\text{GeV}, \tag{2.4}$$

we will assume that

$$1 - \kappa \gg m_h/E. \tag{2.5}$$

Therefore, the kinetic energy of produced particles in our processes would not be arbitrarily small.

It seems useful from the very beginning to elaborate a general point of view on the processes in the VHM domain. This would allow without going into details to estimate the possibility of observation of new phenomena.

2.1. The "Thermodynamical" Limit

We will introduce the generating function:

$$T(s, z) = \sum_{n=1}^{n_{\max}} z^n \sigma_n(s), \tag{2.6}$$

$$s = (p_1 + p_2)^2 \gg m^2, \quad n_{\max} = \sqrt{s}/m.$$

This step is natural, since the number of particles is not conserved in our problem. Thus, the total cross section and the averaged multiplicity will be:

$$\sigma_{\text{tot}}(s) = T(s, 1) = \sum_n \sigma_n(s), \tag{2.7}$$

$$\bar{n}(s) = \sum_n n(\sigma_n(s)/\sigma_{\text{tot}}(s)) = \frac{d}{dz} \ln T(s, z)\bigg|_{z=1}.$$

At the same time, the inverse Mellin transform gives

$$\sigma_n = \frac{1}{n!} \frac{\partial^n}{\partial z^n} T(s, z)\bigg|_{z=0} = \tag{2.8}$$

$$= \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} T(s, z) =$$

$$= \frac{1}{2\pi i} \oint \frac{dz}{z} \exp(-n \ln z + \ln T(s, z)).$$

The essential values of z in this integral are defined by the equation (of state):

$$n = z \frac{\partial}{\partial z} \ln T(z, s). \tag{2.9}$$

Taking into account the definition of the mean multiplicity $\bar{n}(s)$, given in (2.7), we can conclude that the solution of (2.9) z_c is equal to one at $n = \bar{n}(s)$. Therefore, $z > 1$ is essential in the VHM domain.

The asymptotics over n ($n \ll n_{\max}$ is assumed) are governed with exponential accuracy by the smallest solution z_c of (2.9) because of the asymptotic estimation of the integral (2.8):

$$\sigma_n(s) \propto e^{-n \ln z_c(n,s)}. \tag{2.10}$$

Let us assume that in the VHM region and at high energies, $\sqrt{s} \to \infty$, there exists such a value of $z_c(n, s)$ that we can neglect in (2.6) the dependence on the upper boundary $n_{\max}$. This formal trick with the thermodynamical limit allows to consider $T(z, s)$ as the nontrivial function of z for the finite s.

Then, it follows from (2.9) that

$$z_c(n, s) \to z_s \text{ at } n \in \text{VHM}, \tag{2.11}$$

where $z_s(s)$ is the leftmost singularity of $T(z, s)$ in the right half plane of complex z. One can say that the singularity of $T(z, s)$ attracts $z_c(n, s)$ if $n \in$ VHM. We will put this observation in the basis of VHM processes phenomenology.

It must be underlined once more that actually $T(z,s)$ is regular for the arbitrary finite z if s is finite. But $z_c(n,s)$ behaves in the VHM domain as if it is attracted by the (imaginary) singularity z_s. And just this $z_c(n,s)$ defines σ_n in the VHM domain. We want to note that actually the energy $\sqrt{s}$ should be high enough to use such an estimation.

2.2. Classes and Their Physical Content

One can notice from the estimation (2.10) that σ_n weakly depends on the character of the singularity. Therefore it is enough to classify only the possible positions of z_s. We may distinguish the following possibilities:

$$\text{(A)} \quad z_s = 1: \quad \sigma_n > O(e^{-n}), \tag{2.12}$$
$$\text{(B)} \quad z_s = \infty: \quad \sigma_n < O(e^{-n}),$$
$$\text{(C)} \quad 1 < z_s < \infty: \quad \sigma_n = O(e^{-n}),$$

i.e., following this classification, the cross section may decrease slower (A), faster (B), or as (C) an arbitrary power of e^{-n}. It is evident, if all these possibilities may be realized in nature, then we should expect the asymptotics (A).

The cross section σ_n has a meaning of the n particle partition function in the energy representation. Then $T(z,s)$ should be the "big partition function." Taking this interpretation into account, as follows from Lee–Yang theorem, $T(z,s)$ cannot be singular at $|z| < 1$.

At the same time, the direct calculations based on the physically acceptable interaction potentials give the following restriction from above:

$$\text{(D)} \quad \sigma_n < O(1/n). \tag{2.13}$$

This means that σ_n should decrease faster than any power of $1/n$. It should be noted that our classification predicts rough (asymptotic) behavior only.

One may notice (2.10) that

$$-\frac{1}{n}\ln\frac{\sigma_n(s)}{\sigma_{\text{tot}}(s)} = \ln z_c(n,s) + O(1/n). \tag{2.14}$$

Using thermodynamical terminology, the asymptotics of σ_n is governed by the physical value of the activity $z_c(n,s)$. One can introduce also the chemical potential $\mu_c(n,s)$. It defines the work needed for one particle creation, $\ln z_c(n,s) = \beta_c(n,s)\mu_c(n,s)$, where $\bar{\epsilon}(n,s) = 1/\beta_c(n,s)$ is the produced particles mean energy. So, one may introduce the chemical potential if and only if $\beta_c(n,s)$ and $z_c(n,s)$ may be used as the "rough" variables.

Then the above formulated classification has a natural explanation. So, the class (A) may be realized if and only if the system is unstable. In this case $z_c(n,s)$ is the decreasing function of n. (B) means that the system is stable against particles production and the activity $z_c(n,s)$ is the increasing function of n. The asymptotics (C) cannot be realized in equilibrium thermodynamics.

We will show that the asymptotics (B) reflects the multiperipheral processes kinematics: created particles form the jet moving in the CM frame with different velocities along the incoming particles directions, i.e., with restricted transverse momentum. The asymptotics (A) assumes the condensation-like phenomena. The third-type asymptotics (C) is predicted by stationary Markovian processes with the pQCD jets kinematics.

This interpretation of classes (2.12) allows to conclude that we should expect reorganization of production dynamics in the VHM region: the soft channel (B) of particle production should yield to the hard dynamics (C), if the ground state of the investigated system is stable against particle production. Otherwise, we will have asymptotics (A).

Let us consider now in detail the physical content of this classification.

(A) $z_s = 1$. It is known that the singularity $z_s = 1$ reflects the first-order phase transition [14]. To find σ_n for this case, we would adopt Langer's analysis [15]. Introducing the temperature $1/\beta$ instead of total energy $\sqrt{s}$ we can use the isomorphism with Ising model. For this purpose we divide the space volume into the cells and if there is a particle in the cell we will write (-1). In the opposite case it will be $(+1)$. It is the model of "lattice gas" well described by Ising model. We can regulate the number of down-looking spins, i.e., the number of created particles, by the external magnetic field $\mathbf{H}$. Therefore, $z = \exp\{-\beta H\}$ and $\mathbf{H}$ is the chemical potential.

One can find the energy representation using the Fourier transformation:

$$\rho(E,z) = \int_{\Gamma} \frac{d\beta}{2\pi i} e^{E\beta} R(\beta,z), \tag{2.15}$$

where the contour Γ is chosen along the complex axis.

The corresponding partition function in the continuous limit [15] (see also [16]) has the form:

$$R(\beta,z) = \int D\mu \exp\left(-\int dx \times \left\{\frac{1}{2}(\partial\mu)^2 - \epsilon\mu^2 + \alpha\mu^4 - \lambda\mu\right\}\right), \tag{2.16}$$

where $\epsilon \sim (1-\beta_c/\beta)$ and $\lambda \sim H$, with critical temperature $1/\beta_c$.

If $\beta_c > \beta$, there is no phase transition and the potential has one minimum at $\mu = 0$. But if $\beta_c < \beta$, there are two degenerate minima at $\mu_\pm = \pm\sqrt{\epsilon/2\alpha}$ if $\lambda = 0$. Switching on $H < 0$ the left minimum at $\mu_- \sim -\sqrt{\epsilon/2\alpha}$ becomes absolute and the system will tunnel into this minimum (see also [17]). This process describes particles creations as a process of spins overturnings.

The Eq. (2.9) gives at $n \to \infty$

$$\ln z_c \sim n^{-1/3} > 0.$$

As a result,

$$\sigma_n \sim e^{-an^{2/3}} > O(e^{-n}), \qquad a > 0,$$

i.e., decreases slower than e^{-n}. The semiclassical calculation shows that the functional determinant is singular at $\mathbf{H} = 0$. It must be underlined that in the used Ising model description the chemical potential deforms the ground state. Consequently, the semiclassical approximation is applicable since $\ln z_c \ll 1$, i.e., since the processes of spin overturnings are rear at high multiplicity region. It is easy to show in this approximation [15] that the functional determinant is singular at $\mathbf{H} = 0$, i.e., at $z = 1$.

Note that z_c *decreases* to one with n. This unusual phenomenon must be explained. The mechanism of particles creation considered above describes "the fate of false vacuum" [17]. In the process of decay of the unstable state the clusters of the new phase of size X are created. If the cluster has dimension $X > X_c$ its size increases since the volume energy ($\sim X^3$) of the cluster becomes better than the surface tension energy ($\sim X^2$). This condition defines the value of X_c. The "critical" clusters wall will accelerate, i.e., the work needed to add one particle into the cluster decreases with $X > X_c$. This explains the reason why z_c decreases with n. Notice here that, at a given temperature, $\ln z_c$ is proportional to Gibbs free energy per one particle.

The described mechanism of particles creation assumes that we have prepared the *equilibrium* system in the unstable phase at $\mu_+ \sim +\sqrt{\epsilon/2\alpha}$, and going to another state at $\mu_- \sim -\sqrt{\epsilon/2\alpha}$ the system creates the particles. The initial state may be the QGP and final state may be the hadrons system. Therefore, we must describe the way how the quarks system was prepared.

Following the Lee–Yang's picture of the first-order phase transition [14, 16], there is no transition in a finite system (the partition function can not be singular for finite $n_{\max}$). This means that the multiplicity (and the energy) must be high enough to see the described phenomena.

(B) $z_s = \infty$. Let us return to the integral (2.16) to investigate the case $\beta_c > \beta$. In this case the potential has one minimum at $\mu = 0$. The external field $\mathbf{H}$, creates the mean field $\bar\mu = \bar\mu(\mathbf{H})$, and the integral (2.16) should be calculated expanding it near $\mu = \bar\mu$. As a result, in the semiclassical approximation ($\bar\mu$ increase with increasing n),

$$\ln R(\beta, z) \sim (\ln z)^{4/3}.$$

This gives $\ln \bar z \sim n^3$ and $\ln \sigma_n \sim -n^4$, i.e., $\sigma_n < < O(e^{-n})$.

There is also another possibility to interpret the considered case (B). For this case we can put

$$\ln T(z,s)/\sigma_{\rm tot} = \bar n(s)(z-1) + O((z-1)^2) \quad (2.17)$$

at $|z-1| \ll 1$. The experimental distribution of $\ln T(z,s)$ for various energies shows that the contributions of $O((z-1)^2)$ terms increase with energy [1]. It is assumed in the Born approximation that

$$\ln t(z,s) = \bar n(s)(z-1).$$

There are various interpretations of this series, e.g., the multiperipheral model, the Regge pole model, the heavy color strings model, the QCD multiperipheral models, etc. In all these models $\bar n(s) = b_1 + b_2 \ln s$, $b_2 > 0$. The second ingredient of hadrons "Standard Model" is the assumption that the mean value of created particles transfers momentum $\langle k\rangle = \text{const}$, i.e., is the energy (and multiplicity) independent quantity. It can be shown that under these assumptions:

$$\ln T(z,s)/\sigma_{\rm tot} = \sum_n c_n(s)(z-1)^n, \quad (2.18)$$
$$c_1 \equiv \bar n,$$

is *regular* at finite values of z [1] and is able to give predictions confirmed by experiments. Inserting (2.18) into (2.9), we find, taking into account regularity of $T(z,s)$ that $\bar z(n,s)$ is the increasing function of n. Consequently,

$$\sigma_n < O(e^{-n}) \quad (2.19)$$

for hadrons in the "Standard Model."

Notice also that the "Standard Model" has a finite range of validity: beyond $n \sim \bar n^2$ the model must be changed since it is impossible to conserve $\langle k\rangle = \text{const}$ at higher multiplicities [18].

(C) $1 < z_s < \infty$. Let us assume now that at $z > 1$

$$T(z,s) \sim \left(1 - \frac{z-1}{z_c-1}\right)^{-\gamma}, \quad \gamma > 0. \quad (2.20)$$

Then, using the normalization condition, $(\partial T(z,s)/\partial z)|_{z=1} = \bar n_j(s)$ we can find that $z_c(s) = 1 + \gamma/\bar n_j(s)$. The singular structure (2.20) is impossible in the "Standard Model" because of the condition $\langle k\rangle = \text{const}$. But if $|z-1| \ll 1$ we have estimation (2.17). The difference between the

"Standard Model" and (C) is seen only at $1-(z-1)/(z_c-1)\ll 1$, i.e., either in the asymptotics over n or in the asymptotics over energy. The singular structure is familiar for "logistic" equations of QCD jets, e.g., [19].

In the considered case $z_s = z_c + O(\bar{n}_j/n)$ and at high energies ($\bar{n}_j(s)\gg 1$)

$$\sigma_n \sim e^{-\gamma n/\bar{n}_j} = O(e^{-n}). \qquad (2.21)$$

Therefore, *comparing (2.19) and (2.21) we can conclude that at sufficiently high energies, i.e., if $\bar{n}_j \gg \bar{n}$ and $\bar{n}_j \ll n_{max}$, where $\bar{n}$ is the "Standard Model" mean multiplicity, the mechanism* (C) *must dominate in the asymptotics over n.*

It is the general, practically model independent, prediction. From the experimental point of view it has important consequence that at high energies there is a wide range of multiplicities where the "Standard Model" mechanism of hadrons production is negligible. In other words, the cold colored final state of high multiplicity processes is the dynamical consequence of jets and "Standard Model" mechanisms. At transition region between the "soft" of "Standard Model" and "hard" of jets one can expect the "semi-hard" processes of minijets dominance.

The multiplicity distribution in jets has an interesting property noted many decades ago by Volterra in his mathematical theory of populations [20]. In our terms, if the one-jet partition function has the singularity at $z_c^{(1)}(s) = 1+\gamma/\bar{n}_j(s)$ then the two-jet partition function must be singular at

$$z_c^{(2)}(s) = 1 + \frac{\gamma}{\bar{n}_j(s/4)} > z_c^{(1)}(s),$$

and so on. Therefore, at high energies and $n>\bar{n}_j(s)$ the jets number must be minimal (with exponential accuracy). This means that at $n\to\infty$ the processes of hadrons creation have a tendency to be Markovian (with increase of mean transverse momentum $\langle k\rangle$) and only in the last stage the (first order) phase transition (colored plasma) → (hadrons) may be seen.

One can say that in the asymptotics over n we consider the process of thermalization which is so fast that the usual confinement forces are "frozen" and do not play important role in the final colored state creation.

3. THERMALIZATION CONDITIONS

The following sign of the "equilibrium" would be considered. First of all, it is intuitively evident that the thermal equilibrium means the uniform distribution of the energy over all degrees of freedom. Then the system is in a macroscopic *thermal* equilibrium if the energy flows in it are relaxed [1]. On the other hand, the condition of vanishing of the energy correlators and the condition of relaxation of the macroscopic energy flows seem equivalent since the distant points of the macroscopic energy flow should be correlated. Then the relaxation of the flow would lead to the smallness of the "mean" value of the corresponding correlator. This conclusion reminds of the Bogolyubov's principle of vanishing of correlations.

Our idea may be illustrated by the following model. At the very beginning of XX century the couple P. Ehrenfest and T. Ehrenfest offered a model to visualize Boltzmann's interpretation of irreversibility phenomena in statistics. The model is extremely simple and fruitful [21]. It considers two boxes with $2N$ numerated balls. Choosing number $l=1,2,\ldots,2N$ *randomly* one must take the ball with the label l from one box and put it to the other. Starting from the highly "nonequilibrium" state with all balls in one box there is tendency to the equalization of the balls number in the boxes, see [21]. So, irreversible flow toward the preferable (equilibrium) state is seen. One can hope [21] that this model reflects a physical reality of nonequilibrium processes with the initial state very far from equilibrium. A theory of such processes with the (nonequilibrium) flow towards a state with maximal entropy should be sufficiently simple to give definite theoretical predictions.

The early models were based on the assumption that the final state of inelastic hadron processes has maximal entropy $\bar{n}(s)\sim n_{\max}$ [22]. But actually the hidden constraints stop the process of thermalization at the comparably early stages. The result of this is a small value of the hadron mean multiplicity $\bar{n}(s)$, i.e., $\bar{n}(s)\ll n_{\max}$.

3.1. Quantitative Definition of the Equilibrium

Let us define the conditions when the fluctuations in the vicinity of β_c are Gaussian. Firstly, to estimate the integral (2.15) at $z=z_c$ in the vicinity of the extremum, β_c, we should expand $\ln\rho_n(\beta+\beta_c)$ over β:

$$\ln\rho_n(\beta+\beta_c) = \ln\rho_n(\beta_c) - \sqrt{s}\beta + \qquad (3.1)$$
$$+\frac{\beta^2}{2!}\frac{\partial^2\ln\rho_n(\beta_c)}{\partial\beta_c^2} - \frac{\beta^3}{3!}\frac{\partial^3\ln\rho_n(\beta_c)}{\partial\beta_c^3} +$$
$$+\frac{\beta^4}{4!}\frac{\partial^4\ln\rho_n(\beta_c)}{\partial\beta_c^4} - \ldots$$

and, secondly, let us expand the exponent in the integral (2.15) over lth derivatives, $l=3,4,\ldots$, of $\ln\rho_n(\beta_c)$. As a result, if only the third derivative is taken into account then kth term of the perturbation series looks as follows:

$$\rho_{n,k} \sim \left\{\frac{\partial^3\ln\rho_n(\beta_c)/\partial\beta_c^3}{(\partial^2\ln\rho_n(\beta_c)/\partial\beta_c^2)^{3/2}}\right\}^k \Gamma\left(\frac{3k+1}{2}\right). \qquad (3.2)$$

Therefore, because of Euler's $\Gamma((3k+1)/2)$ function, the perturbation theory near β_c leads to the asymptotic series. Considering them as the asymptotic one, we may estimate it by the first term if and only if

$$\partial^3 \ln \rho_n(\beta_c)/\partial\beta_c^3 \ll (\partial^2 \ln \rho_n(\beta_c)/\partial\beta_c^2)^{3/2}. \quad (3.3)$$

One may write this condition as the approximate equality:

$$\partial^3 \ln \rho_n(\beta_c)/\partial\beta_c^3 \approx 0. \quad (3.4)$$

If this condition is satisfied, then the fluctuations are Gaussian with dispersion

$$\sim \left|\partial^2 \ln \rho_n(\beta_c)/\partial\beta_c^2\right|^{1/2},$$

see (3.1).

Let us consider now (3.4) in detail. We will find that this condition means the following approximate equality:

$$\frac{\rho_n^{(3)}}{\rho_n} - 3\frac{\rho_n^{(2)}\rho_n^{(1)}}{\rho_n^2} + 2\frac{(\rho_n^{(1)})^3}{\rho_n^3} \approx 0, \quad (3.5)$$

where $\rho_n^{(k)}$ means the kth derivative over β. For identical particles,

$$\rho_n^{(k)}(\beta_c) = n^k(-1)^k \times \quad (3.6)$$
$$\times \int d\Gamma_n(\beta_c, q_1, q_2, \ldots, q_n) \prod_{i=1}^{k} \epsilon(q_i).$$

The l.h.s. of (3.5) is the 3-point correlator K_3 since $d\Gamma_n(\beta_c, q_1, q_2, \ldots, q_n)$ is a density of states for given β:

$$d\Gamma_n(\beta_c, q_1, q_2, \ldots, q_n) =$$
$$= d\Omega_n(q)|a_n(q_1, q_2, \ldots, q_n)|^2 \prod_{i=1}^{n} e^{-\beta\epsilon(q_i)},$$

$$d\Omega_n = \prod_{i=1}^{n} \frac{d^3 q_i}{(2\pi)^3 \cdot 2\epsilon(q_i)}, \quad (3.7)$$
$$\epsilon(q) = (q^2 + m^2)^{1/2},$$

and a_n is the n-particle amplitude. Then,

$$K_3 = \frac{1}{\rho_n(\beta_c)} \int_{\beta_c} d\Gamma_n \prod_{i=1}^{3} \epsilon(q_i) - \quad (3.8)$$
$$- \frac{3}{\rho_n^2(\beta_c)} \int_{\beta_c} d\Gamma_n \prod_{i=1}^{2} \epsilon(q_i) \int_{\beta_c} d\Gamma_n \epsilon(q_3) +$$
$$+ \frac{2}{\rho_n^3(\beta_c)} \prod_{i=1}^{3} \int_{\beta_c} d\Gamma_n \epsilon(q_i),$$

where the index β_c means that averaging is performed with the Boltzmann factor $\exp\{-\beta_c\epsilon(q)\}$.

As a result, to have all fluctuations in the vicinity of β_c Gaussian, we should have $K_m \approx 0$, $m \geq 3$. Notice that, as follows from (3.3), the set of minimal conditions actually looks as follows:

$$|K_l| \ll |K_2|^{l/2}, \quad l \geq 3. \quad (3.9)$$

If the experiment confirms this conditions then, independently from the number of produced particles, the final-states energy spectrum is defined with high enough accuracy by one parameter β_c and the energy spectrum of particles is Gaussian. In these conditions one may return to the statistical and hydrodynamical models.

But if the inequality is not held then one must take into account the third correlator K_3, forth correlator K_4, etc. The corresponding series is asymptotic, with zero convergence radii. This means that if (3.9) is not held then β_c looses its physical meaning in this case. Therefore if β_c exists, then one may omit the K_l, $l = 3, 4, \ldots$, dependence. Otherwise one must take them into account and the problem becomes "nonintegrable." From all evidence, just this situation is realized at $n \sim \bar{n}(s)$, see the subsequent section.

3.2. Deep Asymptotics over Multiplicity: "Dilute Gas" Approximation

Let us consider the deep asymptotics over multiplicity, when produced particles momentum

$$|q_i| \ll m_h. \quad (3.10)$$

In this case one may ignore the momentum dependence in the amplitudes. This reminds the "dilute gas" approximation considered in statistics.

In the dilute gas approximation

$$\Delta\Gamma_n \sim |a_n|^2 \prod_{i=1}^{n} d\epsilon_i (\epsilon_i^2 - m_h^2)^{1/2} e^{-\beta\epsilon_i}. \quad (3.11)$$

Then

$$K_l(E, n) = \frac{\partial^l}{\partial\beta_c^l} \times \quad (3.12)$$
$$\times \ln\left\{\int \prod_{i=1}^{n} d\epsilon_i (\epsilon_i^2 - m_h^2)^{1/2} e^{-\beta_c(E,n)\epsilon_i}\right\}.$$

The approximation (3.10) means that $0 < (\epsilon_i/m_h) - 1 \ll 1$. Then it is easy to find that the "K_3 to K_2" ratio is small in the dilute gas approximation. For example,

$$R_3 \sim 1/n. \quad (3.13)$$

This result proves our general statement that, at least, in the deep asymptotics over n, the produced particle system *must* obey the property of completely thermalized state.

4. RELAXATION OF CORRELATIONS: MODEL PREDICTIONS

The symmetries may prevent the equilibrium, since they can lead to the nonvanishing distant correlations, if the symmetry is local. Following the terminology of Schwinger [3], there should not be the special correlations among degrees of freedom of the system if the phenomenon of equilibrium is searched.

This question is important because of the hidden constraints of the underlined non-Abelian gauge symmetry. Nevertheless, existence of the multiple production means that the colored partons system is not completely integrable, i.e., that the space–time local non-Abelian gauge and attendant conformal symmetries are unable to produce enough constraints to depress the thermalization process completely. In other words, in hadron dynamics the hidden constraints are "weak" in the sense that they may be "switched off" choosing special external kinematical conditions.

We will illustrate these ideas considering the multiperipheral and deep inelastic scattering kinematics. The symmetry in both cases is realized in a different way but the result is the same: particles are uniformly distributed (over the rapidity in the multiperipheral model, or over the transverse momentum in the deep inelastic scattering kinematics).

4.1. *Multiperipheral Kinematics*

The leading energy asymptotics Pomeron contribution reflects the kinematics, where the longitudinal momentum of produced particles is large and is strictly ordered. So, in terms of rapidities $\xi_i \sim \ln \epsilon_i$ the multiperipheral kinematics means that

$$\xi_1 < \xi_2 < \ldots < \xi_m < \xi. \tag{4.1}$$

At the same time, particles transverse momentum is restricted: $q_\perp^2 \leq 0.2$ GeV^2. The energy conservation law in this kinematics looks as follows:

$$\prod \epsilon_i \sim E: \quad \sum \xi_i = \xi, \tag{4.2}$$

where ξ is the total rapidity. For this reason it is natural to consider the rapidity fluctuations instead of energy. So, we will introduce β as the Lagrange multiplier of the *rapidity* conservation law (4.2).

It was found that the multiperipheral kinematics dominates inclusive cross sections $f(s, p_c)$. Moreover, the created particles spectra do not depend on s at high energies in the multiperipheral region:

$$f(s,p_c) = 2E_c \frac{d\sigma}{d^3p_c} = \\ = \int \frac{dt_1 dt_2 s_1 s_2 \phi_1(t_1)\phi_2(t_2)}{(2\pi)^2 s (t_1 - m^2)^2 (t_2 - m^2)^2},$$

$$s_1 s_2 (-p_{c\perp}^2) = s t_1 t_2.$$

Here, $s_1 = (p_a + p_c)^2$, $s_2 = (p_b + p_c)^2$, $p_c = \alpha_c p_a + \beta_c p_b + p_{c\perp}$, and $\phi_i(t_i)$ are the impact factors of hadrons. So the particle c forgets the details of its creation. It is found experimentally that the ratio

$$\frac{f(\pi^+ p \to \pi^- + \ldots)}{\sigma(\pi^+ p)} = \\ = \frac{f(K^+ p \to \pi^- + \ldots)}{\sigma(K^+ + p)} = \frac{f(pp \to \pi^- + \ldots)}{\sigma(pp)} \tag{4.3}$$

is universal [23].

The total cross section is written in the multiperipheral model in the form:

$$\sigma_{\rm tot}^{ab}(\xi) = g^a P(\xi) g^b, \tag{4.4}$$

where the Pomeron propagator

$$P(\xi) = e^{\Delta \xi} \tag{4.5}$$

and the LLA gives [24]

$$\Delta = \alpha(0) - 1 = \frac{12 \ln 2}{\pi} \alpha_s \approx 0.55, \quad \alpha_s = 0.2. \tag{4.6}$$

But the subsequent correction gives $\Delta \approx 0.2$. The one-particle inclusive spectra of the particle c of the rapidity ξ_1 produced in the collision of particles a and b can be written in the form:

$$f_c^{ab}(\xi, \xi_1) = g^a P(\xi - \xi_1) \psi_c P(\xi_1) g^b = g^a \psi_c g^b e^{\Delta \xi}, \tag{4.7}$$

where the conservation law (4.2) and the definition (4.5) were used.

Omitting the indices, the two-particle spectra look as follows:

$$f_2 = g P(\xi_1) \psi P(\xi_2) \psi P(\xi_3) g, \quad \xi_1 + \xi_2 + \xi_3 = \xi, \quad \xi_i \geq 0. \tag{4.8}$$

Generally,

$$f_k = g \left\{ \prod_{i-1}^{k} P(\xi_i) \psi \right\} P(\xi_{k+1}) g, \quad \sum_{i=1}^{k+1} \xi_i = \xi. \tag{4.9}$$

Noting the normalization condition:

$$\frac{1}{k!} \int \prod_{i=1}^{k+1} d\xi_i \delta\left(\xi - \sum_{i=1}^{k+1} \xi_i\right) f_k = \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} \sigma_n,$$

one may use the Mellin transform to write:

$$T(z,s) = g \mathcal{P}(0, \xi; z) g, \tag{4.10}$$

where the "superpropagator"

$$\mathcal{P}(0,\xi;z) = e^{(z-1)\bar{n}(s)}. \quad (4.11)$$

It is evident that (4.11) leads to the Poisson distribution:

$$\mathcal{P}_n(0,\xi) = e^{-\bar{n}(s)}\frac{\bar{n}(s)^n}{n!}, \quad (4.12)$$
$$\bar{n}(s) = \psi\xi.$$

Therefore, we start the description considering the production of the noncorrelated particles. Indeed, $\omega(\xi,z) = \ln\mathcal{P}(0,\xi;z)$ can be considered as the generating function of particle *number* correlators: $C_l = \partial^l\omega(\xi,z)/\partial z^l$, where if the inclusive correlators are considered then one must take $z=1$ at the very end of calculations. Inserting $\mathcal{P}(0,\xi;z)$ from (4.11), one may find that $C_l = 0$ for all $l > 1$.

But it must be mentioned that, nevertheless, the restrictions (4.1) introduce the produced particles *energy* correlations. We will see as a result that the energy correlators K_l, $l \geq 3$, would be large as compared with $|K_2|$. Therefore, the condition of the *uniform* distribution of particles over the rapidity (4.1) creates strong correlations over rapidities, i.e., over the longitudinal momentum.

One may notice that just the Mellin transform is useful. So,

$$\mathcal{F}^1(\xi,z) = g\mathcal{P}(0,\xi-\xi_1;z)\psi\mathcal{P}(0,\xi_1;z)g = \quad (4.13)$$
$$= g^2\psi\mathcal{P}(0,\xi;z)$$

is the generating function of one-particle exclusive spectrum. The inverse Mellin transform defines the one-particle spectrum in the n-particle environment:

$$\mathcal{F}^1_n(\xi) = g^2\psi\mathcal{P}_n(0,\xi). \quad (4.14)$$

Consequently the two-particle spectrum generating function looks as follows:

$$\mathcal{F}^1(\xi,z) = g\mathcal{P}(0,\xi_3;z)\psi\mathcal{P}(0,\xi_2;z)\psi\mathcal{P}(0,\xi_1;z)g = \quad (4.15)$$
$$= g^2\psi^2\mathcal{P}(0,\xi;z), \qquad \xi_1+\xi_2+\xi_3 = \xi.$$

In conclusion,

$$\mathcal{F}^l(\xi,z) = g^2\psi^l\mathcal{P}(0,\xi;z), \quad (4.16)$$
$$\sum_{i=1}^{l+1}\xi_i = \xi,$$

is the l-particle exclusive spectrum generating functional. Notice the ξ_i independence of $\mathcal{P}(0,\xi;z)$.

Let us calculate now

$$\rho_n^{(l)}(\xi) = \int_0^\infty d\Gamma_n^l(\xi)\prod_{i=1}^{l}\xi_i, \quad (4.17)$$

where, as it is follows from (4.16),

$$d\Gamma_n^l(\xi) = \psi^l\mathcal{P}_n(0,\xi)\prod_{i=1}^{l+1}d\xi_i\delta\left(\sum_{i=1}^{l+1}\xi_i - \xi\right). \quad (4.18)$$

Therefore,

$$\rho_n^{(l)}(\xi) = \frac{\psi^l\xi^{2l}}{(2l)!}\mathcal{P}_n(0,\xi) \quad (4.19)$$

and

$$\rho_n(\beta_c) = \mathcal{P}_n(0,\xi). \quad (4.20)$$

Having (4.19) and (4.20), one can find that, for example,

$$K_2 = \frac{\rho_n^{(2)}}{\rho_n} - \frac{\rho_n^{(1)2}}{\rho_n^2} = \frac{-5\psi^2\xi^4}{4!} \quad (4.21)$$

and

$$K_3 = \frac{\rho_n^{(3)}}{\rho_n} - 3\frac{\rho_n^{(2)}}{\rho_n}\frac{\rho_n^{(1)}}{\rho_n} + 2\frac{\rho_n^{(1)2}}{\rho_n^3} = \frac{316\psi^3\xi^6}{6!}. \quad (4.22)$$

Therefore, the "K_3 to K_2" ratio is large:

$$R_3 = \frac{K_3}{|K_2|^{3/2}} = \frac{316}{6!}\left(\frac{4!}{5}\right)^{3/2} > 1 \quad (4.23)$$

and it is the ξ and n independent number. One may find that

$$R_l > 1 \quad (4.24)$$

for all $l \geq 3$. Therefore the multiperipheral models are not able to show even the tendency to equilibrium.

4.2. Deep Inelastic Scattering Kinematics

The deep inelastic scattering (DIS) structure function $\mathcal{D}_{ab}(x,q^2)$ is described in the LLA by the contribution of the ladder diagrams. From a qualitative point of view this means the approximation of random walk over the coordinate $\ln(1/x)$ and the time is $\ln|q^2|$. The leading contributions, able to compensate the smallness of $\alpha_s(\lambda) \ll 1$, give the integration over a wide range over the "mass" $|k|$ of a real, i.e., time-like gluons. At the same time, the "masses" are strongly ordered:

$$\lambda^2 \ll k_1^2 \ll \ldots \ll k_\nu^2 \ll -q^2, \quad (4.25)$$

where ν is a number of steps (time-like gluons) of the ladder.

If the time needed to capture the parton into the hadron is $\sim(1/\lambda)$ then the gluon should decay if $k_i^2 \gg \lambda^2$. This leads to the creation of the QCD jets. The mean multiplicity $\bar{n}_j$ in the QCD jets is high if the gluon "mass" $|k|$ is high: $\ln\bar{n}_j \sim \sqrt{\ln(k^2/\lambda^2)}$.

Raising the multiplicity may (i) raise the number of jets ν and/or (ii) rise the mean value mass of jets $|\bar{k}_i|$. We will see that the mechanism (ii) would be favorable. But raising the mean value of gluon masses, $|k_i|$, decreases the range of integrability over k_i. For this reason the LLA becomes invalid in the VHM domain and the next-to-leading order corrections should be taken into account.

Indeed, let $\mathcal{F}_{ab}(x,q^2;\omega)$ be the generating functional:

$$\mathcal{F}_{ab}(x,q^2;\omega)=\sum_\nu\int d\Omega_\nu(k)\times$$
$$\times\prod_{i=1}^{\nu}\omega^{r_i}(k_i^2)\left|a_{ab}^{r_1r_2\ldots r_\nu}(k_1,k_2,\ldots,k_\nu)\right|^2,$$

where $a_{ab}^{r_1r_2\ldots r_\nu}$ is the production amplitude of ν partons ($r_i=(q,\bar{q},g)$) with momenta $(k_1,k_2,\ldots,k_\nu)$ in the process of scattering of the parton a on the parton b; $d\Omega_\nu(k)$ is the phase space element; $\omega^r(k^2)$ is the "probe function," i.e., the correlation functions

$$N_{ab}^{r_1r_2\ldots r_\nu}(k_1^2,k_2^2,\ldots,k_\nu^2;x,q^2)=$$
$$=\prod_{i=1}^{\nu}\frac{\delta}{\delta\omega^{r_i}(k_i^2)}\ln\mathcal{F}_{ab}(x,q^2;\omega)\bigg|_{\omega=1}.$$

The generating functional is normalized on the DIS structure function $\mathcal{D}_{ab}(x,q^2)$,

$$\mathcal{F}_{ab}(x,q^2;\omega=1)=\mathcal{D}_{ab}(x,q^2).$$

We will consider the approximation when the cutting line passes only through the steps of the ladder diagram. In this case $\mathcal{D}_{ab}(x,q^2)$ has a meaning of the *probability* to find the parton a in the parton b.

It is useful to consider the Laplace image over $\ln(1/x)$:

$$\mathcal{F}_{ab}(x,q^2;\omega)=\int\frac{dj}{2\pi i}\left(\frac{1}{x}\right)^j\mathbf{F}_{ab}(j,q^2;\omega).\quad(4.26)$$

Then, taking into account the above mentioned conditions, one may find the DGLAP evolution equation:

$$t\frac{\partial}{\partial t}\mathbf{F}_{ab}(j,t;\omega)=\sum_{c,r}\varphi_{ac}^r(j)\omega^r(t)\mathbf{F}_{cb}(j,t;\omega),\quad(4.27)$$

where $t=\ln(|q^2|/\Lambda^2)$,

$$\varphi_{ac}^r(j)\equiv\varphi_{ac}(j)=\int_0^1 dx\,x^{j-1}\,P_{ac}^r(x)$$

and $P_{ac}^r(x)$ are the regular kernels of the Bethe–Salpeter equation for pQCD [18]. The equation (4.27) coincides at $\omega^r=1$ with the habitual equation for Laplace transform of the structure function $\mathcal{D}_{ab}(x,q^2)$. While the Eq. (4.27) was being derived, only one additional assumption had been used for our problem $\omega^r=\omega^r(k^2)$.

The dominance of gluon contributions for the case $x\ll 1$ must be taken into account and for this reason we will omit all parton indices. One may find the solution of (4.27) in terms of the ν-gluon correlation functions $N^{(\nu)}$. Omitting the t dependence in the renormalized constant α_s, let us write:

$$\mathbf{F}(j,t;\omega)=\mathbf{D}(j,t)\times$$
$$\times\exp\left\{\sum_\nu\frac{1}{\nu!}\int\prod_{i=1}^{\nu}dt_i(\omega(t_i)-1)\times\right.$$
$$\left.\times N^{(\nu)}(t_1,t_2,\ldots,t_\nu;x,t)\right\},$$

where $t_i=\ln(k_i^2/\lambda_i^2)$. In the VHM domain, where $x\ll 1$ is important, one must consider $(j-1)\ll 1$. Then

$$N^{(1)}(t_1;j,t)=\varphi(j)\sim\frac{1}{j-1}\gg 1.$$

The second correlator

$$N^{(2)}(t_1,t_2;j,t)=$$
$$=O\left(\max\left\{\left(\frac{t_1}{t}\right)^{\varphi(j)},\left(\frac{t_2}{t}\right)^{\varphi(j)},\left(\frac{t_1}{t_2}\right)^{\varphi(j)}\right\}\right)$$

is negligible at $(j-1)\ll 1$, since, see (4.25), $t_1<$ $<t_2<t$. Therefore, in the LLA,

$$\mathbf{F}(j,t;\omega)=\mathbf{D}(j,t)\exp\left\{\varphi(j)\int_{t_0}^{t}dt_1(\omega(t_1)-1)\right\}.$$

Taking $\omega(t)=$ const, one may find that $\mathbf{F}(j,t;\omega)$ has the Poisson distribution with the "mean multiplicity" $\sim\varphi(j)t$.

If the quantity

$$\omega(t,z),\quad\omega(t,1)=1,\quad t=\ln(k^2/\lambda^2),$$

is the generating function of the preconfinement (close to the mass shell $\simeq\lambda$) partons multiplicity distribution

$$\omega_n(t)=\frac{\partial^n}{\partial z^n}\omega(t,z)\bigg|_{z=0},$$

then, using parton–hadron correspondence idea, as follows from derivation of $\mathbf{F}(j,t;\omega)$, the quantity

$$\mathbf{F}(j,t;\omega)=\mathbf{D}(j,t)\times\quad(4.28)$$
$$\times\exp\left\{\frac{1}{j-1}\int_{t_0}^{t}dt(\omega(t,z)-1)\right\}$$

is the generating function for the hadrons multiplicity distribution in the DIS processes, calculated in the frame of LLA.

Inserting (4.28) into the integral (4.26), one can find that if

$$\bar{\omega}(t, z) \equiv \int_{t_0}^{t} dt\, \omega(t, z),$$

then

$$j - 1 = \{\bar{\omega}(t, z)/\ln(1/x)\}^{1/2}$$

are essential. So, the "mobility"

$$\{\ln(1/x)/\bar{\omega}(t, z)\} \gg 1 \qquad (4.29)$$

decreases with z or, it is the same, with the multiplicity n. This is the reason why the LLA for considered DIS kinematics has a restricted range of validity in the VHM region.

Nevertheless, in the frame of LLA conditions, as follows from (4.28), the generating functional $\mathcal{F}_{ab}(x, t; z)$ has the following estimation:

$$\ln \mathcal{F}_{ab}(x, t; z) \propto \{\ln(1/x)\bar{\omega}(t, z)\}^{1/2}. \qquad (4.30)$$

Therefore, since the coupling is a constant,

$$\ln \mathcal{F}_{ab}(x, t; z = 1) = \ln \mathcal{D}_{ab}(x, t) \propto (t \ln(1/x))^{1/2}.$$

This is a well-known result. Therefore, one should take into account the screening effects, see [18].

Nevertheless, described by (4.30) kinematics cannot predict tendency to equilibrium.

5. CONCLUSION

Summarizing the results, one may conclude that:

(i) If the hadron amplitudes are regular in the zero momentum limit, then, at least, in the deep asymptotics over multiplicity, i.e., in the nonrelativistic limit, one must see the complete thermalization.

(ii) Existing multiple production models are not able to predict even tendency to the thermalization. But it must be noted that the models have finite range of application in the VHM domain. The experimental information in the VHM domain seems crucial for this reason.

(iii) The description of multiple production processes in the finite neighborhood of the mean multiplicity $\bar{n}(s)$ demands a large number of correlators. For this reason the VHM domain, where the correlators vanish, is important.

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REFERENCES

1. J. Manjavidze, Fiz. Elem. Chastits At. Yadra **16**, 101 (1985).
2. J. Manjavidze and A. Sissakian, JINR Rapid Commun. No. 5[31]-88, 5 (1988); G. Chelkov, J. Manjavidze, and A. Sissakian, JINR Rapid Commun., No. 4[96]-99, 35 (1999); hep-ph/9811360.
3. J. Manjavidze and A. Sissakian, Phys. Rep. **346**, 1 (2000).
4. J. Manjavidze and A. Sissakian, Part. Nucl. **31**, 104 (2000).
5. A. Casher, H. Neuberger, S. Nassinov, Phys. Rev. D **20**, 179 (1979); E. Gurvich, Phys. Lett. B **87**, 386 (1979).
6. F. Bloch, Z. Phys. **74**, 295 (1932).
7. T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955).
8. N. P. Landsman and Ch. G. van Weert, Phys. Rep. **145**, 141 (1987).
9. L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
10. A. J. Niemi and G. Semenoff, Ann. Phys. (N. Y.) **152**, 105 (1984).
11. J. Schwinger, J. Math. Phys. A **9**, 2363 (1994).
12. L. V. Keldysh, Sov. Phys. JETP **20**, 1018 (1964).
13. H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo-Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982).
14. T. D. Lee and C. N. Yang, Phys. Rev. **87**, 404, 410 (1952).
15. J. S. Langer, Ann. Phys. (N. Y.) **41**, 108 (1967).
16. M. Kac, G. E. Uhlenbeck, and P. C. Hemmer, J. Math. Phys. (N. Y.) **4**, 216 (1963); **5**, 60 (1964).
17. S. Coleman, Phys.Rev. D **15**, 2929 (1977).
18. L. V. Gribov, E. M. Levin, and M. G. Ryskin, Phys. Rep. **100C**, 1 (1983).
19. K. Konishi, A. Ukawa, and G. Veneziano, Phys. Lett. B **80**, 259 (1979); A. Basseto, M. Ciafaloni, and G. Marchesini, Nucl. Phys. B **163**, 477 (1980).
20. V. Volterra, *Lecons sur la Theoreie Mathematique de la Lutte Pour La Vie* (Paris, 1931).
21. M. Kac, *Probability and Related Topics* (Interscience Publ., London; New York, 1957).
22. L. D. Landau, Izv. Akad. Nauk SSSR, Ser. Fiz. **17**, 85 (1953).
23. M. G. Ryskin, in *Proceedings of the VII Winter School LIYaPh* (Leningrad, 1972).
24. E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, Sov. Phys. JETP **44**, 443 (1976); Zh. Eksp. Teor. Fiz. **71**, 840 (1976); L. N. Lipatov, Sov. J. Nucl. Phys. **20**, 94 (1975); V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys. **15**, 438, 675 (1972); G. Altarelli and G. Parisi, Nucl. Phys. B **126**, 298 (1977).