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Superintegrability in Classical and Quantum Systems

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Two Exactly-Solvable Problems in One-Dimensional Quantum Mechanics on Circle

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ABSTRACT. In this note we establish a relation between two exactly-solvable problems on a circle, namely singular Coulomb and singular oscillator systems.

1. A series of complex mappings $S_{2C} \to S_2$, $S_{4C} \to S_3$ and $S_{8C} \to S_5$ was constructed in a recent paper [**5**]. They extend to spherical geometry the Levi-Civita, Kustaanheimo–Stiefel and Hurwitz transformations, well known for Euclidean space. It was shown that these transformations establish a correspondence between Coulomb and oscillator problems in classical and quantum mechanics for dimensions $(2,2)$, $(3,4)$ and $(5,8)$ on spheres. A detailed analysis of the real mapping on the curved space has been done in [**7**]. It was remarked that in the stereographic projection the relation between Coulomb and oscillator problems functionally coincide with the flat space Levi-Civita and Kustaanheimo–Stiefel transformations. The relation between the quasiradial Schrödinger equations for Coulomb and oscillator problems on the n-dimensional spheres and one- and two-sheeted hyperboloids for $n \geq 2$ was found in the article [**6**].

The present note is devoted to two singular one-dimensional exactly-solvable potentials on circle S_1: $s_0^2 + s_1^2 = R^2$

$$V^{so}(\vec{s}) = \frac{\omega^2 R^2}{2}\frac{s_1^2}{s_0^2} + \frac{1}{2}\frac{k_1^2 - \frac{1}{4}}{s_1^2}, \qquad V^{c}(\vec{s}) = -\frac{\mu}{R}\frac{s_0}{|s_1|} + \frac{1}{2}\frac{p^2 - \frac{1}{4}}{s_1^2}, \tag{0.1}$$

(we consider that $k_1, p > 0$) where s_0, s_1 are Cartesian coordinates in the ambient Euclidean space E_2.

The potentials in (0.1) are the well-known analogs of superintegrable [**4**] and exact-solvable [**8**] potentials restricted to one spatial dimension. Below we will prove that these two systems are connected to each other by a one-dimensional

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This is the final form of the paper.

complex duality transformation [**5**], which is related to the more general class of "gauge transformations" introduced in the paper [**3**].

2. The Schrödinger equation describing the nonrelativistic quantum motion on the circle S_1 in the polar coordinate $\varphi \in [-\pi, \pi]$

$$s_0 = R\cos\varphi, \qquad s_1 = R\sin\varphi$$

has the following form ($\hbar = m = 1$)

$$\frac{d^2\Psi}{d\varphi^2} + 2R^2[E - V(\varphi)]\Psi = 0. \tag{0.2}$$

Substituting the singular oscillator potential $V^{so}(\varphi)$ in (0.2), we obtain a Pöschl–Teller-type equation

$$\frac{d^2\Psi}{d\varphi^2} + \left[\epsilon - \frac{k_0^2 - \frac{1}{4}}{\cos^2\varphi} - \frac{k_1^2 - \frac{1}{4}}{\sin^2\varphi}\right]\Psi = 0 \tag{0.3}$$

where $\epsilon = 2R^2E + \omega^2R^4$ and $k_0^2 = \omega^2R^4 + \frac{1}{4}$. The regular (at points $\varphi = 0$ and $\pi/2$) solution of the above equation maybe chosen in following form [**2**]

$$\Psi_n(\varphi) = \sqrt{\frac{2(2n + k_0 \pm k_1 + 1)\Gamma(n + k_0 + k_1 + 1)\Gamma(n \pm k_1 + 1)}{R[\Gamma(1 \pm k_1)]^2\Gamma(n + k_0 + 1)(n)!}} \tag{0.4}$$
$$\times (\sin\varphi)^{1/2 \pm k_1}(\cos\varphi)^{1/2 + k_0}{}_2F_1(-n, n + k_0 \pm k_1 + 1; 1 \pm k_1; \sin^2\varphi),$$

with

$$\epsilon = (2n \pm k_1 + k_0 + 1)^2, \quad n = 0, 1, 2, \dots . \tag{0.5}$$

The energy spectrum of the one-dimensional singular oscillator is given by

$$E_n(R) = \frac{1}{2R^2}[(2n \pm k_1 + \tfrac{1}{2})^2 + (2k_0 + 1)(2n \pm k_1 + 1)]. \tag{0.6}$$

Let us remark that the wave-functions have been normalized in the domain $[0, \pi/2]$. The positive sign in front of k_1 has to taken whenever $k_1 > \frac{1}{2}$, i.e., the additional term to the oscillator potential is repulsive at the origin and the motion takes place only in the domain $\varphi \in [0, \pi/2]$. If $0 < k_1 \leq \frac{1}{2}$, i.e., the additional term is attractive at the origin, both the positive and negative signs must be taken into account in the solution. The motion in this case takes place in $\varphi \in [-\pi/2, \pi/2]$. This is indicated by the notation $\pm k_1$, in the formulas.

3. Let us write the Schrödinger equation (0.2) for the singular Coulomb potential $V^c(\vec{s})$

$$\frac{d^2\Psi}{d\varphi^2} + \left(2R^2E + 2\mu R\cot|\varphi| - \frac{p^2 - \frac{1}{4}}{\sin^2\varphi}\right)\Psi = 0. \tag{0.7}$$

First we will consider the region $\varphi \in [0, \pi]$. We make now a transformation to the new variable $\theta \in [0, \pi/2]$

$$e^{i\varphi} = \cos\theta, \tag{0.8}$$

which is possible if we continue the variable φ into the complex domain: $\operatorname{Re}\varphi = 0$, $0 \leq \operatorname{Im}\varphi < \infty$ (see Figure 1). We also complexify the coupling constant μ by putting $k = i\mu$ so that

$$\mu\cot\varphi = k(1 - 2\sin^{-2}\theta). \tag{0.9}$$

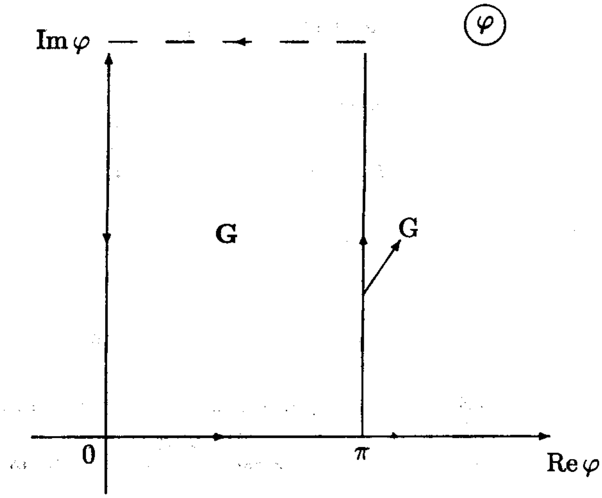

FIGURE 1. Domain $\mathbf{G} = \{0 \le \operatorname{Re}\varphi \le \pi; 0 \le \operatorname{Im}\varphi < \infty\}$ on the complex φ plane.

As a result we obtain the equation

$$\frac{d^2W}{d\theta^2} + \left[\epsilon - \frac{k_0^2 - \frac{1}{4}}{\cos^2\theta} - \frac{k_1^2 - \frac{1}{4}}{\sin^2\theta}\right] W = 0, \tag{0.10}$$

where $W(\theta) = (\cot\theta)^{1/2}\Psi(\theta)$ and

$$\epsilon = 2R^2E + 2kR, \qquad k_0^2 = 2R^2E - 2kR, \qquad k_1^2 = 4p^2. \tag{0.11}$$

From the above equation we see that, up to the substitution (0.11) the equation (0.10) for the one-dimensional singular Coulomb problem coincides with the one-dimensional singular oscillator equation (0.3).

The regular, for $\theta \in [0, \pi/2]$ and $k_0 \ge \frac{1}{4}$, solution of this equation according to (0.4) is

(0.12)

$$\Psi(\theta) = \frac{W(\theta)}{\sqrt{\cot\theta}} = C_n(\sin\theta)^{1\pm k_1}(\cos\theta)^{k_0}{}_2F_1(-n, n + k_0 \pm k_1 + 1; 1 \pm k_1; \sin^2\theta),$$

where C_n is a normalization constant. To compute the constant C_n we require that the wave function (0.12) satisfy the condition

$$R\int_0^{\pi} \Psi_n\Psi_n^{\diamond}\, d\varphi = \frac{1}{2}, \tag{0.13}$$

where the symbol "$\diamond$" means the complex conjugate together with the inversion $\varphi \to -\varphi$, i.e., $\Psi^{\diamond}(\varphi) = \Psi^*(-\varphi)$. [We choose the scalar product involving $\Psi^{\diamond}$ because for $\varphi \in \mathbf{G}$ and real μ, and ϵ, the function $\Psi^{\diamond}(\varphi)$ also belongs to the solution space of (0.7)).] By analogy with Refs. [**5**, **6**], we consider the integral over the contour G in the complex plane of the variable φ (see Figure 1)

$$\oint \Psi_n(\varphi)\Psi_n^\diamond(\varphi)\,d\varphi = \int_0^\pi \Psi_n(\varphi)\Psi_n^\diamond(\varphi)\,d\varphi + \int_\pi^{\pi+i\infty} \Psi_n(\varphi)\Psi_n^\diamond(\varphi)\,d\varphi + \int_{\pi+i\infty}^{i\infty} \Psi_n(\varphi)\Psi_n^\diamond(\varphi)\,d\varphi + \int_{i\infty}^{0} \Psi_n(\varphi)\Psi_n^\diamond(\varphi)\,d\varphi. \tag{0.14}$$

Using the facts that the integrand vanishes as $e^{2ik_0\varphi}$ and that $\Psi_n(\varphi)$ is regular in the domain **G** (see Figure 1), according to the Cauchy theorem we have

$$\int_0^\pi \Psi_n(\varphi)\Psi_n^\diamond(\varphi)\,d\varphi = (1-e^{2i\pi k_0}) \int_0^{i\infty} \Psi_n(\varphi)\Psi_n^\diamond(\varphi)\,d\varphi. \tag{0.15}$$

Making the substitution (0.8) in the right integral of (0.15), we find

$$\int_0^\pi \Psi_n(\varphi)\Psi_n^\diamond(\varphi)\,d\varphi = i(1-e^{2i\pi k_0}) \int_0^{\pi/2} [\Psi_n] \tan\theta\,d\theta. \tag{0.16}$$

and after integration over the angle θ we finally get [1]

$$C_n = \sqrt{\frac{(-ik_0)(2n+k_0\pm k_1+1)\Gamma(n+1\pm k_1)\Gamma(n+k_0\pm k_1+1)}{R[1-e^{2i\pi k_0}](2n\pm k_1+1)n![\Gamma(1\pm k_1)]^2\Gamma(n+k_0+1)}}. \tag{0.17}$$

Let us now consider the two most interesting cases.

3.1. *The case when* $p^2 = \frac{1}{4}$. Then the duality transformation (0.8) establishes the connection between the pure Coulomb problem and the singular oscillator with $k_1^2 = 1$. Comparing eqs. (0.5) and (0.11) and putting $k = i\mu$, we get

$$k_0 = -(n+1) + i\sigma, \qquad \sigma = \frac{\mu R}{n+1} \tag{0.18}$$

and for the energy spectrum

$$E_n(R) = \frac{(n+1)^2}{2R^2} - \frac{\mu^2}{2(n+1)^2}, \quad n = 0, 1, 2, \dots. \tag{0.19}$$

Returning to the variable φ, we obtain that wave function for $0 \le \varphi \le \pi$ has the form

$$\Psi_{n\sigma}(\varphi) = C_n(\sigma)\sin\varphi e^{-i\varphi(n-i\sigma)}{}_2F_1(-n, 1+i\sigma; 2; 1-e^{2i\varphi}), \tag{0.20}$$

where the normalization constant $C_n(\sigma)$ is

$$C_n(\sigma) = e^{\sigma\pi/2}|\Gamma(1+i\sigma)|\sqrt{\frac{(n+1)^2+\sigma^2}{\pi R}}. \tag{0.21}$$

The wave function in the region $-\pi \le \varphi < 0$ ($s_1 < 0$) may be determined from (0.20) by the reflection $\varphi \to -\varphi$. Therefore the general solution of the Schrödinger equation for $\varphi \in [-\pi, \pi]$ can be presented in the form of even and odd functions

$$\Psi_{n\sigma}^{(+)}(\varphi) = C_n(\sigma)\sin|\varphi| e^{-i(n-i\sigma)|\varphi|} F(-n, 1+i\sigma; 2; 1-e^{2i|\varphi|}),$$
$$\Psi_{n\sigma}^{(-)}(\varphi) = C_n(\sigma)\sin\varphi e^{-i(n-i\sigma)|\varphi|} F(-n, 1+i\sigma; 2; 1-e^{2i|\varphi|}).$$

Thus by using the relation between the Coulomb and singular oscillator systems we have constructed the wave functions and energy spectrum for a Coulomb system on the one-dimensional sphere.

3.2. Let us now choose $k_1^2 = \frac{1}{4}$ or equivalently $p^2 - \frac{1}{4} = -3/16$. In this case the centrifugal potential term is attractive at the origin for singular Coulomb systems and the motion take place in the domains $\varphi \in (-\infty, \infty)$. For oscillator system this term equal zero and therefore the duality transformation (0.8) connect singular Coulomb and pure oscillator systems.

Let us introduce the quantity ν which takes two values $\nu = \frac{1}{4}$ and $\nu = \frac{3}{4}$. Making all calculations by analogy to previous case, it is easy to obtain the energy spectrum

$$E_n^\nu(R) = \frac{(n+\nu)^2}{2R^2} - \frac{\mu^2}{2(n+\nu)^2}, \quad n = 0, 1, 2, \dots . \tag{0.22}$$

and wave functions

$$\Psi_{n\sigma}^\nu(\varphi) = e^{\sigma\pi/2} 2^\nu \frac{|\Gamma(\nu + i\sigma)|}{\Gamma(2\nu)} \sqrt{\frac{[(n+\nu)^2 + \sigma^2]\Gamma(n+2\nu)}{4\pi R(n+\nu)n!}} \times (\sin\varphi)^\nu e^{-i\varphi(n-i\sigma)} {}_2F_1(-n, \nu + i\sigma; 2\nu; 1 - e^{2i\varphi}),$$

where $\sigma = \mu R/(n+\nu)$.

In the contraction limit $R \to \infty$, $\varphi \to 0$ and $R\varphi \sim x$ - fixed, we see that

$$\epsilon_n = \lim_{R\to\infty} E_n^\nu(R) = -\frac{\mu^2}{2(n+\nu)^2}, \quad n = 0, 1, \dots, \tag{0.23}$$

and

(0.24)

$$\Phi_n^\nu(y) = \lim_{\substack{R\to\infty \\ \varphi\to 0}} \Psi_{n\sigma}^\nu(\varphi) = \frac{\sqrt{\mu}}{\Gamma(2\nu)} \frac{1}{(n+\nu)} \sqrt{\frac{\Gamma(n+2\nu)}{2n!}} y^\nu e^{-|y|/2} {}_1F_1(-n; 2\nu; y),$$

where $y = 2\mu x/(n+\nu)$. Formulas (0.23) and (0.24) coincides with the formulas for energy levels ϵ_n and up to the factor $\sqrt{2}$ for wave functions $\Phi_n^\nu(y)$ for two type one-dimensional Coulomb anyons with $\nu = \frac{1}{4}$ and $\nu = \frac{3}{4}$ respectively [**9**].

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