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CONTRACTION AND INTERBASES EXPANSIONS ON N-SPHERE

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Lie algebra contractions from o(n+1) to e(n) are used to construct the asymptotic limit of the interbasis expansions between two general subgroup chains $O(n+1) \supset O(n_\alpha+n_\beta)\otimes O(n_\gamma)$ and $O(n+1) \supset O(n_\alpha)\otimes O(n_\gamma+n_\beta)$, where $n+1 = n_\alpha+n_\gamma+n_\beta$. It provides the asymptotic formula of Racah coefficients for $SU(2)$ and $SU(1,1)$ groups.

1 Introduction

In a series of papers[1,2,3,4] we have presented a **new aspect** of the theory of Lie group and Lie algebra contractions: *the relation between separable coordinates systems in curved and flat spaces, related by the contraction of their isometry group.*

In article[1] we considered the two-dimensional sphere S_2. The contractions we use are analytical ones: **the radius of the sphere** R is built into the infinitesimal operators and into the sets of commuting operators, not only into the structure constants. The contractions can be viewed as singular changes of bases, as was the case of the original Inönü-Wigner[5] contractions. For $R \to \infty$ the sphere $S_n \sim O(n+1)/O(n)$ goes into the Euclidean space $E_n \sim E(n)/O(n)$.

In paper[3] the dimension of the space was arbitrary, but only the simplest types of coordinates were considered, **namely subgroup type coordinates.** Furthemore, we introduced a *graphical method* of connecting subgroup–type coordinate systems on the sphere S_n (characterized by tree diagrams) and on the Euclidean space E_n (characterized by cluster diagrams) and gave rules relating the contraction limit $R \to \infty$ of the coordinates, invariant operators, eigenvalues and basis functions.

Vilenkin, Kuznetsov and Smorodinsky[6] (see also[7,8]) developed a graphical method, the "method of trees" to describe subgroup type coordinates on S_n. The corresponding separated eigenfunctions are hyperspherical functions [5-8]. Their relation to subgroup chains and subgroup diagrams was analyzed in

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paper[3], as were their contractions to subgroup type separated basis functions for the groups $E(n)$.

In many-body theories it is often necessary to expand one type of hyperspherical functions in terms of other ones. The expansion coefficients have been called T-coefficients, or overlap functions[9].

The purpose of this contribution is to investigate the $R \to \infty$ contraction limit of the **interbases expansions and overlap functions** for the different spherical and hyperspherical functions on S_n.

A convenient way of calculating the T-coefficients corresponding to a transformation from one tree to another is to introduce a sequence of "intermediate" trees, each differing from the previous one by the transplantation of exactly one branch from one side of a branching point to the other. The T-matrix will be factorized into a product of "elementary T-matrices", corresponding to such elementary transformations. Each elementary T-matrix can be represented by a tree-type diagram consisting of a cell with three ends, each of which can be either open or closed. Therefore there are eight inequivalent diagrams of this type. The T-coefficients for all 8 types of elementary transformations were calculated by Kil'dyushov[9]. They were expressed in terms of generalized hypergeometric functions of argument $x = 1$: ${}_2F_1(1)$, ${}_3F_2(1)$, ${}_4F_3(1)$, Wigner D-functions, or Clebsch-Gordon and Racah coefficients for positive discrete series of representations of the group $SU(1,1)$[7,10].

The T-coefficients representing the general transformation corresponding to the diagram on Fig.1a have the form[9]

$$\left\| \begin{matrix} j_1\, j_2\, j_3 \\ j_{12}\, j\, j_{23} \end{matrix} \right\| = \frac{\sqrt{(2j_{12}+1)(2j_{23}+1)}}{\Gamma(2j_2+2)} \sqrt{\frac{\Gamma(j_2+j_3+j_{23}+2)}{\Gamma(j_{23}-j_2-j_3)}}$$
$$\times \sqrt{\frac{\Gamma(j_2-j_3+j_{23}+1)\Gamma(j-j_{12}-j_3)\Gamma(j_1+j_{12}+j_2+2)}{\Gamma(j_{23}-j_2+j_3+1)\Gamma(j-j_{12}+j_3+1)\Gamma(j_1+j_{12}-j_2+1)}}$$
$$\times \sqrt{\frac{\Gamma(j+j_{12}-j_3+1)\Gamma(j_{12}-j_1+j_2+1)\Gamma(j+j_1+j_{23}+2)}{\Gamma(j+j_{12}+j_3+2)\Gamma(j_{12}-j_1-j_2)\Gamma(j-j_1-j_2+j_3)}}$$
$$\times \sqrt{\frac{\Gamma(j+j_1-j_{23}+1)\Gamma(j-j_1-j_2+j_3)\Gamma(j-j_1-j_2+j_3)}{\Gamma(j+j_1+j_2-j_3+2)\Gamma(j-j_1-j_{23})\Gamma(j-j_1+j_{23}+1)}}$$
$$\times {}_4F_3 \left\{ \begin{matrix} -j_{12}+j_1+j_2+1, j_{12}+j_1+j_2+2, j_{23}+j_2-j_3+1, j_2-j_3-j_{23} \\ 2j_2+2, j+j_1+j_2-j_3+2, j_1+j_2-j_3-j+1 \end{matrix} \middle| 1 \right\}. \quad (1)$$

Here $j_i(j_{ik})$ label the discrete representations of $SU(1,1)$. It was pointed out in[7] that if the number $j_i = -1/4$, or -3/4 is associated with an end point, then one obtains values of the T-matrix for transitions in cells with open ends. The transition matrices for all 8 types of cells can be obtained in this manner.

2 Contractions of the Interbasis Expansion

Let us have two wave functions $\Psi_{j,j_{12}}^{j_1,j_2,j_3}$ and $\Psi_{j,j_{23}}^{j_1,j_2,j_3}$, which are solutions of the Helmholtz equation on S_n and correspond to the two subgroup chains $O(n+1) \supset O(n_\alpha + n_\beta) \otimes O(n_\gamma)$ and $O(n+1) \supset O(n_\alpha) \otimes O(n_\gamma + n_\beta)$, where $n+1 = n_\alpha + n_\gamma + n_\beta$. The transformations between these two functions corresponds to the recoupling of the angular momentum $\{j_1, j_2, j_3, j_{12}, j\}$ and $\{j_1, j_2, j_3, j_{23}, j\}$. The overlap functions expressed in terms of Racah coefficients and the corresponding interbases expansion is

$$\Psi_{j,j_{12}}^{j_1,j_2,j_3}(\theta_1', \theta_2') = \sum_{j_{23}=j_2+j_3+1}^{j-j_1-1} \left\| \begin{matrix} j_1\, j_2\, j_3 \\ \\ j_{12}\, j\, j_{23} \end{matrix} \right\| \Psi_{j,j_{23}}^{j_1,j_2,j_3}(\theta_1, \theta_2), \tag{2}$$

where the angles are related as follows

$$\cos\theta_1 = \cos\theta_1' \cos\theta_2' \qquad \cot\theta_2 = \cot\theta_1' \sin\theta_2',$$

The wave functions in eq. (2) can be constructed following the rules presented in papers[6,7] (see also[4]) and have the form

$$\Psi_{j,j_{12}}^{j_1,j_2,j_3}(\theta_1', \theta_2') = \sqrt{\frac{(2j+1)(2j_{12}+1)\Gamma(j_1+j_2+j_{12}+2)\Gamma(j+j_3+j_{12}+2)}{\Gamma(j_{12}+j_1-j_2+1)\Gamma(j_{12}-j_1+j_2+1)\Gamma(j-j_{12}+j_3+1)}}$$

$$\times \sqrt{\frac{\Gamma(j_{12}-j_1-j_2)\Gamma(j-j_{12}-j_3)}{\Gamma(j+j_{12}-j_3+1)}} (\sin\theta_2')^{2j_2+1} (\cos\theta_2')^{2j_1+1} \tag{3}$$

$$\times (\sin\theta_1')^{2j_3+1} (\cos\theta_1')^{2j_{12}+1} P_{j_{12}-j_1-j_2-1}^{(2j_2+1, 2j_1+1)}(\cos 2\theta_2') P_{j-j_{12}-j_3-1}^{(2j_3+1, 2j_{12}+1)}(\cos 2\theta_1')$$

$$\Psi_{j,j_{23}}^{j_1,j_2,j_3}(\theta_1, \theta_2) = \sqrt{\frac{(2j+1)(2j_{23}+1)\Gamma(j+j_1+j_{23}+2)\Gamma(j_2+j_3+j_{23}+2)}{\Gamma(j_{23}-j_3+j_2+1)\Gamma(j_{23}-j_2+j_3+1)\Gamma(j+j_{23}-j_3+1)}}$$

$$\times \sqrt{\frac{\Gamma(j_{23}-j_3-j_2)\Gamma(j-j_1-j_{23})}{\Gamma(j-j_{23}+j_3+1)}} (\sin\theta_2)^{2j_3+1} (\cos\theta_2)^{2j_2+1} \tag{4}$$

$$\times(\sin\theta_1)^{2j_{23}+1}(\cos\theta_1)^{2j_1+1}\,P^{(2j_3+1,2j_2+1)}_{j_{23}-j_3-j_2-1}(\cos 2\theta_2)P^{(2j_{23}+1,2j_1+1)}_{j-j_{23}-j_1-1}(\cos 2\theta_1)$$

here $P_n^{(\alpha,\,\beta)}(x)$ are Jacobi polynomials.

Consider the contraction limit $R \to \infty$ in the wave functions (3)-(4) and overlap functions (1). For large R we put

$$j \sim kR, \quad j_{12} \sim pR, \quad j_1 \sim qR, \quad \theta_1' \sim \frac{r_{j_2}}{R}, \quad \theta_2' \sim \frac{r_{j_2}}{R}, \quad \theta_1 \sim \frac{r_{j_{23}}}{R}, \tag{5}$$

where $r_{j_{23}} = \sqrt{r_{j_2}^2 + r_{j_3}^2}$, $k_{j_2}^2 = p^2 - q^2$, $k_{j_3}^2 = k^2 - p^2$ and $k_{j_{23}}^2 = k_{j_2}^2 + k_{j_3}^2$. Taking into account the formula [3,4]

$$\lim_{R\to\infty} \frac{\Gamma(j - j_{23} - j_1 - 1)}{\Gamma(j + j_{23} - j_1)}\, P^{(2j_{23}+1,2j_1+1)}_{j-j_{23}-j_1-1}(\cos 2\theta_1) = 2^{2j_{23}+1}\,\frac{J_{2j_{23}+1}(k_{j_{23}}\, r_{j_{23}})}{(k_{j_{23}} r_{j_{23}})^{2j_{23}+1}},$$

where $J_\mu(x)$ is Bessel function, we have

$$\begin{aligned}\lim_{R\to\infty} \Psi^{j_1,j_2,j_3}_{j,j_{12}}(\theta_1',\theta_2') &= \Phi^{j_2,j_3}_{kk_{j_2}k_{j_3}}(r_{j_2}, r_{j_3}) \\ &= 2\sqrt{kp}\, J_{2j_2+1}(k_{j_2} r_{j_2}) J_{2j_3+1}(k_{j_3} r_{j_3}),\end{aligned} \tag{6}$$

$$\begin{aligned}\lim_{R\to\infty} \Psi^{j_1,j_2,j_3}_{j,j_{23}}(\theta_1,\theta_2) &= \Phi^{j_{23},j_2,j_3}_{kk_{j_{23}}}(r_{j_{23}},\theta_2) = 2\sqrt{k}\, J_{2j_{23}+1}(k_{j_{23}}\, r_{j_{23}}) \\ &\times \sqrt{\frac{(2j_{23}+1)\Gamma(j_{23}+j_2+j_3+2)\Gamma(j_{23}-j_2-j_3)}{\Gamma(j_{23}-j_2+j_3+1)\Gamma(j_{23}+j_2-j_3+1)}} \\ &\times (\sin\theta_2)^{2j_3+1}(\cos\theta_2)^{2j_2+1}\, P^{(2j_3+1,2j_2+1)}_{j_{23}-j_3-j_2-1}(\cos 2\theta_2).\end{aligned} \tag{7}$$

Using the asymptotic formulas for ${}_4F_3$ functions, we obtain

$$\begin{aligned}\lim_{R\to\infty} \left\| \begin{matrix} j_1\; j_2\; j_3 \\ \\ j_{12}\; j\; j_{23} \end{matrix} \right\| &= \frac{\sqrt{(2j_{23}+1)}}{\Gamma(2j_{23}+2)} \sqrt{\frac{\Gamma(j_2+j_3+j_{23}+2)\Gamma(j_2-j_3+j_{23}+1)}{\Gamma(j_{23}-j_2-j_3)\Gamma(j_{23}-j_2+j_3+1)}} \\ &\times \sqrt{\frac{2p}{k^2-q^2}} \left(\frac{p^2-q^2}{k^2-q^2}\right)^{j_2+\frac12} \left(\frac{k^2-p^2}{k^2-q^2}\right)^{j_3+\frac12} \\ &\times {}_2F_1\left(-j_3-j_{23}+j_2, -j_3+j_{23}+j_2+1; 2j_2+2; \frac{p^2-q^2}{k^2-q^2}\right)\end{aligned}$$

Using the transformation

$${}_2F_1(a,\ b;\ c;\ z) = (1-z)^{c-a-b} \cdot {}_2F_1(c-a,\ c-b;\ c;\ z)$$

we finally get that

$$\lim_{R\to\infty} \left\| \begin{matrix} j_1\, j_2\, j_3 \\ \\ j_{12}\, j\, j_{23} \end{matrix} \right\| = \sqrt{\frac{p^2(2j_{23}+1)\Gamma(j_2-j_3+j_{23}+1)\Gamma(j_2+j_3+j_{23}+2)}{(k^2-q^2)\Gamma(j_{23}-j_2-j_3)\Gamma(j_{23}-j_2+j_3+1)}}$$

$$\times\, (\cos\phi)^{2j_2+1}(\sin\phi)^{2j_3+1} \cdot P^{(2j_2+1,\, 2j_3+1)}_{j_{23}-j_2-j_3-1}(\cos 2\phi), \qquad (8)$$

where

$$\cos\phi = \left(\frac{p^2-q^2}{k^2-q^2}\right)^{1/2}, \quad \sin\phi = \left(\frac{k^2-p^2}{k^2-q^2}\right)^{1/2}.$$

Taking the contraction limit $R \to \infty$ on both sides of the interbases expansion (2) and using the formulas (6)-(7) and (8), we have $(\theta_2 \equiv \theta)$

$$\Phi^{j_2,j_3}_{kk_{j_2}k_{j_3}}(r_{j_2}, r_{j_3}) = \sum_{j_{23}=j_2+j_3+1}^{\infty} W^{j_{23},j_2,j_3}_{kk_{j_2}k_{j_3}}\ \Phi^{j_{23},j_2,j_3}_{kk_{j_{23}}}(r_{j_{23}}, \theta_2) \qquad (9)$$

or in explicit form

$$J_{2j_2+1}(z\cos\theta\cos\phi)\, J_{2j_3+1}(z\sin\theta\sin\phi) = (\sin\phi\sin\theta)^{2j_3+1}\ (\cos\phi\cos\theta)^{2j_2+1}$$

$$\times \sum_{j_{23}=j_2+j_3+1}^{\infty} \frac{(-1)^{j_{23}-j_2-j_3-1}}{z}\ \frac{(2j_{23}+1)\Gamma(j_{23}+j_2+j_3+2)\Gamma(j_{23}-j_2-j_3)}{\Gamma(j_{23}-j_2+j_3+1)\Gamma(j_{23}+j_2-j_3+1)}$$

$$\times\, P^{(2j_3+1,2j_2+1)}_{j_{23}-j_2-j_3-1}(\cos 2\phi)\ P^{(2j_3+1,\,2j_2+1)}_{j_{23}-j_2-j_3-1}(\cos 2\theta) J_{2j_{23}+1}(z) \qquad (10)$$

where $z \equiv k_{j_{23}} r_{j_{23}}$ and

$$r_{j_2} = r_{j_{23}}\cos\theta, \qquad r_{j_3} = r_{j_{23}}\sin\theta,$$

$$k_{j_2} = k_{j_{23}}\cos\phi, \qquad k_{j_3} = k_{j_{23}}\sin\phi,$$

The last expansion is equivalent to well–known formulas in the theory of Bessel functions[11], namely expansions of the product of two Bessel functions in terms of the product of one Bessel function and two Jacobi polynomials.

The presented procedure of contraction is illustrated on Fig.1. The vertical arrows correspond to the contraction limit (5) from the S_n to E_n. The left tree in Fig.1a contracts to bihyperspherical coordinates; the second to hyperspherical ones. The contraction of the coefficients T is given by eq. (8). This is an asymptotic formula for the Racah coefficients, where the three momenta satisfy $j, j_1, j_{12} \to \infty$. The interbases expansions (9) between two E_n cluster diagrams are obtained from the interbases expansions (2) for the S_n tree diagrams. We thus obtain relations between the bihyperspherical and hyperspherical solutions of the Helmholtz equation on E_n.

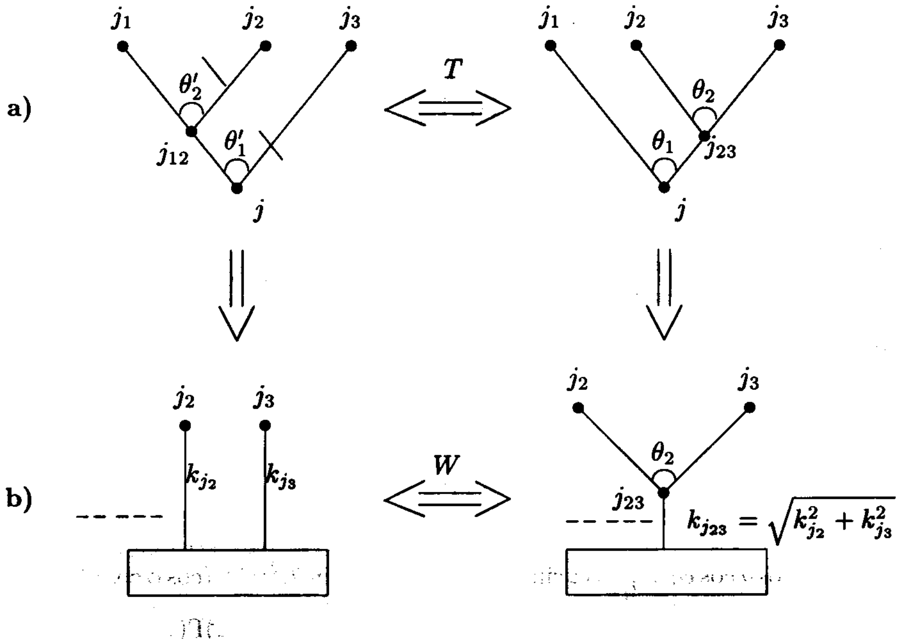

Figure 1: Contractions in interbasis expansion for a general transformation.

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