Symmetries, Variational Principles and Quantum Dynamics

J.Manjavidze
Institute of Physics, Tbilisi, Georgia & JINR, Dubna, Russia, e-mail: joseph@nusun.jinr.ru,
A.Sissakian
JINR, Dubna, Russia, e-mail: sisakian@jinr.ru

Abstract

The role of symmetries in formation of quantum dynamics is discussed. A quantum version of the d'Alambert's principle is proposed to take into account symmetry constrains for quantum case. It is noted that in this approach one can find, in four space-time dimensions, the free of divergences quantum field theory.

1. The problem of "book-keeping" of the quantum degrees of freedom looks till now as a primary task for any modern field theory. The point is that the non-Abelian gauge theory of Yang and Mills, or Einstein gravity, both obey high symmetry and the symmetry itself signifies an existence of a connection among various degrees of freedom, i.e. hence the symmetry reduces the number of dynamical degrees of freedom. So, the "book-keeping" means the procedure of counting of the *independent* degrees of freedom.

This question has a fundamental background. It is known that the symmetry may improve solvability of the quantum problem. The hidden $O(4)$ symmetry solves the Coulomb problem [1] and, therefore, the semiclassical Bohr-Sommerfeld quantization rules are exact for this problem. Another example is the $SL(2C)$ symmetry which solves the 2-dimensional sin-Gordon field theory so that the semiclassical approximation becomes exact [2]. More examples one can find in [3]. So, the symmetry actually reduces the quantum degrees of freedom.

We should keep in mind this marvelous phenomenon although it is hard to hope that we always deal with such simple "completely integrable systems" [4], in which there is enough number of symmetry constrains to prevent perturbations. However, we have to keep in mind such a possibility. What if, e.g., the Einstein gravity is the purely classical theory because of its highest space-time symmetry. Sure, this assumption looks curiously, though, it is most likely that Einstein was thinking it is so. But there is also another side of the "book-keeping" problem.

To note is that general quantum field, due to the constrains, can not be treated always as the dispersive medium, unlike the electromagnetic fields which are described as one, two, and more photon states. In this case the quantum electrodynamics fundamental method of spectral analysis can not be applied. An example is the soliton quantization problem, where ordinary methods are not effective [5].

One needs other formalism, which takes into account the symmetry constrains.

2. The modern quantum theories practically are based on the classical Hamiltonian variational principle. It can be built into the quantum theories keeping in mind the Bohr's "correspondence principle". Formally, this becomes possible thanks to an outstanding result of

Dirac that the action $S(u)$ is a generator of displacements along the $u(t)$ trajectory. Then the transition amplitude of initial state, $|in>$, into the final one, $|out>$, is an exponent: $<in|out>\sim \exp\{iS(u)\}$. The wave nature of quantum dynamics is realized then as a sum over all trajectories $u(t)$.

The simplest way of definition of this sum is based on the Ferma's principle and uses the basis of geometrical optics. A generalization represents an equivalent to the Hamiltonian variational principle. In this frame the dynamics is realized on the actions extremum, i.e. it is supposed that the *physical* trajectory $u_c(t)$ is a solution of:

$$\frac{\delta S(u)}{\delta u(t)} = 0. \tag{1}$$

This approach of "book-keeping" of deviations from u_c is known as the WKB method [6].

A weak point of this variational principle, as it is known from classical mechanics, is that it demands a cumbersome way the (symmetry) *constrains* are introduced into the formalism [7]. The formal problem is to extract the dynamical, i.e. independent, degrees of freedom. In the quantum case this leads to implementation of the Faddeev-Popov *ansatz* [8] which separates the dynamical degrees of freedom from symmetrical ones. Unfortunately, this method is unacceptable for non-Abelian gauge theory (in case the corresponding Yang-Mills fields are sufficiently strong [9], e.g. if the soliton-like excitation is considered) and for gravitational field [10].

3. We propose to use the d'Alambert's variational principle which easily "absorbs" the constrains, as is known from the classics. For that reason, it is of a definite value for us.

Indeed, the d'Alambert's variational principle means that *the virtual deviation* $\varepsilon(t)$ *does not produce any work* $F(u)\varepsilon = \Upsilon$ *since the mechanical motion is time reversible.* So, for mechanical systems $F(u)\varepsilon(t) = 0$ and, as soon as $\varepsilon(t)$ is an arbitrary function, one gets a condition that sum of *all* the forces should be equal zero, $F(u) = 0$. This equality may be treated as the equation of motion for $u(t)$ since $F(u)$ includes also the "force of inertia", $\sim \ddot{u}(t)$. If there are constrains in the system considered, then they have to be included along with other forces into $F(u)$ [4].

It should be noted here that, as follows from d'Alambert's principle, the solution of equation of motion u_c already "absorbs" all the constrains. Therefore, it is enough to count all possible forms of u_c which are fixed by the integration constants.

Therefore, the mechanics built on the d'Alambert's variational principle seems to be most useful for us. The connection among various degrees of freedom, as well as constrains, are naturally included into the prescription that the *mechanical* motion must be time reversible.

The time reversibility may be achieved in quantum theories, for instance, considering loop trajectory with the transition amplitude $R = <in|in>$. In the meantime, to define the integral over particle momentum unambiguously, one needs to implement the $i\epsilon$-prescription of Feynman.

In particle physics the $i\epsilon$-prescription has been chosen so that a wave must disappear in remote future. Note that only such definition conserves a total probability. But it is evident that such a wave process is not time reversible. Therefore, even the simple loop trajectory can not be considered as an example of time reversible wave motion: due to the $i\epsilon$-prescription the loop amplitude is complex, i.e. it is not a singlet of complex conjugation operator.

This example creates an impression that quantum mechanics is time irreversible and, therefore, the d'Alambert's variational principle can not be claimed for it. But then one may ask where is the quantum correspondence principle?

The answer is in fact that the amplitude R is not a measurable quantity while only the product

$$\rho(in, out) =< in|out >< out|in >=< in|out >< in|out >^* = | < in|out > |^2$$

is measured experimentally, i.e. only it has to be time reversible quantity [11]. Note here that to define the amplitude $< in|out >$ the $(+i\epsilon)$-prescription must be used, as well as for $< out|in >=< in|out >^*$ the $(-i\epsilon)$-prescription is applied. Therefore, just $\rho(in, out)$ describes the time reversible process and this removes the contradiction with correspondence principle. Let us consider now what this gives us.

4. Following Dirac, one writes:

$$\rho \sim e^{iS(u_+)} e^{-iS(u_-)}.$$

To note is that u_+ and u_- are two completely independent trajectories and, by definition, the total action $\{S(u_+) - S(u_-)\}$ is defined so that it describes closed-path motion.

To introduce the d'Alambert's variational principle one needs to distinguish between the *physical* trajectory $u(t)$ and the *virtual* deviation, $\varepsilon(t)$, from it. Therefore, it is naturally that

$$u_\pm(t) = u(t) \pm \varepsilon(t)$$

and the integrations over $u(t)$ and $\varepsilon(t)$ are taken independently [11]. Noticing that (i) the closed-path motion is already described and (ii) the end-points, (in, out), can not varied, the integration over $\varepsilon(t)$ must be performed with boundary conditions for initial, t_i, and final t_f, times: $\varepsilon(t_i) = \varepsilon(t_f) = 0$. Then, in the expanding in $\varepsilon(t)$,

$$S(u + \varepsilon) - S(u - \varepsilon) = \{S(u + \varepsilon) - S(u - \varepsilon)\}|_{\varepsilon=0} + 2\mathrm{Re} \int_{t_i}^{t_f} dt \frac{\delta S(u)}{\delta u(t)} \varepsilon(t) + U(u, \varepsilon),$$

one may omit for the moment higher $\varepsilon(t)$-terms noted here by $U(u, \varepsilon)$. The first term is important if the classical trajectory is a periodic function [11]. The resulted integral over $\varepsilon(t)$ gives the functional δ-function:

$$\int \prod_t d\varepsilon(t) \exp\left\{2i\mathrm{Re} \int_{t_i}^{t_f} dt \frac{\delta S(u)}{\delta u(t)} \varepsilon(t)\right\} = {\prod_t}' \delta\left(\frac{\delta S(u)}{\delta u(t)}\right). \tag{2}$$

This equality means that the continuum of contributions with $\delta S(u)/\delta u(t) \neq 0$ are canceled in the integral independently on the shape of the function $u(t)$. It is the result of destructive interference among divergent $e^{+iS(u_+)}$ and convergent $e^{-iS(u_-)}$ waves. Note that this complete cancellation is actually a result of the time reversibility.

The dynamical equilibrium, when $\delta S(u)/\delta u(t) = 0$, is the only surviving and in this case the integral (2) is infinite. This is expressed by δ-function in (2). Therefore, integration over virtual deviation, $\varepsilon(t)$, leads to the Dirac δ-like measure and and the later defines the complete set of contributions in the integrals over the trajectory $u(t)$ [11].

Returning to the d'Alambert's variational principle, one can notice that, formally, the product $\Upsilon = \varepsilon\delta S(u)/\delta u$ in the exponent in (2) represents "virtual work". But in the considered wave process case $\Upsilon \neq 0$, i.e. the quantum virtual deviations can produce some work. However, as follows from (2), the "physical" trajectory is defined by the classical Lagrange equation (1) and, therefore, is the trajectory, where dynamical equilibrium is achieved.

Thus, we get *a quantum version of the d'Alambert's variational principle.* It relies on the general principle of quantum theories that only the time reversible amplitude module is a measurable quantity.

Furthermore, this formalism allows including external forces [12]. Thus, if $U(u,\varepsilon)$ takes into account the higher nonlinear in $\varepsilon(t)$ terms, then the *strict* path integral representation of ρ is given by:

$$\rho = e^{-i\mathbf{K}(j\varepsilon)} \int \prod_t{}' du(t)\delta\left(\frac{\delta S(u)}{\delta u(t)} + j(t)\right) e^{iU(u,\varepsilon)}, \tag{3}$$

where the operator,

$$2\mathbf{K}(j\varepsilon) = \mathrm{Re} \int dt \frac{\delta}{\delta j(t)} \frac{\delta}{\delta \varepsilon(t)}, \tag{4}$$

generates a quantum perturbation series. The interaction functional $U(u,\varepsilon)$ can be easily discovered from the Lagrangian. At the very end one should take in (3) the auxiliary variables j and ε to be equal to zero.

So, one can generate the quantum perturbations including the *random* force $j(t)$ being external to the classical system. This is an authentic indication that our formulating of quantum wave mechanics resembles the classical mechanics built on the d'Alambert's variational principle base.

5. A generalization of the representation (3) to the field theory case is straightforward. One should simply replace $u(t)$ by $u(x,t)$, where x is the space coordinate and t is the time [13, 14]. The formalism does not depends a type of field is considered.

It must be noticed first of all that the functional δ-function in (3) establishes one to one correspondence among quantum and classical description [11]: the quantum case introduces the random force $j(x)$ into the system influence via the Gaussian operator $\mathbf{K}(j\varepsilon)$. Therefore, if we know the *general* solution $u_c \neq 0$, then *all* quantum states can be counted varying the integration constants as the dynamical variables.

This correspondence is extended so that allows involving such a powerful method of classical mechanics as the method of transformation of dynamical variables. It should be underlined that this method can not be adopted on the time irreversible quantum amplitudes level [12, 13].

The most useful variables are the (action,angle)-type ones [14, 15][1]. In this terms the world line of classical system (with enough constrains) moves over (Arnold's) hypertorus with radii being equal to the action variable and its location is defined by the angle variables. In the quantum case, the action of the operator $\exp\{-i\mathbf{K}\}$ leads to the Gaussian thrilling of the hypertorus. This vibration may be described in terms of random fluctuation of the action and angle variables and so that quantum perturbation gets up not in the functional space of field $u(x,t)$.

[1]In definite sense this type of variables play the same role as the "collective coordinates" of Bogolyubov [16].

A transformation to the special type variables, namely to a set of the integration constants, has been proposed in [12]-[15]. To note is that if there is not enough constrains, then the system leaves the hypertorus surface and any mapping becomes meaningless. Such a situation is realized, for example, in quantum electrodynamics, which is gauge invariant but this is not enough for the hypertorus formation. In this case ordinary method of the Faddeev-Popov *ansatz* is effective.

It should be stressed that discussed formulating of the theory is necessary and sufficient as well. The necessity follows from the fact that, in the situation of general position, the dynamics should realized on the Arnold's hypertorus [11]. The sufficiency follows from the fact that the fluctuations are defined on the Gaussian measure and so they cover the (action, angle) phase space completely.

We call such perturbation theory the *topological* theory. Indeed, the topological properties are crucial since if a trajectory on the hypertorus has definite winding number (topological charge), then the probability that after random quantum walking, the system returns to the same (or shifted on a constant winding number) position is equal to zero. Therefore, such systems have to be exactly semiclassical. The quantitative prove of this conclusion is given in [12, 13].

In the case if there is not the topological charge, the system "freely" moves along the hypertorus and can be quantum. It is indeed evident that the Yang-Mills theory in the real-time metric [15], and could be, may be the gravitation theory, are just such kind of theories.

The above described transformation to the (action, angle)-type variables reduces the field-theoretical problem to the quantum mechanical one. Generally, such a mapping is singular but it is possible to isolate these infinities (they look like divergent integrals) and remove them via the normalization. This observation is extremely important since as a result we get the quantum mechanics being theory which is free from divergences. One can mention that isolating the divergences and then canceling them the renormalization was already performed. This property is intriguing since finiteness of the theory was attained without introduction of new fields, e.g. supersymmetry partners.

It is understandable that there is not the necessity to distinguish between the symmetry and dynamical degrees of freedom in new formulation of theory. Indeed, the quantum fluctuations are realized in the (action, angle) factor space and this does not touch the action symmetry. As a result, the perturbation series consists of transparently invariant terms [15].

6. We would like to note the following fact being important from the cognition point of view. It is remarkable that restricting the problem and computing the module of amplitude, i.e. leaving the phase of amplitude undefined, the remarkable progress is shown to be achieved. So, we can say that, for instance, the Yang-Mills theory *exists*[2] as the theory of observables since in this restricted frame (i) it is free from the divergences and (ii) there is no problems [15] with necessity to distinguish between the dynamical and the symmetry degrees of freedom.

However, finally, it must be admitted that one does not know, wether the time reversibility principle, i.e. the quantum d'Alambert's principle, can be applied to the Einstein gravity? If yes, then no gravitational waves are expected to be observed despite the fact that the gravity would be the quantum theory and, moreover, such theory could be free from divergences. The

[2]See **www.claymath.org/prizeproblems** concerning more details for the question of *existence* of the Yang-Mills theory.

point is that, most likely, the gravity does not represent the dispersive media. This follows from the fact that the equation for metrics has a nontrivial solution.

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