CONTRACTIONS OF INTERBASES EXPANSIONS FOR SUBGROUP COORDINATES ON n-DIMENSIONAL SPHERE

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Abstract

Inönü - Wigner contractions from the rotation group $O(n+1)$ to the Euclidean group $E(n)$ are used to obtain the asymptotic relations for matrix elements between the eigenfunctions of the Laplace-Beltrami operator corresponding to separation of variables in the subgroup-type coordinates on the n-dimensional sphere for arbitrary n.

1 Introduction

In the present paper, we continue the series of works [1, 2, 3, 4] devoted to the Inönü-Wigner contactions [5] from rotation groups $O(n+1)$ and $O(n,1)$ to Euclidean groups $E(n)$ and their relation to separation of variables of the Laplace-Beltrami operator equation on $S_n \sim O(n+1)/O(n)$ and $E_n \sim E(n)/O(n)$. In particular, it was shown in [1, 2] that two separable coordinate systems on the sphere $S_2 \sim O(3)/O(2)$ (polar and elliptic) and nine systems on the hyperboloid $H_2 \sim O(2,1)/O(2)$ as $R \to \infty$, where R is the radius of the sphere, can be contracted to four separable systems on the plane E_2. In the paper [3], the case of contraction from the two-dimensional hyperboloid H_2 to the pseudo-Euclidean plane $E_{1,1} \sim E(1,1)/O(2)$ was also investigated.

The contractions on $S_n \sim O(n+1)/O(n)$ for the case of arbitrary dimensions have been described in the article [4], where we considered different type of hyperspherical coordinates, namely, subgroup-type coordinates. It was shown that the contractions related the graphical formalism of "trees" (introduced by Vilenkin, Kuznetsov and Smorodinsky in [6]) on spheres S_n to the "clusters" on the Euclidean space E_n, introduced in the same article [4].

The topologically different sets of bases corresponding to different trees, are related by unitary transformations. The matrix elements of transformations between different bases

*International Center of Advanced Study, Yerevan State University, Armenia (or overlap functions) of represantation for the group $O(n+1)$, i.e., between different types of hyperspherical functions are called "T-coefficients" and have been calculated explicitly in [7].

In the present work, we study the contractions in the limit $R \to \infty$ for T-coefficients and interbases expansions between eigenfunctions of the Laplace-Beltrami operators corresponding to different trees on the S_n sphere. Note that the case of $n = 2, 3$-spheres has already been considered in the paper [8].

2 Method of trees and overlap functions

2.1 Method of trees

Let us consider the Laplace-Beltrami (or Helmholtz) equation on the n-dimensional sphere S_n

$$\Delta_{LB}\Psi = -E\Psi, \qquad \Delta_{LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi_i}\sqrt{g}\, g^{ik}\frac{\partial}{\partial \xi_k}, \quad g = \det g_{ik} \tag{1}$$

where g_{ik} is a metric tensor written in the curvlinear coordinates ξ_i.

The separated eigenfunction of Δ_{LB} can be characterized as common eigenfunctions of a complete set of commuting operators Y_a, $a = 1, 2, .., n$

$$Y_a\Psi = -\lambda_a\Psi, \qquad [Y_a, Y_b] = 0, \qquad \Psi = \prod_{i=1}^{n} \Psi_i(\xi_i, \lambda_1, \lambda_2, ..., \lambda_n). \tag{2}$$

The set of operators $\{Y_a, a = 1, 2, .., n\}$ includes the Laplace-Beltrami operator and consists of second order operators in the enveloping algebra of $o(n+1)$. The simplest types of coordinates are obtained if all operators Y_a in the set are Casimir operators of subalgebras of $o(n+1)$. The corresponding coordinates are called subgroup-type coordinates.

Smorodinsky, Vilenkin and Kuznetsov introduced graphical methods, the "methods of trees", for characterizing different types of subgroup coordinates, or hyperspherical coordinates on S_n. These methods are best presented in the original article [6] and in the books [10] and [9]).

Let us briefly describe the method of trees [6]. Each end point $u_i, i = 0, 1, 2, ..., n$ on the tree corresponds to a Cartesian coordinate in the ambient space E_{n+1}. At each branching point, we introduce an angle θ_j. We move along the tree from the ground upwards to a specific coordinate u_i. At each branching point, we write $\cos\theta_j$, if we go to the left, and $\sin\theta_j$, if we go to the right. In this case, the coordinate u_i may be represented as a product of all the lines coming toward itself. For example, to the tree on Fig.1 there correspond the following polyspherical coordinates:

$$\begin{aligned}
u_0 &= R\cos\theta_1\cos\theta_2, & u_1 &= R\cos\theta_1\sin\theta_2\cos\theta_3,\\
u_2 &= R\cos\theta_1\sin\theta_2\sin\theta_3, & u_3 &= R\sin\theta_1\cos\theta_4\cos\theta_5,\\
u_4 &= R\sin\theta_1\cos\theta_4\sin\theta_5 & u_5 &= R\sin\theta_1\sin\theta_4,
\end{aligned}$$

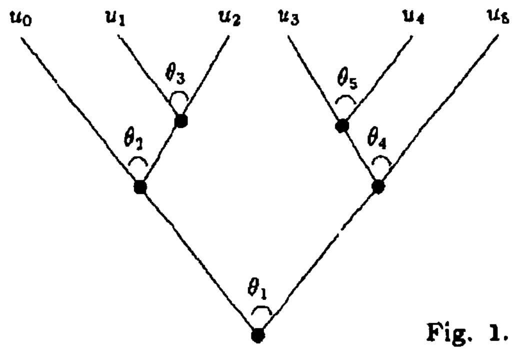

Fig. 1.

To each branching point on the tree diagram there also correspond **non-negative** quantum numbers l_j. It will determine the eigenvalue λ_j of the Laplace-Beltrami operators according to the formula

$$Y_j = R^2\Delta_{LB}\Psi = -\lambda_j\Psi, \qquad \lambda_j = l_j(l_j + k - 2), \tag{3}$$

where k is the dimension of the ambient space above the corresponding vertex on the tree. Only for $k = 2$ we have $l_j = 0, \pm 1, \pm 2, \ldots$. To specify the separated wave function

$$\Psi = \Pi_{j=1}^{n}\Psi_j(\theta_j) \tag{4}$$

on S_n, we follow Refs. [6, 10, 9] and introduce four types of vertices, or "cells" on a tree, as illustrated in Fig.2(1a, 1b, 1b', 1c). Each vertex and each angle θ_j provides a "building block" $\Psi_j(\theta_j)$ for the wave function $\Psi(\theta_1, \ldots, \theta_n)$. Specifically, we have

Cell of type 1a:

$$\Psi_m(\theta_a) = \frac{1}{\sqrt{2\pi}}e^{im\theta_a}; \quad m = 0, \pm 1, \pm 2, \ldots; \; 0 \le \theta_a < 2\pi. \tag{5}$$

Cell of type 1b:

$$\Psi^{c}_{n,l_\beta}(\theta_b) = N_n^{c,c}(\sin\theta_b)^{l_\beta}P_n^{(c,c)}(\cos\theta_b) \tag{6}$$

$$n = l - l_\beta, \; c = l_\beta + \frac{S_\beta}{2}, \; n = 0, 1, 2, \ldots; \; 0 \le \theta_b \le \pi,$$

where $P_n^{(a,b)}(x)$ are the Jacobi polynomials.

Cell of type 1b′:

$$\Psi^{a}_{n,l_\alpha}(\theta_{b'}) = N_n^{a,a}(\cos\theta_{b'})^{l_\alpha}P_n^{(a,a)}(\sin\theta_{b'}) \tag{7}$$

$$n = l - l_\alpha, \; a = l_\alpha + \frac{S_\alpha}{2}, \; n = 0, 1, 2, \ldots; \; -\pi/2 \le \theta_{b'} \le \pi/2.$$

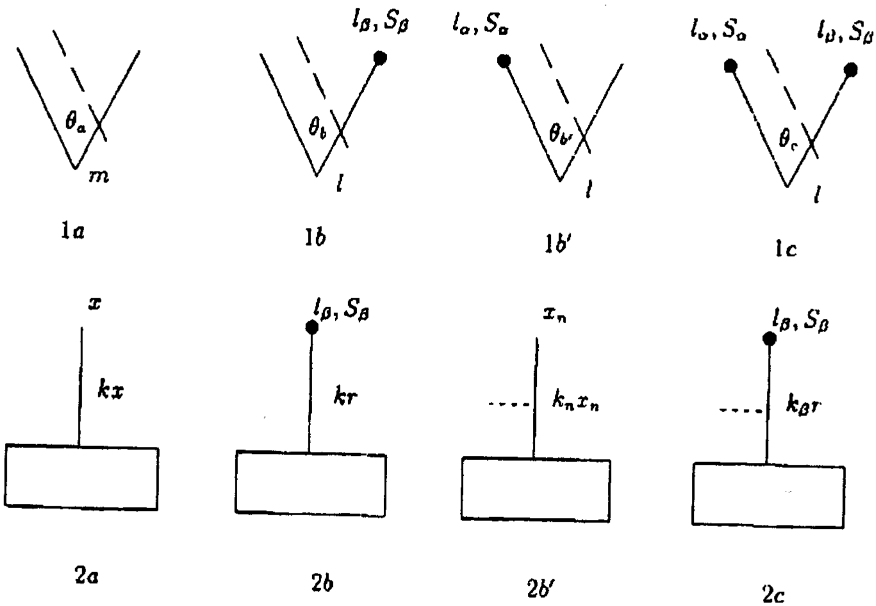

Fig. 2. Elementary cells for S_n (diagrams $1a, \ldots, 1c$) and their contractions to E_n ones (diagrams $2a, \ldots, 2c$)

Cell of type 1c:

$$\Psi^{b,a}_{n,l_\beta,l_\alpha}(\theta_c) = 2^{(b+a)/2+1} N^{b,a}_n (\sin\theta_c)^{l_\beta} (\cos\theta_c)^{l_\alpha} P^{(b,a)}_n(\cos 2\theta_c), \tag{8}$$

$$n = \frac{l - l_\alpha - l_\beta}{2},\ b = l_\beta + \frac{S_\beta}{2},\ a = l_\alpha + \frac{S_\alpha}{2},\ n = 0, 1, 2, \ldots;\ 0 \le \theta_c \le \pi/2.$$

Here, S_α and S_β are the numbers of vertices above the vertex l_α and l_β, respecively. The normalization constants are

$$N^{a,b}_n = \left\{ \frac{(2n + a + b + 1)\Gamma(n + a + b + 1)n!}{2^{a+b+1}\Gamma(n + a + 1)\Gamma(n + b + 1)} \right\}^{1/2}.$$

We mention that the wave functions (6) and (7) can also be expressed in terms of the Gegenbauer polynomials by using the formula [11]:

$$C^\lambda_n(x) = \frac{\Gamma(2\lambda + n)\Gamma(\lambda + 1/2)}{\Gamma(2\lambda)\Gamma(\lambda + n + 1/2)} P^{(\lambda-1/2,\lambda-1/2)}_n(x).$$

2.2 Relations between hyperspherical functions

A convenient way of calculating the T-coefficients corresponding to a transformation from one tree to another, is to introduce a sequence of "*elementary*" trees, each differing from the previous one by the transplantation of exactly one branch from one side of a branching point to the other. The general T-matrix will be factorized into a product of "*elementary T-matrices*" corresponding to such elementary transformations. Each elementary T matrix connects two tree-type diagrams, a cell with 3 ends, each of which can be either open or closed. The "open end" means that the tree above has no branches, on the contrary, the "closed end" means that the tree above has a number of branches, according to the dimension n of the corresponding sphere S_n. All together 8 inequivalent elementary diagrams of this type exist: one with 3 open ends, 3 with 2 open ends, 3 with one open end and one with three closed ones (see Fig.3). The T-coefficients for all 8 types of elementary transformations were calculated by Kil'dyushov [7]. They were expressed in terms of generalized hypergeometric functions of argument $x = 1$: ${}_3F_2(1)$, ${}_4F_3(1)$, Wigner D-functions, or Clebsch-Gordon and Racah coefficients for positive discrete series of representations of the group $SU(1,1)$ [12]. We mention that a relation between the T-coefficients and polynomials of discrete variables has been established [14], namely, the Racah-Wilson, Hahn and Krawtchouk polynomials.

The T coefficient, representing the **general transformation**, corresponds to the diagram with three closed ends on Fig.3(1) [7]:

$$T_{Jlm}^{\alpha\beta\gamma} = \frac{\sqrt{(l + \frac{S_\beta+S_\gamma}{2} + 1)(m + \frac{S_\alpha+S_\beta}{2} + 1)(\frac{J-m-\gamma}{2})!}}{\Gamma(\beta + \frac{S_\beta}{2} + 1)} \frac{\Gamma(\frac{J-\alpha-\beta+\gamma+S_\gamma}{2} + 1)}{\Gamma(\frac{J+\alpha+\beta-\gamma+S_\alpha+S_\beta}{2} + 2)}$$

$$\times \left\{ \frac{\Gamma(\frac{l+\beta+\gamma+S_\beta+S_\gamma}{2} + 1)\Gamma(\frac{l+\beta-\gamma+S_\beta}{2} + 1)\Gamma(\frac{J+\alpha+l+S_\alpha+S_\beta+S_\gamma}{2} + 2)}{\Gamma(\frac{m+\alpha-\beta+S_\alpha}{2} + 1)\Gamma(\frac{l-\beta+\gamma+S_\gamma}{2} + 1)\Gamma(\frac{J+m+\gamma+S_\alpha+S_\beta+S_\gamma}{2} + 2)} \right\}^{1/2}$$

$$\times \left\{ \frac{\Gamma(\frac{J+\alpha-l+S_\alpha}{2} + 1)\Gamma(\frac{J+m-\gamma+S_\alpha+S_\beta}{2} + 2)\Gamma(\frac{m+\alpha+\beta+S_\alpha+S_\beta}{2} + 1)\Gamma(\frac{m-\alpha+\beta+S_\beta}{2} + 1)}{(\frac{m-\alpha-\beta}{2})!(\frac{l-\beta-\gamma}{2})!(\frac{J-l-\alpha}{2})!\Gamma(\frac{J+l-\alpha+S_\beta+S_\gamma}{2} + 2)\Gamma(\frac{J-m+\gamma+S_\gamma}{2} + 1)} \right\}^{1/2}$$

$$\times {}_4F_3 \left\{ \begin{matrix} -\frac{m-\alpha-\beta}{2}, & \frac{m+\alpha+\beta+S_\alpha+S_\beta}{2} + 1, & \frac{l-\gamma+\beta+S_\beta}{2} + 1, & -\frac{l-\beta+\gamma+S_\gamma}{2}; \\ \beta + \frac{S_\beta}{2} + 1, & \frac{J-\gamma+\alpha+\beta+S_\alpha+S_\beta}{2} + 2, & -\frac{J-\alpha-\beta+\gamma+S_\gamma}{2} & \end{matrix} \middle| 1 \right\}. \tag{9}$$

Here, for brevity we replace the numbers $(l_\alpha, l_\beta, l_\gamma)$ by (α, β, γ) and S_{α_j} $(\alpha_j = \alpha, \beta, \gamma)$ is the number of knots above the knot α_j.

It was pointed out in Refs. [14, 13] that the numbers $S_{\alpha_j} = -1$, $\alpha_j = 0, 1$ determine the appropriate open end. In this case, one obtains values of the T-matrix for transitions in cells with one open end. The transition matrices, practically for all 8 types of T-"cells" in Fig.3 (up to a phase factor!), can be obtained in this manner and presented in the appendix.

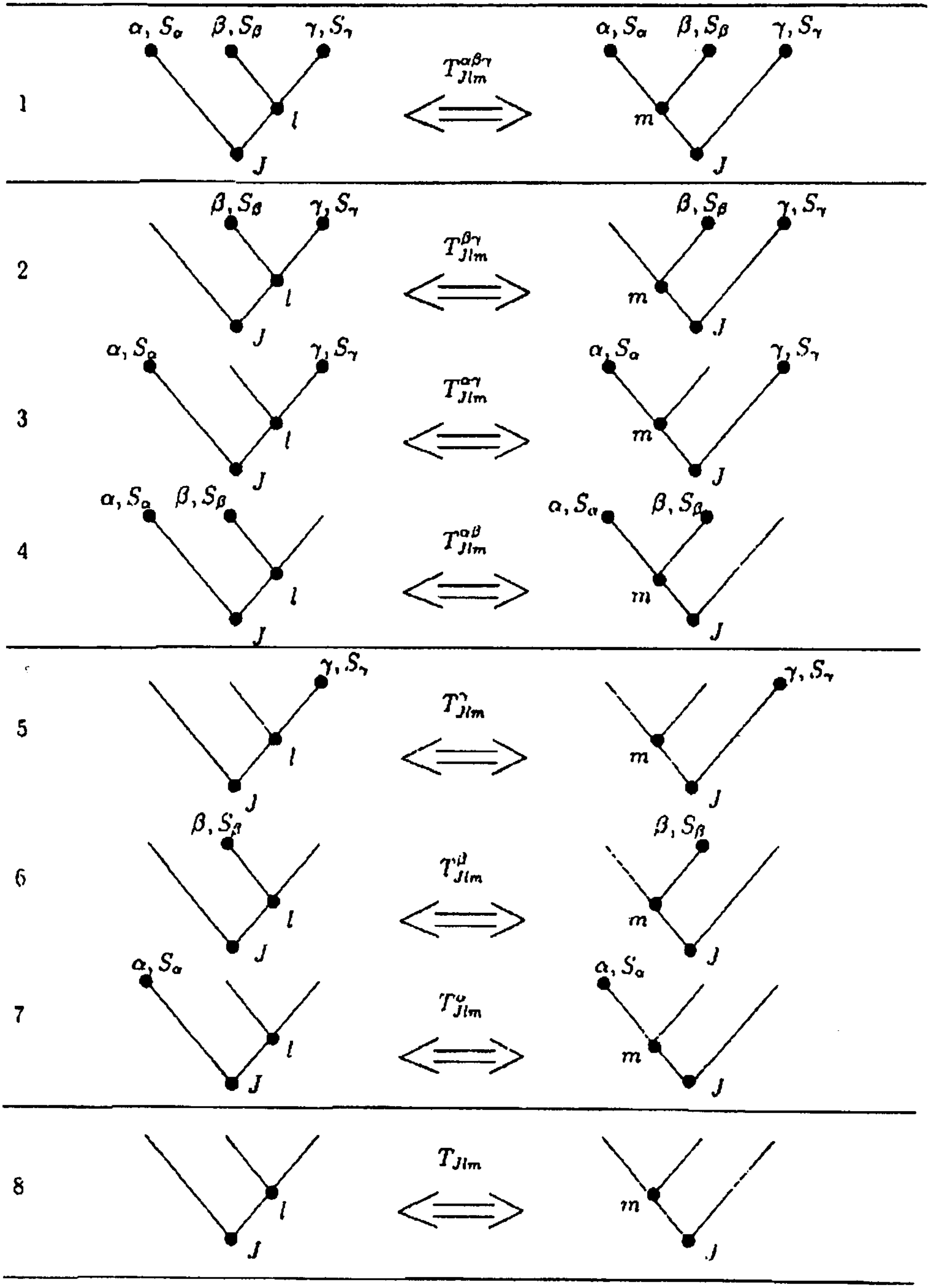

Fig. 3. Diagrams representing elementary"transformations" between trees

3 Contractions of the basis functions

Let us consider the n-dimensional sphere S_n: $u_0^2 + \sum_{\nu=1}^n u_\nu^2 = R^2$, $R > 0$, where u_μ are Cartesian coordinates in the Euclidean ambient space E_{n+1}. The isometry group is $O(n+1)$. We choose a standard basis $L_{\mu\nu}$ for the Lie algebra $o(n+1)$

$$L_{ik} = (u_i\partial_k - u_k\partial_i) \tag{10}$$

$$[L_{ik}, L_{mn}] = \delta_{km}L_{in} + \delta_{in}L_{km} - \delta_{im}L_{kn} - \delta_{kn}L_{im}. \qquad 0 \le i,k,m,n \le n \tag{11}$$

The Laplace-Beltrami operator on S_n is

$$\Delta_{LB} = \frac{1}{R^2}\sum_{0\le i<k\le n} L_{ik}^2 \tag{12}$$

We shall use R^{-1} as a contraction parameter. To realize the contraction explicitly, let us introduce Beltrami coordinates on the sphere S_n putting

$$y_i = R\frac{u_i}{u_0} = u_i\left(1 - \frac{1}{R^2}\sum_{k=1}^n u_k^2\right)^{-1/2}, \quad i = 1,2,3,\ldots,n. \tag{13}$$

Then, the $O(n+1)$ generators can be expressed as

$$\frac{L_{0i}}{R} \equiv \pi_i = p_i + \frac{y_i}{R^2}\sum_{k=1}^n (y_k p_k), \tag{14}$$

$$L_{ik} \equiv y_i p_k - y_k p_i = y_i\pi_k - y_k\pi_i; \quad i,k = 1,2,\ldots,n, \tag{15}$$

where $p_i = \partial/\partial y_i$. The commutation relations now are

$$[L_{ik}, L_{mn}] = \delta_{km}L_{in} + \delta_{in}L_{km} - \delta_{im}L_{kn} - \delta_{kn}L_{im}, \tag{16}$$

$$[\pi_i, L_{kj}] = \delta_{ik}\pi_j - \delta_{ij}\pi_k, \quad [\pi_i, \pi_k] = \frac{L_{ik}}{R^2}, \tag{17}$$

so that as $R \to \infty$ the $o(n+1)$ algebra contracts to the Euclidean $e(n)$ one. The Beltrami coordinates y_i (13) contract to the Cartesian coordinates on E_n, and we have

$$y_i \to x_i, \quad \pi_i \to p_i = \frac{\partial}{\partial x_i}, \tag{18}$$

so that the rotation generators L_{0i} turn into the translations p_i. The $o(n+1)$ Laplace-Beltrami operator (12) contracts to the $e(n)$ one

$$\Delta_{LB} = \sum_{i=1}^n \pi_i^2 + \sum_{i,k=1}^n \frac{L_{ik}^2}{2R^2} \to \Delta = p_1^2 + p_2^2 + \cdots + p_n^2. \tag{19}$$

Recently, in [4] we have introduced the *graphical methods* of connecting the subgroup-type coordinates systems on S_n (characterized by tree diagrams) and E_n (characterized by cluster diagrams) and gave the rules relating the contraction limit $R \to \infty$ of the coordinates, invariant operators, eigenvalues and basis functions.

Graphically, the contraction $R \to \infty$ corresponds to the cut line off the ground along the u_0 branch by the dashed line, as represented for the general S_n tree diagram in Fig.4(a). The dashed line then becomes the ground for the corresponding cluster E_n diagram of Fig.4(b), and the ambient space coordiantes $(u_0, u_1, ..., u_n)$ for S_n are transformed to the Cartesian coordinates $(x_1, x_2, ..., x_n)$. The angles $\theta_1, \theta_2, \ldots, \theta_j$ and the angular momentum quantum numbers $l_1, l_2, ..., l_j$ leading to the branches, cut off by the dotted line, satisfy $\theta_i \to 0$ and $l_i \to \infty$ in the contraction and are replaced by the radial coordinates r_i, or Cartesian coordinates x_m (if the survived branch leads directly to a single coordinate on S_n and E_n) and some constants k_i. We have

$$\theta_j \sim \frac{r_j}{R}, \quad l_j \sim k_j R, \quad R \to \infty. \tag{20}$$

When we cut off the branches of a tree as in Fig.4(a), the cutting line intersects an elementary cell at the branch (see Fig.2) and each elementary cell in Fig.2(2a) then goes into an elementary trunk, as indicated on Fig.2(2b). The limiting procedure for cells is always the same as in eq. (20).

Let us now run the contraction of basis functions through the individual cells in Fig.2.

Cell 1a to 2a:

In the contraction limit $R \to \infty$, $m \sim kR$, $\theta_a \sim x/R$ we have

$$\Phi_k(x) = \lim_{R\to\infty} \Psi_m(\theta_a) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} e^{im\theta_a} = \frac{1}{\sqrt{2\pi}} e^{ikx}. \tag{21}$$

Cell 1b to 2b:

In the contraction limit $l \sim kR$, $\theta_b \sim r/R$ we have

$$\begin{aligned}\Phi^c_{k,l_\beta}(r) = \lim_{R\to\infty} \frac{1}{\sqrt{R^{S_\beta+1}}} \Psi^c_{n,l_\beta}(\theta_b) &= \lim_{R\to\infty} \frac{N_{l-l_\beta}^{l_\beta+\frac{S_\beta}{2}, l_\beta+\frac{S_\beta}{2}}}{\sqrt{R^{S_\beta+1}}} (\sin\theta_b)^{l_\beta} P_{l-l_\beta}^{(l_\beta+\frac{S_\beta}{2}, l_\beta+\frac{S_\beta}{2})}(\cos\theta_b) \\ &= \sqrt{\frac{k}{r^{S_\beta}}} J_{l_\beta+\frac{S_\beta}{2}}(kr).\end{aligned} \tag{22}$$

Cell 1b′ to 2b′:

In the limit $R \to \infty$ and $\theta_{b'} \sim x_n/R$, $l \sim kR$, $l_\alpha \sim pR$ we have

$$\begin{aligned}\Phi^a_{k,p}(x_n) &= \lim_{R\to\infty} (-1)^{\frac{l-l_\alpha}{2}} \Psi^a_{n,l_\alpha}(\theta_{b'}) = \lim_{R\to\infty} (-1)^{\frac{l-l_\alpha}{2}} N_{l-l_\alpha}^{l_\alpha+\frac{S_\alpha}{2}, l_\alpha+\frac{S_\alpha}{2}} (\cos\theta_{b'})^{l_\alpha} \\ &\times P_{l-l_\alpha}^{(l_\alpha+\frac{S_\alpha}{2}, l_\alpha+\frac{S_\alpha}{2})}(\sin\theta_{b'}) = \sqrt{\frac{2k}{\pi k_n}} \begin{cases} \cos(k_n x_n) & (l-l_\alpha) - \text{even}, \\ -i\sin(k_n x_n) & (l-l_\alpha) - \text{odd}, \end{cases}\end{aligned} \tag{23}$$

where $k^2 = k_n^2 + p^2$.

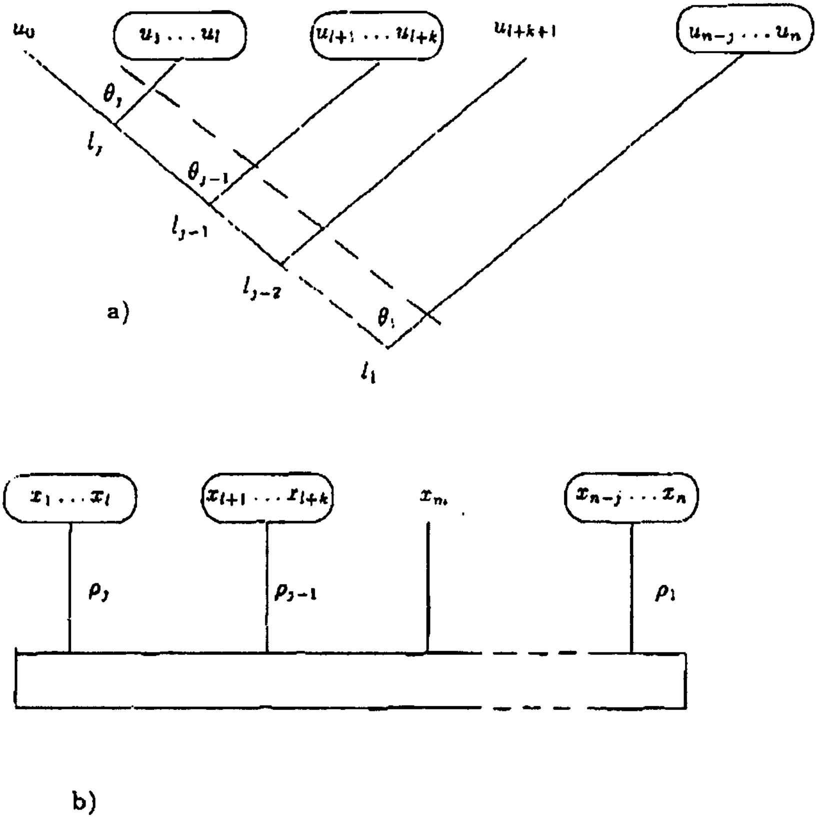

Fig. 4. Contractions of tree diagrams into cluster ones for S_n

Cell 1c to 2c:

In the limit $R \to \infty$ $l \sim kR$, $l_\alpha \sim k_\alpha R$ and $\theta_c \sim r/R$, we have

$$\Phi^{l_\beta}_{k,k_\beta,k_\alpha}(r) = \lim_{R\to\infty} \frac{\Psi^{b,a}_{n,l_\beta,l_\alpha}(\theta_c)}{\sqrt{R^{S_\beta+1}}} = \lim_{R\to\infty} \sqrt{\frac{2^{l_\alpha+l_\beta+(S_\alpha+S_\beta)/2+2}}{R^{S_\beta+1}}}\, N^{l_\beta+\frac{S_\beta}{2},\, l_\alpha+\frac{S_\alpha}{2}}_{\frac{(l-l_\alpha-l_\beta)}{2}} (\sin\theta_c)^{l_\beta}$$

$$\times \quad (\cos\theta_c)^{l_\alpha} P^{(l_\beta+\frac{S_\beta}{2},\, l_\alpha+\frac{S_\alpha}{2})}_{\frac{l-l_\alpha-l_\beta}{2}}(\cos 2\theta_c) = \sqrt{\frac{2k}{r^{S_\beta}}}\, J_{l_\beta+\frac{S_\beta}{2}}(k_\beta r), \qquad (24)$$

where $k^2 = k_\alpha^2 + k_\beta^2$ and the parameters a, b, and c are determined by formulae (6), (7), and (8).

Thus, using these contractions for basis functions of the elementary cells we go from (1a, ..., 1c) to the (2a, ..., 2c) (see Fig.2) and determine the general contractions for hyperspherical functions corresponding to any tree on the spheres S_n (see Fig.4).

4 Contractions of the interbases expansions

Having the explicit form of the T coefficients for all eight types of "elementary" trees transitions we shall now consider the contraction limit $R \to \infty$ for the interbasis expansions in Fig.5.

1. Contraction of Racah coefficients

The tree on the left-hand side of Fig.5(a) corresponds to the subgroup chains $O(n+1) \supset O(n_\alpha + n_\beta) \otimes O(n_\gamma)$ while the tree on the right-hand side of Fig.5(a) corresponds to the chain $O(n+1) \supset O(n_\alpha) \otimes O(n_\gamma + n_\beta)$, where $n+1 = n_\alpha + n_\gamma + n_\beta$.

The interbasis expansion corresponding to the transformations between trees on Fig.5(a) has the form

$$\Psi^{\alpha\beta\gamma}_{Jm}(\theta_1', \theta_2') = \sum_{l=\beta+\gamma, \beta+\gamma+2, \dots}^{J-\alpha} T^{\alpha\beta\gamma}_{Jlm} \, \Psi^{\alpha\beta\gamma}_{Jl}(\theta_1, \theta_2), \tag{25}$$

where

$$\cos\theta_1 = \cos\theta_1' \cos\theta_2' \qquad \cot\theta_2 = \cot\theta_1' \sin\theta_2'.$$

The T coefficients are given by formula (9), and the wave functions Ψ can be obtained with the help of the rules of section 2 [eq. (8)].

Consider now the contraction limit $R \to \infty$ in the interbasis expansion (25). For large R we put

$$J \sim kR, \qquad m \sim pR, \qquad \alpha \sim qR, \qquad \theta_1' \sim \frac{r_\gamma}{R}, \qquad \theta_2' \sim \frac{r_\beta}{R}, \qquad \theta_1 \sim \frac{r_{\beta\gamma}}{R}. \tag{26}$$

where $r_{\beta\gamma} = \sqrt{r_\beta^2 + r_\gamma^2}$, $k_\beta^2 = p^2 - q^2$, $k_\gamma^2 = k^2 - p^2$ and $k_{\beta\gamma}^2 = k_\beta^2 + k_\gamma^2$ and we have

$$\lim_{R\to\infty} \Psi^{\alpha\beta\gamma}_{Jm}(\theta_1', \theta_2') = \Phi^{\beta\gamma}_{kk_\beta k_\gamma}(r_\beta, r_\gamma) = \frac{2\sqrt{kp}}{(r_\beta)^{\frac{S_\beta}{2}} (r_\gamma)^{\frac{S_\gamma}{2}}} J_{\beta+\frac{S_\beta}{2}}(k_\beta r_\beta) \, J_{\gamma+\frac{S_\gamma}{2}}(k_\gamma r_\gamma), \tag{27}$$

$$\lim_{R\to\infty} \Psi^{\alpha\beta\gamma}_{Jl}(\theta_1, \theta_2) = \Phi^{l\beta\gamma}_{kk_{\beta\gamma}}(r_{\beta\gamma}, \theta_2) = \frac{\sqrt{2k(2l + S_\beta + S_\gamma + 2)}}{(r_{\beta\gamma})^{1+\frac{S_\beta+S_\gamma}{2}}} J_{l+\frac{S_\beta+S_\gamma}{2}+1}(k_{\beta\gamma} r_{\beta\gamma})$$

$$\times \sqrt{\frac{\Gamma(\frac{l+S_\beta+S_\gamma+\beta+\gamma}{2}+1)(\frac{l-\beta-\gamma}{2}+1)!}{\Gamma(\frac{l-\beta+\gamma+S_\gamma}{2}+1)\Gamma(\frac{l+\beta-\gamma+S_\beta}{2}+1)}} (\sin\theta_2)^\gamma (\cos\theta_2)^\beta \, P^{(\gamma+\frac{S_\gamma}{2}, \beta+\frac{S_\beta}{2})}_{\frac{l-\beta-\gamma}{2}}(\cos 2\theta_2). \tag{28}$$

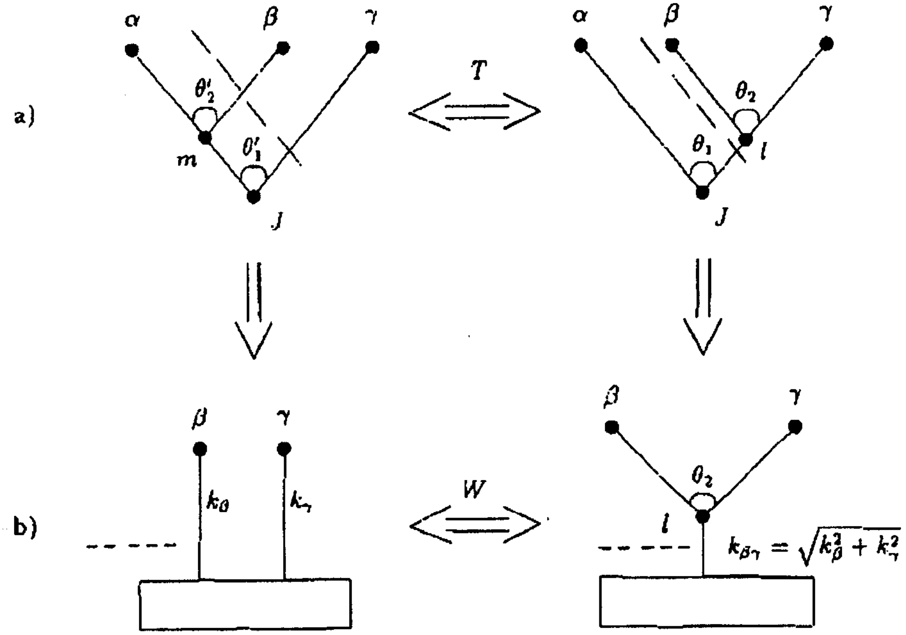

Fig. 5. Contractions of the general case Fig. 3(1)

Using now the asymptotic formulae for the ${}_4F_3$ functions and Γ functions [11] in eq. (9), we obtain

$$\lim_{R\to\infty} T^{\alpha\beta\gamma}_{Jlm} = W^{l\beta\gamma}_{kk_{\beta\gamma}k_\beta k_\gamma} = \sqrt{\frac{2p(2l+S_\beta+S_\gamma+2)\Gamma(\frac{l+\beta+\gamma+S_\beta+S_\gamma}{2}+1)\Gamma(\frac{l+\beta-\gamma+S_\beta+S_\gamma}{2}+1)}{(\frac{l-\beta-\gamma}{2})!\Gamma(\frac{l-\beta+\gamma+S_\gamma}{2}+1)[\Gamma(\beta+\frac{S_\beta}{2}+1)]^2}}$$

$$\frac{k_\beta^{\beta+\frac{S_\beta}{2}} k_\gamma^{\gamma+\frac{S_\gamma}{2}}}{k_{\beta\gamma}^{\beta+\gamma+\frac{S_\beta+S_\gamma}{2}+1}} \; {}_2F_1\left(-\frac{l-\beta-\gamma}{2}, \frac{l+\beta+\gamma+S_\gamma+S_\beta}{2}+1; \beta+\frac{S_\beta}{2}+1; \frac{k_\beta^2}{k_\beta^2+k_\gamma^2}\right)$$

$$= (-1)^{\frac{l-\beta-\gamma}{2}} \sqrt{\frac{2p(2l+S_\gamma+S_\beta+2)(\frac{l-\beta-\gamma}{2})!\Gamma(\frac{l+\beta+\gamma+S_\beta+S_\gamma}{2}+1)}{k_{\beta\gamma}^2\Gamma(\frac{l+\beta-\gamma+S_\beta}{2}+1)\Gamma(\frac{l-\beta+\gamma+S_\gamma}{2}+1)}} \times$$

$$\times (\cos\phi)^{\beta+\frac{S_\beta}{2}} (\sin\phi)^{\gamma+\frac{S_\gamma}{2}} P^{(\gamma+\frac{S_\gamma}{2},\beta+\frac{S_\beta}{2})}_{\frac{l-\beta-\gamma}{2}}(\cos 2\phi), \tag{29}$$

Taking now the contraction limit $R \to \infty$ in (25), we get $(\theta_2 \equiv \theta)$

$$\Phi^{\beta\gamma}_{kk_\beta k_\gamma}(r_\beta, r_\gamma) = \sum_{l=\beta+\gamma,\beta+\gamma+2,\ldots}^{\infty} W^{l\beta\gamma}_{kk_{\beta\gamma}k_\beta k_\gamma} \Phi^{l\beta\gamma}_{kk_{\beta\gamma}}(r_{\beta\gamma}, \theta) \tag{30}$$

Using the orthogonality condition for the Jacobi polynomials

$$\int_0^{k_{\beta\gamma}} W^{l\beta\gamma}_{kk_{\beta\gamma}k_\beta k_\gamma} W^{l'\beta\gamma *}_{kk_{\beta\gamma}k_\beta k_\gamma} \frac{k_\beta}{\sqrt{k^2 - k_\gamma^2}} dk_\beta = \delta_{ll'},$$

we get the inverse expansion

$$\Phi^{l\beta\gamma}_{kk_{\beta\gamma}}(r_{\beta\gamma}, \theta) = \int_0^{k_{\beta\gamma}} W^{l'\beta\gamma *}_{kk_{\beta\gamma}k_\beta k_\gamma} \Phi^{\beta\gamma}_{kk_\beta k_\gamma}(r_\beta, r_\gamma) \frac{k_\beta}{\sqrt{k^2 - k_\gamma^2}} dk_\beta \tag{31}$$

Putting now the functions (27) - (28) and interbasis coefficients (29) into the expansions (30) and (31), we obtain

$$J_{\beta+\frac{S_\beta}{2}}(z\cos\theta\cos\phi)\, J_{\gamma+\frac{S_\gamma}{2}}(z\sin\theta\sin\phi) = (\sin\phi\sin\theta)^{\gamma+\frac{S_\gamma}{2}} (\cos\phi\cos\theta)^{\beta+\frac{S_\beta}{2}}$$

$$\times \sum_{l=\beta+\gamma,\beta+\gamma+2,\ldots}^{\infty} (-1)^{\frac{l-\beta-\gamma}{2}} \frac{(2l+S_\beta+S_\gamma+2)}{z} \frac{\Gamma(\frac{l+S_\beta+S_\gamma+\beta+\gamma}{2}+1)(\frac{l-\beta-\gamma}{2}+1)!}{\Gamma(\frac{l-\beta+\gamma+S_\gamma}{2}+1)\Gamma(\frac{l+\beta-\gamma+S_\beta}{2}+1)}$$

$$\times P^{(\gamma+\frac{S_\gamma}{2},\beta+\frac{S_\beta}{2})}_{\frac{l-\beta-\gamma}{2}}(\cos 2\phi)\; P^{(\gamma+\frac{S_\gamma}{2},\beta+\frac{S_\beta}{2})}_{\frac{l-\beta-\gamma}{2}}(\cos 2\theta) J_{l+\frac{S_\beta+S_\gamma}{2}+1}(z) \tag{32}$$

and

$$\frac{(-1)^{\frac{l-\beta-\gamma}{2}}}{z} J_{l+\frac{S_\beta+S_\gamma}{2}+1}(z)\, (\sin\theta)^{\gamma+\frac{S_\gamma}{2}} (\cos\theta)^{\beta+\frac{S_\beta}{2}} P^{(\gamma+\frac{S_\gamma}{2},\beta+\frac{S_\beta}{2})}_{\frac{l-\beta-\gamma}{2}}(\cos 2\theta) = \int_0^{\frac{\pi}{2}} (\sin\phi)^{\gamma+\frac{S_\gamma}{2}+\frac{1}{2}}$$

$$\times J_{\beta+\frac{S_\beta}{2}}(z\cos\theta\cos\phi)\, J_{\gamma+\frac{S_\gamma}{2}}(z\sin\theta\sin\phi)(\cos\phi)^{\beta+\frac{S_\beta}{2}+\frac{1}{2}} P^{(\gamma+\frac{S_\gamma}{2},\beta+\frac{S_\beta}{2})}_{\frac{l-\beta-\gamma}{2}}(\cos 2\phi)\, d\phi \tag{33}$$

where $z \equiv k_{\beta\gamma} r_{\beta\gamma}$ and

$$r_\beta = r_{\beta\gamma}\cos\theta, \qquad r_\gamma = r_{\beta\gamma}\sin\theta,$$

$$k_\beta = k_{\beta\gamma}\cos\phi, \qquad k_\gamma = k_{\beta\gamma}\sin\varphi,$$

The last two expansions are equivalent to the well-known formulae in the theory of Bessel functions [11], namely, expansions of two Bessel functions through the Bessel function and two Jacobi polynomials, and vice versa.

The entire procedure of contraction is illustraded in Fig.5. The vertical arrows correspond to the contraction (26) from the S_n trees to the E_n clusters. The first tree in Fig.5(a) contracts to the bihyperspherical coordinates; the second, to hyperspherical ones. The contraction of the coefficients T or overlap functions is given by eq. (29): the asymptotic formula for the Racah coefficients where the three momenta $J, m, \alpha \to \infty$. The interbasis expansion (30) and its inverse integral expansion (31) between two E_n cluster diagrams (see Fig.5(b)) are obtained from the interbasis expansions (25) of the S_n trees diagrams, i.e., between the bihyperspherical and hyperspherical bases for the Helmholtz equation on E_n.

The contraction limit $R \to \infty$ of the interbasis expansion corresponding to Fig.3(2) (with the open α end) can be obtained from eqs.(30) and (31) with the substitutions $q^2 = 0$, $r = r_{\beta\gamma}$ and $k = k_{\beta\gamma}$.

4.2 Further contraction of Racah coefficients

The two trees in Fig.6 correspond to two subgroup reductions: $O(n+1) \supset O(n_\alpha + 1) \otimes O(n_\gamma)$ on the left side and $O(n+1) \supset O(n_\alpha) \otimes O(n_\gamma + 1)$ on the right. Since the overlap functions are again expressed in terms of the Racah coefficients (see eq. (4)). The corresponding interbasis expansion is

$$\Psi^{\alpha\gamma}_{Jm}(\theta'_1, \theta'_2) = \sum_{l=\gamma,\gamma+1}^{J-\alpha} T^{\alpha\gamma}_{Jlm} \, \Psi^{\alpha\gamma}_{Jl}(\theta_1, \theta_2), \tag{34}$$

where the T coefficients are given by (4) and the hyperspherical functions Ψ in both the sides of eq. (34) can be written with the help of the formulae of section 2. Since the quantum numbers $J - m - \gamma$ and $J - l - \alpha$ are even, $l - \gamma + m - \alpha$ are also even, and in the expansion (34) we have $l = \gamma, \gamma + 2, ...J - \alpha$ for the $m - \alpha$ even parameter and $l = \gamma + 1, \gamma + 3, ...J - \alpha$ for the $m - \alpha$ odd parameter.

As in the previous case, the contraction will involve three quantum numbers: J, m, and α. In the contraction limit $R \to \infty$,

$$J \sim kR, \qquad m \sim pR, \qquad \alpha \sim qR, \qquad \theta'_1 \sim \frac{r_\gamma}{R}, \qquad \theta'_2 \sim \frac{x}{R}, \qquad \theta_1 \sim \frac{\sqrt{r_\gamma^2 + x^2}}{R}, \tag{35}$$

where $k_\gamma^2 = k^2 - p^2$, $k_x^2 = p^2 - q^2$, $k_{x\gamma}^2 = k_x^2 + k_\gamma^2$, we obtain

$$\lim_{R\to\infty} (-1)^{\frac{m+\alpha}{2}} \, \Psi^{\alpha\gamma}_{Jm}(\theta'_1, \theta'_2) = \Phi^{\gamma}_{kk_x k_\gamma}(r_\gamma, x) = \sqrt{\frac{k}{\pi k_x}} \frac{2(k^2 - k_\gamma^2)^{\frac{1}{4}}}{r_\gamma^{\frac{S_\gamma}{2}}}$$

$$\times \; J_{\gamma + \frac{S_\gamma}{2}}(k_\gamma r_\gamma) \begin{cases} \cos k_x x, & (m-\alpha) - \text{even}, \\ -i \sin k_x x, & (m-\alpha) - \text{odd}, \end{cases} \tag{36}$$

$$\lim_{R\to\infty} R^{-\frac{S_\gamma+1}{2}} \, \Psi^{\alpha\gamma}_{Jl}(\theta_1, \theta_2) = \Phi^{l\gamma}_{kk_{x\gamma}}(\sqrt{r_\gamma^2 + x^2}, \theta_2) = \sqrt{\frac{k(2l + S_\gamma + 1)(l-\gamma)!}{\pi(l + S_\gamma + \gamma)!}} \frac{\Gamma(\gamma + \frac{S_\gamma+1}{2})}{(r_\gamma^2 + x^2)^{\frac{S_\gamma+1}{4}}}$$

$$\times 2^{\gamma+\frac{S_\gamma}{2}} (\sin\theta_2)^\gamma \, J_{l+\frac{S_\gamma+1}{2}}\left(k_{x\gamma}\sqrt{r_\gamma^2 + x^2}\right) P_{l-\gamma}^{(\gamma+\frac{S_\gamma}{2},\gamma+\frac{S_\gamma}{2})}(\cos\theta_2). \tag{37}$$

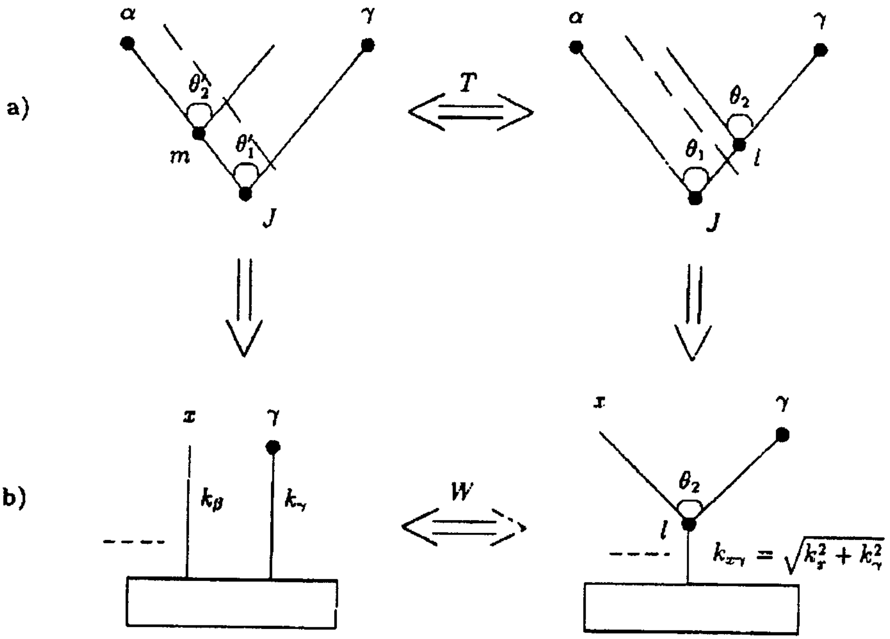

Fig. 6. Contractions of the case Fig. 3(3)

For the contractions of the interbasis coefficients T, we get

$$\lim_{R\to\infty} (-1)^{\frac{m-\alpha}{2}} \, T_{Jlm}^{\alpha\gamma} = W_{kk_{x\gamma}k_xk_\gamma}^{l\gamma} = \sqrt{\frac{2(2l+S_\gamma+1)}{\pi k_{x\gamma} k_x}} \, (k^2 - k_\gamma^2)^{\frac{1}{4}} \left(\frac{k_\gamma}{k_{x\gamma}}\right)^{\gamma+\frac{S_\gamma}{2}}$$

$$\times \begin{cases} \mathcal{A}^{\frac{1}{2}} \cdot {}_2F_1\left(-\frac{l-\gamma}{2}, \frac{l+\gamma+S_\gamma+1}{2}; \frac{1}{2}; \frac{k_x^2}{k_{x\gamma}^2}\right), & (l-\gamma) - \text{even}, \\ -2i\,\mathcal{A}^{-\frac{1}{2}} \left(\frac{k_x}{k_{x\gamma}}\right) \cdot {}_2F_1\left(-\frac{l-\gamma-1}{2}, \frac{l+\gamma+S_\gamma}{2}+1; \frac{3}{2}; \frac{k_x^2}{k_{x\gamma}^2}\right), & (l-\gamma) - \text{odd}, \end{cases} \tag{38}$$

where

$$\mathcal{A} = \frac{\Gamma\left(\frac{l+\gamma+S_\gamma+1}{2}\right)\Gamma\left(\frac{l-\gamma+1}{2}\right)}{\Gamma(\frac{l+\gamma+S_\gamma}{2}+1)\Gamma(\frac{l-\gamma}{2}+1)}.$$

Using now the connection between hypergeometrical functions ${}_2F_1$ and the Gegenbauer polynomials [11], we get

$$W^{l\gamma}_{kk_{z\gamma}k_zk_\gamma} = \frac{(-1)^{\frac{l-\gamma}{2}}2^{\gamma+\frac{S_\gamma+1}{2}}}{(k^2-k_\gamma^2)^{-\frac{1}{4}}}\Gamma\left(\gamma+\frac{S_\gamma+1}{2}\right)\sqrt{\frac{(2l+S_\gamma+1)(l-\gamma)!}{\pi k_z k_\gamma(l+\gamma+S_\gamma)!}}$$
$$\times \quad (\sin\phi)^{\gamma+\frac{S_\gamma+1}{2}}C_{l-\gamma}^{\gamma+\frac{S_\gamma+1}{2}}(\cos\phi), \qquad \cos\phi = \frac{k_z}{k_{z\gamma}}. \tag{39}$$

Multiplying the interbasis expansion (34) by the factor $R^{-\frac{S_\gamma+1}{2}}$ and taking the contraction limit $R\to\infty$, we obtain ($\theta_2 \equiv \theta$)

$$\Phi^\gamma_{kk_zk_\gamma}(r_\gamma, z) = \sum_{l=\gamma,\gamma+1}^{\infty}{}' W^{l\gamma}_{kk_{z\gamma}k_zk_\gamma}\Phi^{l\gamma}_{kk_{z\gamma}}(\sqrt{r_\gamma^2+z^2},\theta). \tag{40}$$

We use the orthogonality condition for the Gegenbauer polynomials [11]

$$\int_{-k_{z\gamma}}^{k_{z\gamma}} W^{l\gamma}_{kk_{z\gamma}k_zk_\gamma}W^{l'\gamma*}_{kk_{z\gamma}k_zk_\gamma}\frac{k_z dk_z}{\sqrt{k^2-k_\gamma^2}} = 4\delta_{ll'} \tag{41}$$

to get the inverse expansion

$$\Phi^{l\gamma}_{kk_{z\gamma}}(\sqrt{r_\gamma^2+z^2},\theta) = \frac{1}{4}\int_{-k_{z\gamma}}^{k_{z\gamma}} W^{l\gamma*}_{kk_{z\gamma}k_zk_\gamma}\Phi^\gamma_{kk_zk_\gamma}(r_\gamma,z)\frac{k_z dk_z}{\sqrt{k^2-k_\gamma^2}}. \tag{42}$$

Thus, the interbasis expansion (34) transforms into the expansion between the hypercylindrical and hyperspherical bases for Helmholtz equation.

Substituting formulae (36), (37) and (39) into expansions (42) and (42) and putting

$$k_z = k_{z\gamma}\cos\phi, \qquad k_\gamma = k_{z\gamma}\sin\phi,$$
$$z = \sqrt{r_\gamma^2+z^2}\cos\theta_2, \qquad r_\gamma = \sqrt{r_\gamma^2+z^2}\sin\theta_2,$$

we have ($z \equiv k_{z\gamma}\sqrt{r_\gamma^2+z^2}$)

$$J_{\gamma+\frac{S_\gamma}{2}}(z\sin\phi\sin\theta)\begin{Bmatrix}\cos(z\cos\phi\cos\theta)\\ \sin(z\sin\phi\sin\theta)\end{Bmatrix} = \frac{2^{2\gamma+S_\gamma-1}}{\sqrt{\pi z}}\Gamma^2(\gamma+\frac{S_\gamma+1}{2})\,(\sin\phi\sin\theta)^{\gamma+\frac{S_\gamma}{2}}$$

$$\times \sum_{l=\gamma,\gamma+1}^{\infty}\frac{(2l+S_\gamma+1)(l-\gamma)!}{(l+\gamma+S_\gamma)!}C_{l-\gamma}^{\gamma+\frac{S_\gamma+1}{2}}(\cos\phi)\,C_{l-\gamma}^{\gamma+\frac{S_\gamma+1}{2}}(\cos\theta)\,J_{l+\frac{S_\gamma+1}{2}}(z), \tag{43}$$

and

$$J_{l+\frac{S_\gamma+1}{2}}(z)\ (\sin\theta)^{\gamma+\frac{S_\gamma}{2}}\ C_{l-\gamma}^{\gamma+\frac{S_\gamma+1}{2}}(\cos\theta) = \sqrt{\frac{z}{2\pi}}\int_0^\pi d\phi\,(\sin\phi)^{\gamma+\frac{S_\gamma}{2}+1}$$

$$\times\, C_{l-\gamma}^{\gamma+\frac{S_\gamma+1}{2}}(\cos\phi)\ J_{\gamma+\frac{S_\gamma}{2}}(z\sin\phi\sin\theta)\ \left\{\begin{matrix}\cos(z\cos\phi\cos\theta)\\ \sin(z\sin\phi\sin\theta)\end{matrix}\right\}.$$

The contraction limit $R\to\infty$ of the interbasis expansion corresponding to Fig.3(5) can be presented by formulae (41) and (42) with the substitutions $q^2=0$ and therefore $r^2=r_\gamma^2+x^2$ and $k^2=k_x^2-k_\gamma^2$.

4.3 Contraction of interbasis expansions in Fig. 7

In this case, the left tree corresponds to the subgroup chains $O(n+1)\supset O(n_\alpha+n_\beta)$ and the right one to $O(n+1)\supset O(n_\alpha)\otimes O(n_\beta+1)$. The overlap functions (60) are again expressed in terms of the Racah coefficients. The expansion corresponding to Fig.7 has the form

$$\Psi_{Jm}^{\alpha\beta}(\theta_1',\theta_2') = \sum_{l=\beta,\beta+1}^{J-\alpha} T_{Jlm}^{\alpha\beta}\,\Psi_{Jl}^{\alpha\beta}(\theta_1,\theta_2), \tag{44}$$

where the T coefficient is given by eq. (60) and the wave functions Ψ may be constructed by using the rules given in section 2. As in the previous case the quantum number l runs $l=\beta,\beta+2,...,J-\alpha$ or $l=\beta+1,\beta+3,...,J-\alpha$ depending on $J-m$ being even or odd.

In the contraction limit $R\to\infty$

$$J\sim kR,\qquad m\sim pR,\qquad \alpha\sim qR,\qquad \theta_1'\sim\frac{x}{R},\qquad \theta_2'\sim\frac{r_\beta}{R},\qquad \theta_1\sim\frac{\sqrt{r_\beta^2+x^2}}{R}$$

we obtain

$$\begin{aligned}\Phi_{kk_xk_\beta}^{\beta}(x,r_\beta) &= \lim_{R\to\infty}(-1)^{\frac{J-m}{2}}\,\Psi_{Jm}^{\alpha\beta}(\theta_1',\theta_2')\\ &= \frac{2\sqrt{kp}}{\sqrt{\pi k_x}}(r_\beta)^{-\frac{S_\beta}{2}}\,J_{\beta+\frac{S_\beta}{2}}(k_\beta r_\beta)\left\{\begin{matrix}\cos k_x x, & (J-m)-\text{even},\\ -i\sin k_x x, & (J-m)-\text{odd},\end{matrix}\right.\end{aligned} \tag{45}$$

$$\begin{aligned}\Phi_{kk_{x\beta}}^{l\beta}(\sqrt{r_\beta^2+x^2},\theta_2) &= \lim_{R\to\infty}\Psi_{Jl}^{\alpha\beta}(\theta_1,\theta_2) = \frac{2^{\beta+\frac{S_\beta}{2}}\Gamma(\beta+\frac{S_\beta+1}{2})}{(r_\beta^2+x^2)^{\frac{S_\beta+1}{4}}}\sqrt{\frac{k(2l+S_\beta+1)(l-\beta)!}{\pi(l+\beta+S_\beta)!}}\\ &\times\ J_{l+\frac{S_\beta+1}{2}}(k_{x\beta}\sqrt{r_\beta^2+x^2})\ (\cos\theta_2)^\beta\,C_{l-\beta}^{\beta+\frac{S_\beta+1}{2}}(\sin\theta_2),\end{aligned} \tag{46}$$

where $k_\beta^2=p^2-q^2$, $k_x^2=k^2-p^2$ and $k_{x\beta}^2=k_x^2+k_\beta^2$.

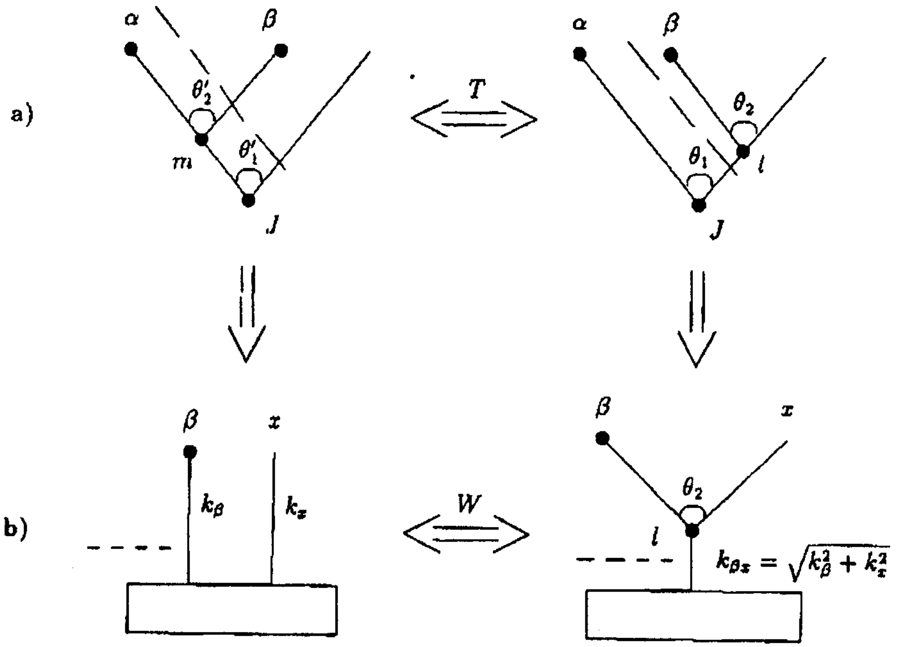

Fig. 7. Contractions of the case Fig. 3(4)

For the T coefficients we get

$$W^{l\beta}_{kk_{x\beta}k_xk_\beta} = \lim_{R\to\infty} T^{\alpha\beta}_{Jlm} = \frac{1}{2^{\beta+\frac{S_\beta-1}{2}}}\sqrt{\frac{(2l+S_\beta+1)(l+\beta+S_\beta)!}{(l-\beta)![\Gamma(\beta+\frac{S_\beta}{2}+1)]^2}}\sqrt{\frac{p}{k_xk_\beta}}$$

$$\times\left(\frac{k_x}{k_{x\beta}}\right)^l\left(\frac{k_\beta}{k_x}\right)^\beta\left(\frac{k_\beta}{k_{x\beta}}\right)^{\frac{S_\beta}{2}}{}_2F_1\left(-\frac{l-\beta}{2},-\frac{l-\beta-1}{2};\beta+\frac{S_\beta}{2}+1;-\frac{k_\beta^2}{k_x^2}\right)$$

$$=\frac{2^{\beta+\frac{S_\beta+1}{2}}\Gamma(\beta+\frac{S_\beta+1}{2})}{\sqrt{\pi k_xk_{x\beta}}}\left\{\frac{p(2l+S_\beta+1)(l-\beta)!}{(l+\beta+S_\beta)!}\right\}^{\frac{1}{2}}(\cos\phi)^{\beta+\frac{S_\beta}{2}}C_{l-\beta}^{\beta+\frac{S_\beta+1}{2}}(\sin\phi),$$

where $\cos\phi = k_\beta/k_{x\beta}$.

Multiplying the expansion (44) by the factor $R^{-\frac{S_\beta+1}{2}}$ and taking the contraction limit $R\to\infty$, we go into the flat space expansion ($\theta_2\equiv\theta$)

$$\Phi^{\beta}_{kk_xk_\beta}(x,r_\beta)=\sum_{l=\beta,\beta+1}^{\infty} W^{l\beta}_{kk_{x\beta}k_xk_\beta}\Phi^{l\beta}_{kk_{x\beta}}(\sqrt{r_\beta^2+x^2},\theta), \tag{47}$$

Using the orthogonality condition for the Gegenbauer polynomials [11], we have

$$\int_{-k_{x\beta}}^{k_{x\beta}} W^{l\beta}_{kk_{x\beta}k_xk_\beta}W^{l'\beta*}_{kk_{x\beta}k_xk_\beta}\frac{k_x dk_x}{\sqrt{k^2-k_\beta^2}}=4\delta_{ll'} \tag{48}$$

and the inverse expansion has the following form:

$$\Phi^{l\beta}_{kk^2_{x\beta}}(\sqrt{r_\beta^2+x^2},\theta)=\frac{1}{4}\int_{-k_{x\beta}}^{k_{x\beta}} W^{l\beta*}_{kk_{x\beta}k_xk_\beta}\Phi^{\beta}_{kk_xk_\beta}(r_\beta,x)\frac{k_x dk_x}{\sqrt{k^2-k_\beta^2}}. \tag{49}$$

Putting

$$k_x=k_{x\beta}\sin\phi,\qquad k_\gamma=k_{x\beta}\cos\phi,$$
$$x=\sqrt{r_\beta^2+x^2}\sin\theta_2,\qquad r_\beta=\sqrt{r_\beta^2+x^2}\cos\theta_2,$$

we finally obtain the expansion of the product of Bessel and cos or sin functions over the two Gegenbauer polynomials and Bessel function

$$\sqrt{z}\,J_{\beta+\frac{S_\beta}{2}}(z\cos\theta\cos\phi)\left\{\begin{matrix}\cos(z\sin\theta\sin\phi)\\ \sin(z\sin\theta\sin\phi)\end{matrix}\right\}=\frac{2^{2\beta+S_\beta}\Gamma^2(\beta+\frac{S_\beta+1}{2})}{\pi}(\cos\theta\cos\phi)^{\beta+\frac{S_\beta}{2}}$$

$$\times\sum_{l=\beta,\beta+1}^{\infty}\frac{(2l+S_\beta+1)(l-\beta)!}{(l+\beta+S_\beta)!}C_{l-\beta}^{\beta+\frac{S_\beta+1}{2}}(\sin\phi)\,C_{l-\beta}^{\beta+\frac{S_\beta+1}{2}}(\sin\theta_2)\,J_{l+\frac{S_\beta+1}{2}}(z), \tag{50}$$

where ($z\equiv k_{x\beta}\sqrt{r_\beta^2+x^2}$) and the top line on the left-hand side corresponds to a summation over $l=\beta,\beta+2,...,J-\alpha$ and the bottom one to a summation over $l=\beta+1,\beta+3,...,J-\alpha$. The expansion (50) is related to (43) by the substitutions $\theta\to\pi/2-\theta$, $\phi\to\pi/2-\phi$ and $\gamma\to\beta$, $S_\gamma\to S_\beta$.

Note that the contraction limit $R\to\infty$ in the interbasis expansion, corresponding to Fig.3(6), can be obtained from the expansion (50) by the substitutions $q^2=0$ and $k^2=k_{x\beta}^2$ and $r^2=r_\beta^2+x^2$.

4.4 The contractions of interbasis expansion in Fig. 8

The tree on the left side of Fig.8 corresponds to the subgroup chains $O(n+1)\supset O(n_\alpha+1)\supset O(n_\alpha)$ and the right one to $O(n+1)\supset O(n_\alpha)\otimes O(2)$. The overlap functions (63) are expressed in terms of the Clebsh-Gordan coefficients of the SU(2) group and the interbasis expansion is

$$\Psi^{\alpha}_{Jm}(\theta_1',\theta_2')=\sum_{l=-(J-\alpha)}^{(J-\alpha)}(-i)^{m-\alpha+l}(-1)^{\frac{|l|-l}{2}}\,C^{m+\frac{S_\alpha}{2},\alpha+\frac{S_\alpha}{2}}_{\frac{J}{2}+\frac{S_\alpha}{4},\frac{\alpha+l}{2}+\frac{S_\alpha}{4};\frac{J}{2}+\frac{S_\alpha}{4},\frac{\alpha-l}{2}+\frac{S_\alpha}{4}}\,\Psi^{\alpha}_{Jl}(\theta_1,\theta_2), \tag{51}$$

where the wave functions Ψ can be written out by using the formulae in section 2, and l has the same parity as $(J-\alpha)$.

In the contraction limit $R \to \infty$

$$J \sim kR, \qquad m \sim pR, \qquad \alpha \sim qR, \qquad \theta_1 \sim \frac{r}{R}, \qquad \theta_2' \sim \frac{x_1}{R}, \qquad \theta_1' \sim \frac{x_2}{R},$$

where $k_1^2 = p^2 - q^2$, $k_2^2 = k^2 - p^2$, $k_r^2 = k_1^2 + k_2^2$, we obtain $(\theta \equiv \theta_2)$

$$\lim_{R\to\infty} \frac{1}{\sqrt{R}} \Psi^{\alpha}_{Jl}(\theta_1, \theta_2) = \Phi^{l}_{kk_r}(r, \theta) = \sqrt{\frac{k}{\pi}}\, J_{|l|}(k_r r)\, e^{il\theta} \tag{52}$$

$$\lim_{R\to\infty} (-1)^{\frac{J-\alpha}{2}}\, \Psi^{\alpha}_{Jm}(\theta_1', \theta_2') = \Phi_{kk_1k_2}(x_1, x_2) = \sqrt{\frac{4pk}{\pi^2 k_1 k_2}}$$
$$\times \begin{cases} \cos(k_1x_1)\cos(k_2x_2), & (J-m) - \text{even}, \quad (m-\alpha) - \text{even}, \\ -i\cos(k_1x_1)\sin(k_2x_2), & (J-m) - \text{odd}, \quad (m-\alpha) - \text{even}, \\ -i\sin(k_1x_1)\cos(k_2x_2), & (J-m) - \text{even}, \quad (m-\alpha) - \text{odd}, \\ -\sin(k_1x_1)\sin(k_2x_2), & (J-m) - \text{odd}, \quad (m-\alpha) - \text{odd}, \end{cases} \tag{53}$$

$$\lim_{R\to\infty} (-1)^{-\frac{J-\alpha}{2}} \sqrt{R}\, (-i)^{m-\alpha+l} (-1)^{\frac{|l|-l}{2}}\, C^{m+\frac{S_\alpha}{2},\, \alpha+\frac{S_\alpha}{2}}_{\frac{J}{2}+\frac{S_\alpha}{4},\, \frac{\alpha+l}{2}+\frac{S_\alpha}{4};\, \frac{J}{2}+\frac{S_\alpha}{4},\, \frac{\alpha-l}{2}+\frac{S_\alpha}{4}} = W^{l}_{kk_1k_2}$$
$$= (i)^l (-1)^{\frac{|l|-l}{2}} \sqrt{\frac{4p}{\pi k_1 k_2}} \begin{cases} {}_2F_1\left(-\frac{|l|}{2}, \frac{|l|}{2}; \frac{1}{2}; -\frac{k_2^2}{k_r^2}\right) & (J-m) - \text{even}, \\ -il\frac{k_2}{k_r}\; {}_2F_1\left(\frac{1-|l|}{2}, \frac{1+|l|}{2}; \frac{3}{2}; -\frac{k_2^2}{k_r^2}\right) & (J-m) - \text{odd}, \end{cases}$$
$$= (i)^l (-1)^{\frac{|l|-l}{2}} \sqrt{\frac{4pk_r}{\pi k_1 k_2}} \begin{cases} \cos|l|\phi, & (J-m) - \text{even}, \\ -i\sin|l|\phi, & (J-m) - \text{odd}, \quad \cos\phi = k_1/k_r. \end{cases} \tag{54}$$

Multiplying the expansion (51) by the factor $(-1)^{\frac{J-\alpha}{2}}$ and taking the contraction limit $R \to \infty$, we obtain

$$e^{-ik_1x_1} \begin{Bmatrix} \cos(k_2x_2) \\ \sin(k_2x_2) \end{Bmatrix} = \sum_{l=-\infty}^{\infty} (i)^l (-1)^{\frac{|l|-l}{2}} \sqrt{k_r} \begin{Bmatrix} \cos|l|\phi \\ \sin|l|\phi \end{Bmatrix} J_{|l|}(k_r r)\, e^{il\theta}, \tag{55}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $\tan\theta = \frac{x_2}{x_1}$. The inverse expansion is

$$J_{|l|}(k_r r)\, e^{il\theta} = \frac{(i)^l (-1)^{\frac{|l|-l}{2}}}{2\pi} \int_0^{2\pi} e^{il\phi - ik_r r\cos(\theta-\phi)} d\phi. \tag{56}$$

For $\theta = 0$ the last formula is equivalent to the well-known formula in the theory of Bessel functions [11].

Note that the contraction limit $R \to \infty$ for the interbasis expansion between the trees with open ends (see Fig.3(8)) can be obtained from (55) by the substitution $q = 0$.

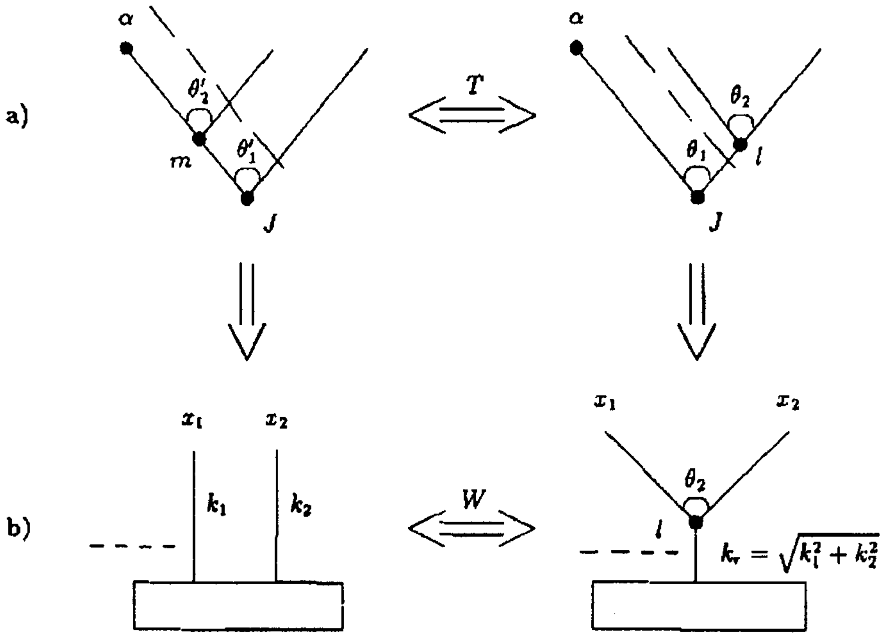

Fig. 8. Contractions of the case Fig. 3(7)

Acknowledgements

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Appendix

Here, we consider all particular cases of the overlap functions T. The substitution $S_{\alpha_j} = -1$ $\alpha_j = 0, 1$ in the general formula (9), where $\alpha_i = \alpha, \beta, \gamma$, gives us two variants of the T coefficients. Using several times the formula for the hypergeometric function ${}_4F_3$ [15]

$$
{}_4F_3\left\{\begin{matrix} -n,\ c,\ d,\ b; \\ e,\ f,\ g \end{matrix}\Bigg|1\right\} = \frac{(f-b)_n(g-b)_n}{(f)_n(g)_n}\,{}_4F_3\left\{\begin{matrix} -n,\ b,\ e-c,\ e-d; \\ e,\ 1+b-n-f,\ 1+b-n-g \end{matrix}\Bigg|1\right\}, \quad (57)
$$

$$
-n+b+c+d=-1+e+f+g,
$$

one can show that these two variants can be united into one formula. We mention here that some formulae for the T-coefficients [7, 10] must be prepared for contractions by using formula (57).

Let us list seven formulae for the T-coefficients prepared to contractions.

I. **Open end** α (see Fig.3(2)). Putting $S_\alpha=-1$ and $\alpha=0,1$ in formula (9) and using the transformation (57), we obtain

$$
T^{\beta\gamma}_{Jlm} = \left[\frac{1+(-1)^{J-l+m-\beta}}{2}\right]\frac{2^{l-\beta+\frac{S_\gamma}{2}}}{\Gamma(\beta+\frac{S_\beta}{2}+1)}\sqrt{\frac{(2l+S_\beta+S_\gamma+2)(2m+S_\beta+1)(J-l)!}{\Gamma(\frac{J+m+\gamma+S_\beta+S_\gamma+3}{2})\Gamma(\frac{J+m-\gamma+S_\beta+3}{2})}}
$$

$$
\times\left\{\frac{(\frac{J-m-\gamma}{2})!(m+\beta+S_\beta)!\Gamma(\frac{J-m+\gamma+S_\gamma}{2}+1)\Gamma(\frac{l+\beta+\gamma+S_\beta+S_\gamma}{2}+1)\Gamma(\frac{l+\beta-\gamma+S_\beta}{2}+1)}{\Gamma(\frac{l-\beta+\gamma+S_\gamma}{2}+1)(\frac{l-\beta-\gamma}{2})!(m-\beta)!(J+l+S_\beta+S_\gamma+2)!}\right\}^{1/2}
$$

$$
\times\frac{\Gamma(\frac{J+m+l-\beta+S_\beta+S_\gamma+3}{2})}{\Gamma(\frac{J-l-m+\beta}{2}+1)}\,{}_4F_3\left(\begin{matrix} -\frac{m-\beta}{2},\ -\frac{m-\beta-1}{2},\ -\frac{l-\beta+\gamma+S_\gamma}{2},\ -\frac{l-\beta-\gamma}{2}; \\ \beta+\frac{S_\beta}{2}+1,\ -\frac{J+l+m-\beta+S_\beta+S_\gamma+1}{2},\ \frac{J-l-m+\beta}{2}+1 \end{matrix}\Bigg|1\right). \quad (58)
$$

II. **Open end** β (see Fig.3(3)). Choosing the parameters $S_\beta=-1$ and $\beta=0,1$ in formula (9), we arrive at the following form of the coefficients $T^{\alpha\gamma}_{Jlm}$:

$$
T^{\alpha\gamma}_{Jlm} = \frac{[1+(-1)^{l-\gamma+m-\alpha}]}{4\sqrt{\pi}}(-1)^{\frac{m-\alpha}{2}}\sqrt{(2m+S_\alpha+1)(2l+S_\gamma+1)}
$$

$$
\times\left\{\frac{\Gamma(\frac{J-m-\gamma}{2}+1)\Gamma(\frac{J+\alpha+l+S_\alpha+S_\gamma+3}{2})\Gamma(\frac{J+\alpha-l+S_\alpha}{2}+1)\Gamma(\frac{J+m-\gamma+S_\alpha+3}{2})}{\Gamma(\frac{J-m+\gamma+S_\gamma}{2}+1)\Gamma(\frac{J-\alpha-l}{2}+1)\Gamma(\frac{J-\alpha+l+S_\gamma+3}{2})\Gamma(\frac{J+m+\gamma+S_\alpha+S_\gamma+3}{2})}\right\}^{1/2}
$$

$$
\times\begin{cases} A\cdot{}_4F_3\left(\begin{matrix} -\frac{m-\alpha}{2},\ \frac{m+\alpha+S_\alpha+1}{2},\ \frac{l-\gamma+1}{2},\ -\frac{l+\gamma+S_\gamma}{2}; \\ \frac{1}{2},\ \frac{J+\alpha-\gamma+S_\alpha+3}{2},\ -\frac{J-\alpha+\gamma+S_\gamma}{2} \end{matrix}\Bigg|1\right), & (m-\alpha)-\text{even}, \\ -2iB\cdot{}_4F_3\left(\begin{matrix} -\frac{m-\alpha-1}{2},\ \frac{m+\alpha+S_\alpha}{2}+1,\ \frac{l-\gamma}{2}+1,\ -\frac{l+\gamma+S_\gamma-1}{2}; \\ \frac{3}{2},\ \frac{J+\alpha-\gamma+S_\alpha}{2}+2,\ -\frac{J-\alpha+\gamma+S_\gamma-1}{2} \end{matrix}\Bigg|1\right), & (m-\alpha)-\text{odd}, \end{cases} \quad (59)
$$

where

$$A=\frac{\Gamma\left(\frac{J-\alpha+\gamma+S_\gamma}{2}+1\right)}{\Gamma\left(\frac{J+\alpha-\gamma+S_\alpha+3}{2}\right)}\left\{\frac{\Gamma(\frac{l+\gamma+S_\gamma+1}{2})\Gamma(\frac{l-\gamma+1}{2})\Gamma(\frac{m+\alpha+S_\alpha+1}{2})\Gamma(\frac{m-\alpha+1}{2})}{\Gamma(\frac{l+\gamma+S_\gamma}{2}+1)\Gamma(\frac{l-\gamma}{2}+1)\Gamma(\frac{m+\alpha+S_\alpha}{2}+1)\Gamma(\frac{m-\alpha}{2}+1)}\right\}^{1/2},$$

$$B=\frac{\Gamma\left(\frac{J-\alpha+\gamma+S_\gamma+1}{2}\right)}{\Gamma\left(\frac{J+\alpha-\gamma+S_\alpha}{2}+2\right)}\left\{\frac{\Gamma(\frac{l+\gamma+S_\gamma}{2}+1)\Gamma(\frac{l-\gamma}{2}+1)\Gamma(\frac{m+\alpha+S_\alpha}{2}+1)\Gamma(\frac{m-\alpha}{2}+1)}{\Gamma(\frac{l+\gamma+S_\gamma+1}{2})\Gamma(\frac{l-\gamma+1}{2})\Gamma(\frac{m+\alpha+S_\alpha+1}{2})\Gamma(\frac{m-\alpha+1}{2})}\right\}^{1/2}.$$

III. **Open end** γ (see Fig.3(4)). In this case, the coefficient $T^{\alpha\beta}_{Jlm}$ differs from the coefficient $T^{\beta,\gamma}_{Jlm}$ in (58) by the substitutions: $m \Longleftrightarrow l$, $\gamma\to\alpha$, $S_\gamma\to S_\alpha$ and by the phase factor $(-1)^{\frac{J-|m|-l+\beta}{2}}$:

$$T^{\alpha\beta}_{Jlm}=\left[\frac{1+(-1)^{J-m+l-\beta}}{2}\right](-1)^{\frac{J-|m|-l+\beta}{2}}\frac{2^{m-\beta+\frac{S_\alpha}{2}+1}\Gamma(\frac{J+l+m-\beta+S_\alpha+S_\beta+3}{2})}{\Gamma(\beta+\frac{S_\beta}{2}+1)\Gamma(\frac{J-l-m+\beta}{2}+1)}\times$$

$$\left\{\frac{\Gamma(\frac{m+\beta+\alpha+S_\alpha+S_\beta}{2}+1)\Gamma(\frac{m+\beta-\alpha+S_\beta}{2}+1)(J-m)!(\frac{J-l-\alpha}{2})!(l+\beta+S_\beta)!\Gamma(\frac{J-l+\alpha+S_\alpha}{2}+1)}{\Gamma(\frac{J+l-\alpha+S_\beta+3}{2})\Gamma(\frac{m-\beta+\alpha+S_\alpha}{2}+1)(\frac{m-\beta-\alpha}{2})!(l-\beta)!(J+m+S_\alpha+S_\beta+2)!}\right\}^{\frac{1}{2}}$$

$$\sqrt{\frac{(m+\frac{S_\alpha+S_\beta}{2}+1)(l+\frac{S_\beta+1}{2})}{\Gamma(\frac{J+l+\alpha+S_\alpha+S_\beta+3}{2})}}\;{}_4F_3\left(\begin{matrix}-\frac{l-\beta}{2},\ -\frac{l-\beta-1}{2},\ -\frac{m-\beta+\alpha+S_\alpha}{2},\ -\frac{m-\alpha-\beta}{2};\\ \beta+\frac{S_\beta}{2}+1,\ -\frac{J+l+m-\beta+S_\beta+S_\alpha+1}{2},\ \frac{J-l-m+\beta}{2}+1\end{matrix}\middle|1\right).\quad(60)$$

IV. **Ends** α **and** β **are open** (see Fig.3(5)). The corresponding T^{γ}_{Jlm} coefficient has the following form [7]:

$$T^{\gamma}_{Jlm}=(i)^{l-\gamma}(-1)^{\frac{J-l-\gamma}{2}}\,C^{l+\frac{S_\gamma}{2},\gamma+\frac{S_\gamma}{2}}_{\frac{J}{2}+\frac{S_\gamma}{4},\frac{\gamma+m}{2}+\frac{S_\gamma}{4};\frac{J}{2}+\frac{S_\gamma}{4},\frac{\gamma-m}{2}+\frac{S_\gamma}{4}}$$

$$=(-i)^{l-\gamma}\left\{\frac{\Gamma(\frac{J+\gamma-m}{2}+\frac{S_\gamma}{2}+1)(\frac{J-\gamma+m}{2})!(l+\gamma+S_\gamma)!(2l+S_\gamma+1)}{\Gamma(\frac{J+\gamma+m}{2}+\frac{S_\gamma}{2}+1)(\frac{J-\gamma-m}{2})!(J-l)!(J+l+S_\gamma+1)!(l-\gamma)!}\right\}^{1/2}$$

$$\times\frac{\Gamma(J+\frac{S_\gamma}{2}+1)}{\Gamma(\gamma+\frac{S_\gamma}{2}+1)}\;{}_3F_2\left(\begin{matrix}-\frac{J-m-\gamma}{2},\ -l-\frac{S_\gamma}{2},\ l+\frac{S_\gamma}{2}+1;\\ -J-\frac{S_\gamma}{2},\ \gamma+\frac{S_\gamma}{2}+1\end{matrix}\middle|1\right),\qquad(61)$$

where $C^{\gamma l,\gamma}_{a,\alpha;b,\beta}$ are the Clebsch-Gordan coefficients for the SU(1,1) group, if S_γ is odd, and SU(2) group for even S_γ.

V. **Ends α and γ are open** (see Fig.3(6)). Putting $S_\gamma = -1$ and $\gamma = 0,1$ in formula (60), we obtain two values for the coefficient T^{β}_{Jlm} depending on parity of $(m-\beta)$. Using several times the transformation (57), we get

$$T^{\beta}_{Jlm} = \left[\frac{1+(-1)^{J-l+m-\beta}}{2}\right](-1)^{\frac{J-m-l+\beta}{2}} 2^{l+m-2\beta}\frac{\sqrt{(2l+S_\beta+1)(2m+S_\beta+1)}}{\Gamma(\frac{J-l-m+\beta}{2}+1)}$$
$$\times\frac{\Gamma(\frac{J+l+m-\beta+S_\beta}{2}+1)}{\Gamma(\beta+\frac{S_\beta}{2}+1)}\left\{\frac{(l+\beta+S_\beta)!(J-l)!(J-m)!(m+\beta+S_\beta)!}{(m-\beta)!(l-\beta)!(J+l+S_\beta+1)!(J+m+S_\beta+1)!}\right\}^{1/2}$$
$$\times{}_4F_3\left(\begin{matrix}-\frac{m-\beta}{2},\ -\frac{m-\beta-1}{2},\ -\frac{l-\beta}{2},\ -\frac{l-\beta-1}{2};\\ \beta+\frac{S_\beta}{2}+1,\ -\frac{J+l+m-\beta+S_\beta}{2},\ \frac{J-l-m+\beta}{2}+1\end{matrix}\Bigg|1\right). \tag{62}$$

VI. **Ends β and γ are open** (see Fig.3(7)). The corresponding T^{α}_{Jlm} coefficient has the form

$$T^{\alpha}_{Jlm} = (-i)^{m-\alpha+l}(-1)^{\frac{|l|-l}{2}}\, C^{m+\frac{S_\alpha}{2},\,\alpha+\frac{S_\alpha}{2}}_{\frac{J}{2}+\frac{S_\alpha}{4},\,\frac{\alpha+l}{2}+\frac{S_\alpha}{4};\ \frac{J}{2}+\frac{S_\alpha}{4},\,\frac{\alpha-l}{2}+\frac{S_\alpha}{4}}. \tag{63}$$

Expression, of the Clebsch-Gordan coefficients in terms of the ${}_3F_2$ function is not convenient for taking the contraction limit. Instead, we use the following integral representation [16]:

$$C^{J,M}_{j,m_1,j,m_2} = (i)^{J-M}(-1)^{j-m_1}\left\{\frac{(J+M)!(j-m_1)!(j-m_2)!}{(J-M)!(j+m_1)!(j+m_2)!}\right\}^{1/2}$$
$$\times\frac{\sqrt{(2J+1)(2j-J)!(2j+J+1)!}}{2^{J+M+2}\Gamma(2j+3/2)}\frac{1}{\sqrt{\pi}}\int_0^{2\pi}(\sin\phi)^{J-M}P^{(J+\frac12,J+\frac12)}_{2j-J}(\cos\phi)e^{i(m_2-m_1)\phi}d\phi$$

and the formulae [11]

$$P^{(\alpha,\alpha)}_n(\cos\phi) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!}\times\begin{cases}{}_2F_1(-\frac{n}{2},\frac{n+1}{2}+\alpha;\alpha+1;\sin^2\phi), & n-\text{even},\\ \cos\phi\,{}_2F_1(-\frac{n-1}{2},\frac{n}{2}+\alpha+1;\alpha+1;\sin^2\phi), & n-\text{odd}.\end{cases}$$

After integrating over ϕ, we obtain a representation of the Clebsch-Gordan coefficients in terms of the hypergeometric function ${}_4F_3$

$$C^{m+\frac{S_\alpha}{2},\,\alpha+\frac{S_\alpha}{2}}_{\frac{J}{2}+\frac{S_\alpha}{4},\,\frac{\alpha+l}{2}+\frac{S_\alpha}{4};\ \frac{J}{2}+\frac{S_\alpha}{4},\,\frac{\alpha-l}{2}+\frac{S_\alpha}{4}} = (i)^{m-\alpha}(-1)^{\frac{J-\alpha}{2}-|l|}\frac{\sqrt{2m+S_\alpha+1}}{2^{2m+S_\alpha+1}}\sqrt{(m-\alpha)!(m+\alpha+S_\alpha)!}$$
$$\sqrt{\frac{(J+m+S_\alpha+1)!(\frac{J-\alpha-|l|}{2})!\Gamma(\frac{J+\alpha-|l|}{2}+\frac{S_\alpha}{2})}{(J-m)!(\frac{J-\alpha+|l|}{2})!\Gamma(\frac{J+\alpha+|l|}{2}+\frac{S_\alpha}{2})}}$$

$$\times \begin{cases} \frac{\Gamma(\frac{J-m+1}{2})[\Gamma(\frac{J+m+S_a+3}{2})]^{-1}}{\Gamma(1+\frac{m+\alpha-|l|+S_a}{2})\Gamma(1+\frac{m-\alpha-|l|}{2})} \, {}_4F_3 \left(\begin{matrix} -\frac{|l|}{2}, -\frac{|l|-1}{2}, \frac{J+m+S_a}{2}+1, -\frac{J-m}{2}; \\ \frac{1}{2}, 1+\frac{m-\alpha-|l|}{2}, 1+\frac{m+\alpha-|l|+S_a}{2} \end{matrix} \middle| 1 \right), \\ \frac{-i|l|\Gamma(\frac{J-m}{2}+1)[\Gamma(\frac{J+m+S_a+2}{2})]^{-1}}{\Gamma(\frac{m+\alpha+S_a-|l|+3}{2})\Gamma(\frac{m-\alpha-|l|+3}{2})} \, {}_4F_3 \left(\begin{matrix} -\frac{|l|-1}{2}, -\frac{|l|-2}{2}, \frac{J+m+S_a+3}{2}, -\frac{J-m-1}{2}; \\ \frac{3}{2}, \frac{m-\alpha-|l|+3}{2}, \frac{m+\alpha-|l|+3}{2} \end{matrix} \middle| 1 \right), \end{cases} \tag{64}$$

where the top line corresponds to an even $(J-m)$ and the bottom one to an odd $(J-m)$.

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