SEPARATION OF VARIABLES
IN CLASSICAL AND QUANTUM MECHANICS

ON A GENERALIZED OSCILLATOR: INVARIANCE ALGEBRA AND INTERBASIS EXPANSIONS

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This article deals with a quantum-mechanical system which generalizes the ordinary isotropic harmonic oscillator system. We give the coefficients connecting the polar and Cartesian bases for $D = 2$ and the coefficients connecting the Cartesian and cylindrical bases as well as the cylindrical and spherical bases for $D = 3$. These interbasis expansion coefficients are found to be analytic continuations to real values of their arguments of the Clebsch–Gordan coefficients for the group $SU(2)$. For $D = 2$, the superintegrable character for the generalized oscillator system is investigated from the points of view of a quadratic invariance algebra.

1. INTRODUCTION

During the last 30 years, superintegrable dynamical systems have been the object of considerable interest (see [1–10] and references therein). In particular, numerous works have been devoted to the search for dynamical invariance algebras (especially quadratic algebras) of nonrelativistic systems with potentials presenting singularities. Such systems are important in various fields (e.g., Aharonov–Bohm effect, Dirac or Schwinger monopoles, confining problems, supersymmetry, etc.).

It is the aim of this paper to investigate the system with the potential

$$V = \sum_{a=1}^{D} V_a,$$
$$V_a = \frac{1}{2}\Omega^2 x_a^2 + \frac{1}{2}P\frac{1}{x_a^2}, \quad P = k_a^2 - \frac{1}{4}, \tag{1}$$

where $\Omega > 0$ and $k_a^2 > 0$ $(a = 1, 2, \ldots, D)$. This system was already discussed for $D = 2$ by the late Prof. Smorodinsky and his collaborators [1] from a classical and quantum-mechanical point of view. We shall be concerned here mainly with $D = 2$ and 3 for which the spectrum of the Schrödinger equation

$$H\Psi = E\Psi, \quad H = -\frac{1}{2}\Delta + V \tag{2}$$

will be given. Emphasis will be put on interbasis expansions in terms of analytic continuation of Clebsch–Gordan coefficients (CGc's) for the group $SU(2)$. As another important result, we shall introduce a quadratic invariance algebra in the $D = 2$ case.

2. D-DIMENSIONAL CASE

We briefly consider here the D-dimensional case in Cartesian coordinates. We start with $D = 1$ and look for a solution of the one-dimensional equation (2) for the potential V_1, see (1), with $x_1 \equiv x$ and $k_1 \equiv k$. The resolution of this equation, with the conditions $\Psi(x) \longrightarrow 0$ as $x \longrightarrow 0$ and ∞, leads to the normalized wave function

$$\Psi_n(x; \pm k) = \sqrt{\frac{\Omega^{\frac{1}{2}} n!}{\Gamma(n \pm k + 1)}} (\sqrt{\Omega x^2})^{\frac{1}{2} \pm k} \times \exp\left(-\frac{\Omega}{2}x^2\right) L_n^{\pm k}(\Omega x^2), \quad n \in \mathbf{N}, \tag{3}$$

where L_n^{ν} is an associated Laguerre polynomial [5]. The normalization is such that

$$2\int_0^{\infty} \Psi_{n'}(x; \pm k)^* \Psi_n(x; \pm k) dx = \delta_{n'n}. \tag{4}$$

The discrete energy spectrum is given by

$$E = \Omega(2n \pm k + 1).$$

Only the sign + may be taken in front of k when $k > 1/2$. For $0 < k < 1/2$, both the signs + and – are admissible. For $k = (1/2)^-$, due to the connecting formulas [11] between the (even and odd) Hermite polynomials $\mathcal{H}_p(x)$ and the Laguerre polynomials $L_n^{(\pm 1/2)}(x^2)$ and by putting $p = 2n + 1$ for the sign + and $p = 2n$ for the sign –, we immediately have

$$\Psi_n\left(x; \pm\frac{1}{2}\right) = \left(\frac{\Omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^p p!}} \exp\left(-\frac{\Omega}{2}x^2\right) \mathcal{H}_p(\sqrt{\Omega x^2}).$$

We now deal with the D-dimensional case. In this case, the Cartesian wave function, that vanishes when

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$x_a \longrightarrow 0$ and ∞ ($a = 1, 2, \dots, D$), is

$$\Psi_n(\mathbf{x}; \mathbf{k}) = \prod_{a=1}^{D} \Psi_{n_a}(x_a; \pm k_a),$$

where $\mathbf{n} = n_1, \dots, n_D$ with $n_a \in \mathbf{N}$, $\mathbf{x} = x_1, \dots, x_D$ and $\mathbf{k} = \pm k_1, \dots, \pm k_D$. The energy is

$$E = \Omega\left[2n + D + \sum_{a=1}^{D} (\pm k_a)\right],$$

where $n = n_1 + n_2 + \dots + n_D$ is the principal quantum number.

3. TWO-DIMENSIONAL CASE

3.1. Cartesian basis

In Cartesian coordinates ($x_1 \equiv x$, $x_2 \equiv y$), the wave function is

$$\Psi_{n_1 n_2}(x, y; \pm k_1, \pm k_2) = \Psi_{n_1}(x; \pm k_1)\Psi_{n_2}(y; \pm k_2), \quad (5)$$

where Ψ_{n_a} (with $a = 1, 2$) are given by (3). Note that we have the new constant of motion

$$N = \frac{1}{4\Omega}\left(D_{xx} - D_{yy} + \frac{k_1^2 - \frac{1}{4}}{x^2} - \frac{k_2^2 - \frac{1}{4}}{y^2}\right) \quad (6)$$

(in addition to the energy), where $D_{\alpha\beta} = -\partial_{\alpha\beta} + \Omega^2\alpha\beta$ is the Demkov tensor [12].

3.2. Polar basis

In polar coordinates (ρ, φ), the potential (1) reads

$$V = \frac{1}{2}\Omega^2\rho^2 + \frac{1}{2\rho^2}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right),$$

for which equation (2) may be separated by seeking a solution in the form $R(\rho)\Phi(\varphi)$. This leads to the system of coupled differential equations

$$\left(d_{\varphi\varphi} + A^2 - \frac{k_1^2 - \frac{1}{4}}{\cos^2\varphi} - \frac{k_2^2 - \frac{1}{4}}{\sin^2\varphi}\right)\Phi = 0,$$
$$\left[\frac{1}{\rho}d_\rho(\rho d_\rho) + 2E - \Omega^2\rho^2 - \frac{A^2}{\rho^2}\right]R = 0, \quad (7)$$

where A is a polar separation constant.

The solution $\Phi(\varphi) \equiv \Phi_m(\varphi; \pm k_1, \pm k_2)$ of the angular equation in (7) with the conditions

$$\Phi(0) = \Phi\left(\frac{\pi}{2}\right) = 0 \quad (8)$$

is easily found to be

$$\Phi(\varphi) = \sqrt{\frac{(2m \pm k_1 \pm k_2 + 1)m!\Gamma(m \pm k_1 \pm k_2 + 1)}{2\Gamma(m \pm k_1 + 1)\Gamma(m \pm k_2 + 1)}} \times$$
$$\times (\cos\varphi)^{\frac{1}{2} \pm k_1}(\sin\varphi)^{\frac{1}{2} \pm k_2} P_m^{(\pm k_2, \pm k_1)}(\cos 2\varphi), \quad (9)$$

where $m \in \mathbf{N}$ and $P_n^{(\alpha,\beta)}$ denotes a Jacobi polynomial. The normalization is such that

$$4\int_0^{\pi/2} \Phi_{m'}(\varphi; \pm k_1, \pm k_2)^*\Phi_m(\varphi; \pm k_1, \pm k_2)d\varphi = \delta_{m'm}. \quad (10)$$

Then, the separation constant A is quantized as

$$A = 2m \pm k_1 \pm k_2 + 1. \quad (11)$$

The radial solution $R(\rho) \equiv R_{n_\rho m}(\rho; \pm k_1, \pm k_2)$ in (7) is

$$R(\rho) = \sqrt{\frac{2\Omega n_\rho!}{\Gamma(n_\rho + 2m \pm k_1 \pm k_2 + 2)}} \times$$
$$\times (\sqrt{\Omega\rho^2})^A \exp\left(-\frac{\Omega}{2}\rho^2\right)L_{n_\rho}^A(\Omega\rho^2), \quad (12)$$

where $n_\rho \in \mathbf{N}$ is the radial quantum number. The function R satisfies the orthogonality relation

$$\int_0^\infty R_{n'_\rho m}(\rho; \pm k_1, \pm k_2)R_{n_\rho m}(\rho; \pm k_1, \pm k_2)\rho d\rho = \delta_{n'_\rho n_\rho}.$$

The energy E corresponding to the $n + 1$ wave functions

$$\Psi_{n_\rho m}(\rho, \varphi; \pm k_1, \pm k_2) \equiv R(\rho)\Phi(\varphi)$$

(with $n = n_\rho + m$ fixed) is

$$E = \Omega(2n \pm k_1 \pm k_2 + 2), \quad n \in \mathbf{N}, \quad (13)$$

where n is the principal quantum number. Note that only the sign + in front of k_1 and k_2 has to be taken when $k_1 > 1/2$ and $k_2 > 1/2$. In the case $0 < k_a < 1/2$ (with $a = 1, 2$), equation (9) shows that for each n we have four levels corresponding to ($\pm k_1$, $\pm k_2$). The degeneracy of the level with the principal quantum number n is $n + 1$. This degeneracy is identical to the one of the isotropic oscillators in two dimensions, for which the degeneracy group is $SU(2)$.

For $k_1 = k_2 = \left(\frac{1}{2}\right)^-$, we have $A\left(\frac{1}{2}, \frac{1}{2}\right) = 2m + 2$, $A\left(-\frac{1}{2}, -\frac{1}{2}\right) = 2m$ and $A\left(\frac{1}{2}, -\frac{1}{2}\right) = A\left(-\frac{1}{2}, \frac{1}{2}\right) = 2m + 1$. Then, by using the connecting formulas [11] between Jacobi and Chebychev polynomials, we obtain the four following wave functions [3]

$$\Psi_{2n, 2m}(\rho, \varphi) = \frac{1}{\sqrt{\pi}}R_{2n, 2m}(\rho)\cos 2m\varphi, \quad \tilde{n} = 2n; \quad (14)$$

$$\Psi_{2n+2,\,2m+2}(\rho,\varphi) = \frac{1}{\sqrt{\pi}}R_{2n+2,\,2m+2}(\rho)\sin(2m+2)\varphi, \quad (15)$$
$$\tilde{n} = 2n+2;$$

$$\Psi_{2n+1,\,2m+1}(\rho,\varphi) = \frac{1}{\sqrt{\pi}}R_{2n+1,\,2m+1}(\rho)\cos(2m+1)\varphi, \quad (16)$$
$$\tilde{n} = 2n+1;$$

$$\Psi_{2n+1,\,2m+1}(\rho,\varphi) = \frac{1}{\sqrt{\pi}}R_{2n+1,\,2m+1}(\rho)\sin(2m+1)\varphi, \quad (17)$$
$$\tilde{n} = 2n+1$$

corresponding to the energy $E = \Omega(\tilde{n}+1)$. In equations (14)–(17), we have

$$R_{p,t}(\rho) = \sqrt{\frac{2\Omega\left(\frac{p-t}{2}\right)!}{\left(\frac{p+t}{2}\right)!}}\left(\sqrt{\Omega\rho^2}\right)^t \exp\left(-\frac{\Omega}{2}\rho^2\right)L^t_{\frac{p-t}{2}}(\Omega\rho^2)$$

to be compared with the corresponding result for the ordinary circular oscillator.

To close this section, let us mention that

$$M = \frac{1}{4}\left(-\partial_{\varphi\varphi} + \frac{k_1^2-\frac{1}{4}}{\cos^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right) = \frac{1}{4}\left[L_z^2 + (x^2+y^2)\left(\frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right)\right] \quad (18)$$

is a polar constant of motion, the eigenvalues of which are A (see (11)).

3.3. Connecting Cartesian and polar bases

According to first principles in quantum mechanics, we have

$$\Psi_{n_1 n_2} = \sum_{m=0}^{n} W^m_{n_1 n_2}(\pm k_1, \pm k_2)\Psi_{n_\rho m}, \quad (19)$$

where $n_\rho + m = n_1 + n_2 = n$. In equation (19), it is understood that the wave functions both in the left- and right-hand sides are written in polar coordinates (ρ, φ). Furthermore, by using the asymptotic formula for the associated Laguerre polynomials, equation (19) yields an equation that depends only on the variable φ. Thus, by using the orthonormality property of the function Φ with respect to the quantum number m, we obtain

$$W^m_{n_1 n_2}(\pm k_1, \pm k_2) = (-1)^m B^m_{n_1 n_2}(\pm k_1, \pm k_2) E^m_{n_1 n_2}(\pm k_1, \pm k_2), \quad (20)$$

where

$$E^m_{n_1 n_2}(\pm k_1, \pm k_2) = 2\int_0^{\pi/2}(\sin\varphi)^{2n_2+1\pm 2k_2} \times (\cos\varphi)^{2n_1+1\pm 2k_1} P_m^{(\pm k_2, \pm k_1)}(\cos 2\varphi)\,d\varphi \quad (21)$$

and

$$B^m_{n_1 n_2}(\pm k_1, \pm k_2) = \sqrt{2m \pm k_1 \pm k_2 + 1}\sqrt{\frac{(n-m)!\,m!\,\Gamma(m \pm k_1 \pm k_2 + 1)\Gamma(n+m \pm k_1 \pm k_2 + 2)}{n_1!\,n_2!\,\Gamma(m \pm k_1 + 1)\Gamma(m \pm k_2 + 1)\Gamma(n_1 \pm k_1 + 1)\Gamma(n_2 \pm k_2 + 1)}}. \quad (22)$$

By making the change of variable $x = \cos 2\varphi$ and by using the Rodrigues formula [11] for the Jacobi polynomial, equations (20)–(22) lead to the integral representation

$$W^m_{n_1 n_2}(\pm k_1, \pm k_2) = \sqrt{\frac{(2m \pm k_1 \pm k_2 + 1)(n-m)!\,\Gamma(m \pm k_1 \pm k_2 + 1)\Gamma(n+m \pm k_1 \pm k_2 + 2)}{n_1!\,n_2!\,m!\,\Gamma(m \pm k_1 + 1)\Gamma(m \pm k_2 + 2)\Gamma(n_1 \pm k_1 + 1)\Gamma(n_2 \pm k_2 + 1)}} \times \frac{1}{2^{n_1+n_2+m \pm k_1 \pm k_2 + 1}}\int_{-1}^{1}(1-x)^{n_2}(1+x)^{n_1}\frac{d^m}{dx^m}[(1-x)^{m \pm k_2}(1+x)^{m \pm k_1}]\,dx \quad (23)$$

for the interbasis expansion coefficients $W^m_{n_1 n_2}(\pm k_1, \pm k_2)$.

Equation (23) can be compared with the integral representation [13] for the CGc's $\langle ab\alpha\beta|c\gamma\rangle$ of the group $SU(2)$. This yields

$$W^m_{n_1 n_2}(\pm k_1, \pm k_2) = (-1)^{n_1 - n_\rho}\langle ab\alpha\beta|c\gamma\rangle \quad (24)$$

with $2a = n_1 + n_2 \pm k_1$, $2b = n_1 + n_2 \pm k_2$, $2c = 2m \pm k_1 \pm k_2$, $2\alpha = n_1 - n_2 \pm k_1$ and $2\beta = n_2 - n_1 \pm k_2$. Since the quantum numbers in (24) are not necessarily integers or half of odd integers, the coefficients for the expansion of the Cartesian basis in terms of the polar basis may be considered as analytical continuation of the $SU(2)$ CGc's.

The inverse of equation (19), viz.,

$$\Psi_{n_\rho m} = \sum_{n_1=0}^{n} \tilde{W}^{n_1}_{n_\rho m}(\pm k_1, \pm k_2)\Psi_{n_1 n_2} \tag{25}$$

follows from the orthonormality property of the $SU(2)$ CGc's. Thus, the relation

$$\tilde{W}^{n_1}_{n_\rho m}(\pm k_1, \pm k_2) = W^{m}_{n_1 n_2}(\pm k_1, \pm k_2)$$

gives the expansion coefficients in (25). The $SU(2)$ CGc's can be expressed [13] in terms of the hypergeometric function ${}_3F_2(1)$, so that equation (24) can be rewritten as

$$W^{m}_{n_1 n_2}(\pm k_1, \pm k_2) = (-1)^{n_2}\frac{n!\Gamma(n_2+m\pm k_2+1)}{\sqrt{n_\rho!\Gamma(m\pm k_1+1)\Gamma(m\pm k_2+1)}}\times$$

$$\times\sqrt{(2m\pm k_1\pm k_2+1)\frac{\Gamma(n_1\pm k_1+1)\Gamma(m\pm k_1\pm k_2+1)}{n_1!n_2!m!\Gamma(n_2\pm k_2+1)\Gamma(n+m\pm k_1\pm k_2+2)}}\;{}_3F_2\left(\left.\begin{matrix}-n-m\mp k_1\mp k_2-1, -n_2, -m\\ -n_1-n_2, -n_2-m\mp k_2\end{matrix}\right|1\right).$$

By using symmetry properties for ${}_3F_2(1)$, we arrive at the expression

$$\begin{aligned}W^{m}_{n_1 n_2}(\pm k_1, \pm k_2) &= \frac{(-1)^m n!}{\Gamma(1\pm k_2)}\times\\ &\times\sqrt{(2m\pm k_1\pm k_2+1)\frac{\Gamma(m\pm k_1\pm k_2+1)\Gamma(m\pm k_2+1)}{n_1!n_2!m!n_\rho!\Gamma(m\pm k_1+1)}}\times\\ &\times\sqrt{\frac{\Gamma(n_1\pm k_1+1)\Gamma(n_2\pm k_2+1)}{\Gamma(n+m\pm k_1\pm k_2+2)}}\;{}_3F_2\left(\left.\begin{matrix}-m, m\pm k_1\pm k_2+1, -n_2\\ 1\pm k_2, -n_1-n_2\end{matrix}\right|1\right).\end{aligned} \tag{26}$$

Alternatively, by using the formula [14] connecting the Hahn polynomial $h_n^{(\alpha,\beta)}$ and the function ${}_3F_2(1)$, we obtain

$$W^{m}_{n_1 n_2}(\pm k_1, \pm k_2) = (-1)^m\sqrt{(2m\pm k_1\pm k_2+1)\frac{m!n_\rho!\Gamma(m\pm k_1\pm k_2+1)}{n_1!n_2!\Gamma(m\pm k_1+1)\Gamma(m\pm k_2+1)}}\times$$

$$\times\sqrt{\frac{\Gamma(n_1\pm k_1+1)\Gamma(n_2\pm k_2+1)}{\Gamma(n+m\pm k_1\pm k_2+2)}}h_m^{(\pm k_2, \pm k_1)}(n_1, n_1+n_2+1)$$

in terms of Hahn polynomials.

3.4. *Invariance algebra*

Let us consider the following realization of the $SU(1, 1)$ generators

$$J_0^{(a)} = \frac{1}{4\Omega}\left(-\partial_{x_a x_a}+\Omega^2 x_a^2+\frac{k_a^2-\frac{1}{4}}{x_a^2}\right),$$

$$J_1^{(a)} = -J_0^{(a)}+\frac{1}{2}\Omega x_a^2,\quad J_2^{(a)} = \frac{i}{2}\left(x_a\partial_{x_a}+\frac{1}{2}\right).$$

We thus have two copies (for $a = 1, 2$) of the Lie algebra $SU(1, 1)$ given by

$$[J_0^{(a)}, J_1^{(a)}] = iJ_2^{(a)},\quad [J_1^{(a)}, J_2^{(a)}] = -iJ_0^{(a)},$$
$$[J_2^{(a)}, J_0^{(a)}] = iJ_1^{(a)}$$

with the Casimir operator

$$Q_a = [J_0^{(a)}]^2-[J_1^{(a)}]^2-[J_2^{(a)}]^2 = \frac{1}{4}(k_a^2-1). \tag{27}$$

Introducing the raising and lowering operators $J_\pm^{(a)} = J_1^{(a)} \pm iJ_2^{(a)}$, we get

$$[J_0^{(a)}, J_\pm^{(a)}] = \pm J_\pm^{(a)},\quad [J_-^{(a)}, J_+^{(a)}] = 2J_0^{(a)}$$

and $Q_0 = [J_0^{(a)}]^2 - J_0^{(a)} - J_+^{(a)}J_-^{(a)}$.

As an irreducible representation of $SU(1, 1)$, the positive discrete series consists of an infinite number of states. Each of these states will be denoted as $|j_a m_a\rangle$, where $m_a = j_a + n_a$ ($n_a = 0, 1, 2, \ldots$). The eigenvalue of the Casimir operator is

$$Q_a = j_a(j_a - 1),$$

so that from (27) we have $j_a = \frac{1}{2}(1 \pm k_a)$. The matrix elements of the generators of the group $SU(1, 1)$ may be obtained through

$$J_0^{(a)}|j_a m_a\rangle = m_a|j_a m_a\rangle,$$
$$J_\pm^{(a)}|j_a m_a\rangle = \sqrt{(m_a \pm j_a)(m_a \mp j_a \pm 1)}|j_a m_a \pm 1\rangle \quad (28)$$

with $J_-^{(a)}|j_a j_a\rangle = 0$. Let us now define

$$C_0 = J_0^{(1)} + J_0^{(2)}, \quad C_\pm = J_\pm^{(1)} + J_\pm^{(2)}. \quad (29)$$

Equation (29) corresponds to the direct sum of the two $SU(1, 1)$ algebras for $a = 1, 2$. The coupled basis $|jm\rangle$ satisfies

$$C_0|jm\rangle = m|jm\rangle = (j + n)|jm\rangle,$$
$$Q|jm\rangle = j(j-1)|jm\rangle.$$

Given the values j_1 and j_2, the parameter j can take the discrete values

$$j = j_1 + j_2 + q, \quad q \in \mathbf{N}.$$

The Clebsch–Gordan decomposition yields

$$|jm\rangle = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2|jm\rangle|j_1 m_1\rangle \otimes |j_2 m_2\rangle,$$
$$m = m_1 + m_2$$

with $2j_a = 1 \pm k_a$, $2m_a = 2n_a + 1 \pm k_a$ and $2j = 2q + 2 \pm \pm k_1 \pm k_2$. By using the connection between the $SU(1, 1)$ CGc and the ${}_3F_2(1)$ function [15], one can obtain the same hypergeometric function as in (26).

Note that the Hamiltonian H of our two-dimensional oscillator system is

$$H = 2\Omega C_0.$$

From (28) and (29), we recover the spectrum of the system as given by (13) with $n = n_1 + n_2$.

Let us consider the two following operators:

$$N = J_0^{(1)} - J_0^{(2)},$$

$$M = Q_1 + Q_2 + 2J_0^{(1)}J_0^{(2)} - J_+^{(1)}J_-^{(2)} - J_-^{(1)}J_+^{(2)} + \frac{1}{4}.$$

They commute with H. Indeed, they are nothing but the integrals of motion (6) and (18). Moreover, let us define a third operator T via $T = [N, M]$. We have

$$T = 2[J_-^{(1)}J_+^{(2)} - J_+^{(1)}J_-^{(2)}]$$

or

$$T = -\frac{1}{4\Omega}(D_{xx} - D_{yy}) - \frac{i}{2\Omega}D_{xy}L_z + \frac{k_1^2 - \frac{1}{4}}{2\Omega x^2}\left(y\partial_y + \frac{1}{2}\right) - \frac{k_2^2 - \frac{1}{4}}{2\Omega y^2}\left(x\partial_x + \frac{1}{2}\right).$$

The operators N, M, T and H span a closed quadratic algebra since

$$[M, T] = -2(MN + NM) + \frac{k_1^2 - k_1^2}{2\Omega}H - N,$$

$$[T, N] = -2N^2 + \frac{1}{2\Omega^2}H^2 - 4M - k_1^2 - k_2^2 - 1$$

hold in addition to $[N, M] = T$, $[N, H] = 0$ and $[M, H] = 0$.

In the limiting case $k_1 = k_2 = 1/2$, we obtain a quadratic algebra too. In this case

$$N = \frac{1}{4\Omega}(D_{xx} - D_{yy}), \quad M = \frac{1}{4}L_z^2,$$

$$T = -\frac{1}{4\Omega}(D_{xx} - D_{yy}) - \frac{i}{2\Omega}D_{xy}L_z.$$

Instead of N, L_z^2 and T, we can consider N, L_z and $[N, L_z]$. In this regard, by putting

$$P_1 = N, \quad P_2 = \frac{1}{2}L_z, \quad P_3 = \frac{1}{i}[P_1, P_2] = \frac{1}{2}\Omega D_{xy}$$

we end up with the Lie algebra corresponding to the commutation relations

$$[P_k, P_l] = i\varepsilon_{klm}P_m, \quad k, l, m \in \{1, 2, 3\}.$$

Finally, going back to the generic case for k_1 and k_2, we define

$$L_0 = -\frac{1}{4}(2N \mp k_1 \pm k_2)$$

and

$$L_+ = \frac{J_-^{(1)}J_+^{(2)}}{\sqrt{(n_1 \pm k_1)(n_2 \pm k_2 + 1)}},$$

$$L_- = \frac{J_+^{(1)}J_-^{(2)}}{\sqrt{(n_2 \pm k_2)(n_1 \pm k_1 + 1)}}.$$

They act on the eigenfunctions (5) of the Hamiltonian H as

$$L_0\Psi_{n_1 n_2} = \frac{1}{2}(n_2 - n_1)\Psi_{n_1 n_2},$$

$$L_\pm\Psi_{n_1 n_2} = \sqrt{\left(n_1 + \frac{1}{2} \mp \frac{1}{2}\right)\left(n_2 + \frac{1}{2} \mp \frac{1}{2}\right)}\Psi_{n_1 \mp 1\, n_2 \pm 1}.$$

The operators L_0, L_+ and L_- generate the Lie algebra $SU(2)$ with

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2L_0$$

and are closely connected to our integrals of motion.

4. THREE-DIMENSIONAL CASE

4.1. Spherical basis

In spherical coordinates (r, θ, φ), the potential (1) can be rewritten as

$$V = \frac{1}{2}\Omega^2 r^2 + \frac{1}{2r^2}\left[\frac{1}{\sin^2\theta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{k_3^2-\frac{1}{4}}{\cos^2\theta}\right].$$

Looking for a solution of equation (2) in the form $R(r)\Theta(\theta)\Phi(\varphi)$, we are left with the system

$$\left(d_{\varphi\varphi}+A^2-\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}-\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)\Phi = 0, \tag{30}$$

$$\left[\frac{1}{\sin\theta}d_\theta(\sin\theta d_\theta) + J(J+1)-\frac{A^2}{\sin^2\theta}-\frac{k_3^2-\frac{1}{4}}{\cos^2\theta}\right]\Theta = 0, \tag{31}$$

$$\left[\frac{1}{r^2}d_r(r^2 d_r)+2E-\Omega^2 r^2+\frac{J(J+1)}{r^2}\right]R = 0. \tag{32}$$

The solution $\Phi(\varphi) \equiv \Phi_m(j; \pm k_1, \pm k_2)$ of equation (30), satisfying the boundary conditions (8) and the normalization condition (10), is given by (9). The separation constant A in (30) and (31) is quantized according to (11).

The solution $\Theta(\theta) \equiv \Theta_{qm}(\theta; \pm k_1 \pm k_2 \pm k_3)$ of (31) is (see [5])

$$\Theta(\theta) = \sqrt{\frac{[2(m+q+1)\pm k_1 \pm k_2 \pm k_3]q!\Gamma(q+2m\pm k_1\pm k_2\pm k_3+2)}{\Gamma(q\pm k_3+1)\Gamma(q+2m+2\pm k_1\pm k_2)}}(\cos\theta)^{\frac{1}{2}\pm k_3}(\sin\theta)^A P_q^{(A,\pm k_3)}(\cos 2\theta),$$

which satisfies the boundary condition

$$\Theta(0) = \Theta\left(\frac{\pi}{2}\right) = 0$$

and the normalization condition

$$2\int_0^{\pi/2}\Theta_{q'm}(\theta; \pm k_1, \pm k_2, \pm k_3)^* \times \Theta_{qm}(\theta; \pm k_1, \pm k_2, \pm k_3)\sin\theta d\theta = \delta_{q'q}.$$

The spherical separation constant J in (31) and (32) is

$$J = 2q + A \pm k_3 + \frac{1}{2} = 2q+2m\pm k_1 \pm k_2 \pm k_3 + \frac{3}{2}.$$

The solution $R(r) \equiv R_{n_r qm}(r; \pm k_1, \pm k_2, \pm k_3)$ of equation (32) is

$$R(r) = \sqrt{\frac{2\Omega^{\frac{3}{2}}n_r!}{\Gamma(n_r+2q+2m\pm k_1\pm k_2\pm k_3+3)}} \times (\sqrt{\Omega r^2})^J \exp\left(-\frac{\Omega}{2}r^2\right)L_{n_r}^{J+\frac{1}{2}}(\Omega r^2)$$

with

$$\int_0^\infty R_{n_r' qm}(r; \pm k_1, \pm k_2, \pm k_3) \times R_{n_r qm}(r; \pm k_1, \pm k_2, \pm k_3) r^2 dr = \delta_{n_r' n_r},$$

where $n_r \in \mathbf{N}$ is the radial quantum number.

The energy of the system is

$$E = \Omega\left(2n_r + J + \frac{3}{2}\right) = \Omega(2n \pm k_1 \pm k_2 \pm k_3 + 3), \quad n \in \mathbf{N};$$

where $n = n_r + q + m$ is the principal quantum number. It corresponds to the wave functions

$$\Psi_{nqm}(r, \theta, \varphi; \pm k_1, \pm k_2, \pm k_3) \equiv R(r)\Theta(\theta)\Phi(\varphi)$$

with n fixed.

4.2. Cylindrical Basis

In cylindrical coordinates (ρ, φ, z), we have

$$V = \frac{1}{2}\Omega^2\rho^2 + \frac{1}{2\rho^2}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right) + \frac{1}{2}\left(\Omega^2 z^2 + \frac{k_3^2-\frac{1}{4}}{z^2}\right).$$

The corresponding Schrödinger equation may be solved by looking for a solution in the form $R(\rho)\Phi(\varphi)Z(z)$. By combining the results of Sections 2 and 3, we get

$$Z(z) \equiv \Psi_{n_3}(z; \pm k_3), \quad \Phi(\varphi) \equiv \Phi_m(\varphi; \pm k_1, \pm k_2),$$

$$R(\rho) \equiv R_{n_\rho m}(\rho; \pm k_1, \pm k_2),$$

as given by (3), (9) and (12), respectively. The energy

$$E = \Omega(2n \pm k_1 \pm k_2 \pm k_3 + 3)$$

corresponds to the wave functions

$$\Psi_{n_\rho m n_3}(\rho, \varphi, z; \pm k_1, \pm k_2, \pm k_3) \equiv R(\rho)\Phi(\varphi)Z(z)$$

for which the principal quantum number $n = n_\rho + m + n_3$ is fixed.

4.3. Connecting Cartesian, Cylindrical and Spherical Bases

In the three-dimensional case, we have

$$\Psi_{n_1 n_2 n_3} = \sum_{m=0}^{n_1+n_2} W^m_{n_1 n_2}(\pm k_1, \pm k_2)\Psi_{n_\rho m n_3},$$

$$\Psi_{n_\rho m n_3} = \sum_{q=0}^{n_\rho+n_3} V^q_{n_\rho n_3}(\pm k_1, \pm k_2, \pm k_3)\Psi_{n_r q m},$$

where $n_1 + n_2 = m + n_\rho$ and $n_r + q = n_\rho + n_3$. For the expansion of the Cartesian basis over the spherical basis, we have

$$\Psi_{n_1 n_2 n_3} = \sum_{mq} C^{mq}_{n_1 n_2 n_3}(\pm k_1, \pm k_2, \pm k_3)\Psi_{n_r q m}, \qquad (33)$$

where $n_1 + n_2 + n_3 = n_r + q + m$. The coefficient $W^m_{n_1 n_2}(\pm k_1, \pm k_2)$ is identical to the one found in the two-dimensional case. It is given by (24). Similarly, it is easy to obtain

$$V^q_{n_\rho n_3}(\pm k_1, \pm k_2, \pm k_3) = (-1)^{n_\rho - q}\langle a'b'\alpha'\beta'|c'\gamma'\rangle, \quad (34)$$

where $2a' = n_3 + n_\rho \pm k_3$, $2b' = n_3 + n_\rho + 2m + 1 \pm k_1 \pm k_2$, $2c' = 2q + 2m + 1 \pm k_1 \pm k_2 \pm k_3$, $2\alpha' = n_3 - n_\rho \pm k_3$ and $2\beta' = 2m + n_\rho - n_3 + 1 \pm k_1 \pm k_2$. The expansion coefficients in (33) are given by the formula

$$C^{mq}_{n_1 n_2 n_3}(\pm k_1, \pm k_2, \pm k_3) = W^m_{n_1 n_2}(\pm k_1, \pm k_2) V^q_{n_\rho n_3}(\pm k_1, \pm k_2, \pm k_3). \qquad (35)$$

The value of the right-hand side of (35) follows from (24) and (34).

The authors would like to thank V.M. Ter-Antonyan for interesting discussions.

The work of one of the authors (G.P.) has been supported in part by the Russian Foundation for Basic Research under grant 98-01-00330.

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