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FINITE DIMENSIONAL INTEGRABLE SYSTEMS

8D OSCILLATOR AS A HIDDEN SU(2)-MONOPOLE

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In this report, in the framework of an analytical approach and with help of the generalized version of the Hurwitz transformation the five-dimensional $SU(2)$-monopole model is constructed from the eight-dimensional quantum oscillator. The Clebsh–Gordan expansion stimulated by the space gauge coupling, the hyperangle and the radial parts of the total wave function, the energy spectrum of the charge-monopole bound system and the corresponding degeneracy are calculated.

1. INTRODUCTION

This paper deals with the problem of monopole generation from oscillator-like systems, i.e. systems with a potential chosen as "oscillator + anything". In turn, the mentioned problem is connected with the search for the electromagnetic duality (ED) in the structure of Quantum Mechanics (QM). The existence of QM-duality seems important for two reasons. First, QM is a mathematically more simple theory than the gauge theories, so we have an excellent polygon for experience in ED. Second, there appears a wide range of applications because of ED pretentions to realize accurate calculations outside perturbation theory: according to ED, strongly coupled gauge theories can be formulated in the form of weakly coupled magnetic monopoles [1].

During the last years, the following machinery has been developed for a monopole generation: Hurwitz-like transformations applied to $2D$, $4D$ and $8D$ quantum oscillators transfer them into the charge-monopole bound systems in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^5$, respectively [2–4]. In two space dimensions the oscillator model was also constructed which can be transformed into a charge-monopole bound system with fractional statistics, interpolating the bosonic and fermionic limits [5]. Thus, the important extension of ED to the world of anyons is achieved.

Recently, the algebraic approach has been developed to clarify the relation between the $8D$ quantum oscillator and the charge-dyon bound system with the $SU(2)$-monopole [6][1]. This approach is exhaustive, but rather abstract. We make here an attempt to fulfill this gap[2] by presenting the analytical approach that is more explicit and hence more acceptable for understanding. Special attention is given to the space-gauge coupling and to the spectroscopy of the charge-dyon bound system.

[1] The particular version of the problem was previously considered in [7].

[2] See also [2] where the analytical approach to the same problem was presented more concisely.

2. SU(2)-MONOPOLE

Let us recall the way used for passage from the $8D$ oscillator to the $5D$ $SU(2)$-monopole. The initial system is governed by the equation

$$\frac{\partial^2\psi}{\partial u_\mu^2} + \frac{2m}{\hbar^2}\left(E - \frac{m\omega^2 u^2}{2}\right)\psi = 0, \qquad (1)$$

where $u_\mu \in \mathbb{R}^8$, $\mu = 0, 1, \ldots, 7$; $u^2 = u_\mu u_\mu$.

With the help of the special transformation[3]

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2, \\ x_1 &= 2(u_0u_4 + u_1u_5 - u_2u_6 - u_3u_7), \\ x_2 &= 2(u_0u_5 - u_1u_4 + u_2u_7 - u_3u_6), \\ x_3 &= 2(u_0u_6 + u_1u_7 + u_2u_4 + u_3u_5), \\ x_4 &= 2(u_0u_7 - u_1u_6 - u_2u_5 + u_3u_4), \end{aligned} \qquad (2)$$

$$\alpha_T = \frac{i}{2}\ln\frac{(u_0 + iu_1)(u_2 - iu_3)}{(u_0 - iu_1)(u_2 + iu_3)} \in [0, 2\pi),$$

$$\beta_T = 2\arctan\left(\frac{u_2^2 + u_3^2}{u_0^2 + u_1^2}\right)^{1/2} \in [0, \pi], \qquad (3)$$

$$\gamma_T = \frac{i}{2}\ln\frac{(u_0 + iu_1)(u_2 + iu_3)}{(u_0 - iu_1)(u_2 - iu_3)} \in [0, 4\pi)$$

we present $\mathbb{R}^8$ as a direct product $\mathbb{R}^5 \otimes S^3$ of the new configuration space $\mathbb{R}^5$ with the Cartesian coordinates $x_j \in (-\infty, \infty)$ and the intrinsic space S^3 with the coordinates α_T, β_T and γ_T. In the new coordinates, equation (1) can be led to the form

$$\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x_j} - \hbar A_j^a \hat{T}_a\right)^2\psi + \frac{\hbar^2}{2mr^2}\hat{T}^2\psi - \frac{e^2}{r}\psi = \epsilon\psi. \qquad (4)$$

[3] Formulae (2) are known as the Hurwitz transformation [8]; (3) is copied from [9].

Here $r = (x_j x_j)^{1/2}$ and $j = 0, 1, \ldots, 4$

$$\epsilon = -m\omega^2/8, \quad e^2 = E/4. \tag{5}$$

The operators $\hat{T}_a$ are the generators of the $SU(2)$ group and have the form

$$\hat{T}_1 = i\left(\cos\alpha_T \cot\beta_T \frac{\partial}{\partial\alpha_T} + \sin\alpha_T \frac{\partial}{\partial\beta_T} - \frac{\cos\alpha_T}{\sin\beta_T}\frac{\partial}{\partial\gamma_T}\right),$$

$$\hat{T}_2 = i\left(\sin\alpha_T \cot\beta_T \frac{\partial}{\partial\alpha_T} - \cos\alpha_T \frac{\partial}{\partial\beta_T} - \frac{\sin\alpha_T}{\sin\beta_T}\frac{\partial}{\partial\gamma_T}\right), \tag{6}$$

$$\hat{T}_3 = -i\frac{\partial}{\partial\alpha_T}.$$

The $5D$ vectors $\mathbf{A}^a$ have the form

$$\mathbf{A}^1 = \frac{1}{r(r+x_0)}(0, -x_4, -x_3, x_2, x_1),$$

$$\mathbf{A}^2 = \frac{1}{r(r+x_0)}(0, x_3, -x_4, -x_1, x_2),$$

$$\mathbf{A}^3 = \frac{1}{r(r+x_0)}(0, x_2, -x_1, x_4, -x_3).$$

Every term of the triplet A_j^a coincides with the vector potential of $5D$ Dirac monopole[4)] with a unit topological charge and the line of singularity along the nonpositive part of the x_0-axis. The vectors A_j^a are orthogonal to each other,

$$A_j^a A_j^b = \frac{1}{r^2}\frac{(r-x_0)}{(r+x_0)}\delta_{ab}$$

and also to the vector $x = (x_0, x_1, x_2, x_3, x_4)$.

We see that equation (4) describes the charge-dyon system with the $SU(2)$-monopole.

By using the orthogonality condition for vectors A_j^a we can transform equation (4) into

$$\left(\Delta_5 - 2iA_j^a \hat{T}_a \frac{\partial}{\partial x_j} - \frac{2}{r(r+x_0)}\hat{T}^2\right)\psi + \\ + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\psi = 0. \tag{7}$$

3. *LT*-COUPLING

Let us note that

$$iA_j^a \frac{\partial}{\partial x_j} = \frac{2}{r(r+x_0)}\hat{L}_a,$$

where

$$\hat{L}_1 = \frac{i}{2}[D_{41}(x) + D_{32}(x)],$$

$$\hat{L}_2 = \frac{i}{2}[D_{13}(x) + D_{42}(x)],$$

$$\hat{L}_3 = \frac{i}{2}[D_{12}(x) + D_{34}(x)]$$

and

$$D_{ij}(x) = -x_i\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial x_i}.$$

Using these formulae we can transform equation (7) into

$$\left(\Delta_5 - \frac{4}{r(r+x_0)}\hat{\vec{L}}\hat{\vec{T}} - \frac{2}{r(r+x_0)}\hat{T}^2\right)\psi + \\ + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\psi = 0. \tag{8}$$

We see that equation (8) contains the LT-coupling term demonstrating that we have no way to separate the wave function dependence on $\mathbb{R}^5$ and S^3.

Let us introduce in $\mathbb{R}^5$ the hypershperical coordinates $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, $\gamma \in [0, 4\pi)$ according to

$$x_0 = r\cos\theta,$$

$$x_1 + ix_2 = r\sin\theta\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}},$$

$$x_3 + ix_4 = r\sin\theta\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}}.$$

In these coordinates

$$\Delta_5 = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right) - \frac{4}{r^2\sin^2\theta}\hat{L}^2,$$

where

$$\hat{L}_1 = i\left(\cos\alpha\cot\beta\frac{\partial}{\partial\alpha} + \sin\alpha\frac{\partial}{\partial\beta} - \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right),$$

$$\hat{L}_2 = i\left(\sin\alpha\cot\beta\frac{\partial}{\partial\alpha} - \cos\alpha\frac{\partial}{\partial\beta} - \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right),$$

$$\hat{L}_3 = -i\frac{\partial}{\partial\alpha}$$

and

[4)] The $SU(2)$-monopole theory in $\mathbb{R}^5$ was constructed by Yang [10].

$$\hat{L}^2 = -\left[\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + \right.$$

$$\left. + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta\frac{\partial^2}{\partial\alpha\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right)\right].$$

Let us introduce $\hat{J}_a = \hat{L}_a + \hat{T}_a$. Since $\hat{J}_2 = \hat{L}_2 + \hat{T}_2 + 2\hat{L}_a\hat{T}_a$, equation (8) can be rewritten as

$$\left(\Delta_{r\theta} - \frac{\hat{L}^2}{r^2\sin^2(\theta/2)} - \frac{\hat{J}^2}{r^2\cos^2(\theta/2)}\right)\psi + \tag{9}$$

$$+ \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\psi = 0,$$

where

$$\Delta_{r\theta} = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right).$$

Emphasize that

$$[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc}\hat{L}_c, \quad [\hat{T}_a, \hat{T}_b] = i\epsilon_{abc}\hat{T}_c,$$

$$[\hat{J}_a, \hat{J}_b] = i\epsilon_{abc}\hat{J}_c.$$

Introduce the separation ansatz

$$\psi = \Phi(r, \theta)G(\alpha, \beta, \gamma; \alpha_T, \beta_T, \gamma_T),$$

where G are the eigenfunctions of $\hat{L}^2$, $\hat{T}^2$ and $\hat{J}^2$ with the eigenvalues $L(L+1)$, $T(T+1)$ and $J(J+1)$. If this is substituted into equation (9), the differential equation for the function $\Phi(r, \theta)$ immediately follows

$$\left(\Delta_{r\theta} - \frac{L(L+1)}{r^2\sin^2(\theta/2)} - \frac{J(J+1)}{r^2\cos^2(\theta/2)}\right)\Phi + \tag{10}$$

$$+ \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\Phi = 0.$$

Because of an LT-interaction, we look for the function G in the form

$$G = \sum_{M=m+t}(JM|L, m; T, t)D^L_{mm'}(\alpha, \beta, \gamma)D^T_{tt'}(\alpha_T, \beta_T, \gamma_T),$$

where $(JM|L, m; T, t)$ are the Clebsh–Gordan coefficients and $D^L_{mm'}$ and $D^T_{tt'}$ are the Wigner functions.

4. HYPERMOMENTUM

Pick up the function $\Phi(r, \theta)$ of the form

$$\Phi(r, \theta) = R(r)Z(\theta).$$

Then, equation (10) is separated into

$$\frac{1}{\sin^3\theta}\frac{d}{d\theta}\left(\sin^3\theta\frac{dZ}{d\theta}\right) - \frac{2L(L+1)}{1-\cos\theta}Z - \tag{11}$$

$$- \frac{2J(J+1)}{1+\cos\theta}Z + \lambda(\lambda+3)Z = 0$$

and a purely radial equation

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4\frac{dR}{dr}\right) - \frac{\lambda(\lambda+3)}{r^2}R + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)R = 0 \tag{12}$$

with the separation constant $\lambda(\lambda+3)$ equal to the non-negative eigenvalues of the hypermomentum operator.

It is convenient to make in equation (11) a change of variables, $y = (1-\cos\theta)/2$ and write

$$Z(y) = y^L(1-y)^J W(y).$$

Substituting this into equation (11), we obtain the hypergeometric equation

$$y(1-y)\frac{d^2W}{dy^2} + [c - (a+b+1)y]\frac{dW}{dy} - abW = 0$$

with $a = -\lambda + L + J$, $b = \lambda + L + J + 3$, $c = 2L + 2$.

Thus, we find that

$$Z(\theta) = (1-\cos\theta)^L(1+\cos\theta)^J \times$$

$$\times {}_2F_1\left(-\lambda + J + L, \lambda + J + L + 3; 2L + 2; \frac{1-\cos\theta}{2}\right).$$

This solution is well behaved at $\theta = \pi$ if the series ${}_2F_1$ terminates, i.e.

$$-\lambda + J + L = -n_\theta,$$

where $n_\theta = 0, 1, 2, \ldots$

5. ENERGY LEVELS

Let us now turn to the radial equation and introduce the function

$$f(r) = e^{\kappa r}r^{-\lambda}R(r).$$

It is easy to verify that the equation for $f(r)$ has the form of the confluent hypergeometric equation

$$z\frac{d^2f}{dz^2} + (c - z)\frac{df}{dz} - af = 0,$$

where $z = 2\kappa r$, $\kappa = \sqrt{-2m\epsilon/\hbar^2}$, $c = 2\lambda + 4$, $a = \lambda + 2 - 1/\kappa r_0$ and $r_0 = \hbar^2/me^2$. For the bound state solutions ($\epsilon < 0$)

$$\lambda + 2 - 1/\kappa r_0 = -n_r = 0, -1, -2, \ldots$$

and therefore

$$\epsilon^T_N = -\frac{me^4}{2\hbar^2(N/2+2)^2},$$

where

$$N = 2(n_r + \lambda) = 2(n_r + n_\theta + J + L).$$

6. DEGENERACY

For fixed T, the energy levels ϵ_N^T do not depend on L, J and λ, i.e. are degenerate. The total degeneracy is

$$g_N^T = (2T+1)\sum_\lambda \sum_L (2L+1)\sum_J (2J+1).$$

Since $\lambda = n_\theta + J + L$ and $N = 2(n_r + \lambda)$, it follows that (for fixed N and T) $\lambda = T, T+1, \ldots, N/2$. Then, $L_{\max} = \lambda - J_{\min}$ ($L_{\max}$ is fixed) and therefore $L_{\max} = \lambda - (L_{\max} - T)$ or $L_{\max} = (\lambda + T)/2$. Thus,

$$g_N^T = (2T+1)\sum_{\lambda=T}^{N/2} \sum_{L=0,1/2}^{\frac{\lambda-T}{2}} (2L+1)\sum_J (2J+1).$$

Now, comparing $|L-T| \le J \le L+T$ and $J \le \lambda - L$ we conclude that

$$J = |L-T|, |L-T|+1, \ldots, L+T, \quad \text{for}$$
$$L = 0, \frac{1}{2}, \ldots, \frac{\lambda-T}{2},$$

$$J = |L-T|, |L-T|+1, \ldots, \lambda - L, \quad \text{for}$$
$$L = \frac{\lambda-T+1}{2}, \ldots, \frac{\lambda+T}{2}$$

and rewrite the formula for g_N^T in a more explicit form

$$g_N^T = (2T+1)\sum_{\lambda=T}^{N/2}\left\{\sum_{L=0,1/2}^{\frac{\lambda-T}{2}}(2L+1)\sum_{J=|L-T|}^{L+T}(2J+1) + \right.$$
$$\left. + \sum_{L=\frac{\lambda-T+1}{2}}^{\frac{\lambda+T}{2}}(2L+1)\sum_{J=|L-T|}^{\lambda-L}(2J+1)\right\}.$$

Finally, after some tedious calculations we obtain the following result:

$$g_N^T = \frac{1}{12}(2T+1)^2\left(\frac{N}{2}-T+1\right)\left(\frac{N}{2}-T+2\right)\times$$
$$\times\left\{\left(\frac{N}{2}-T+2\right)\left(\frac{N}{2}-T+3\right)+2T(N+5)\right\}.$$

For $T=0$ and $N=2n$ (even) the r.h.s. of the last formula is equal to $(n+1)(n+2)^2(n+3)/12$, i.e. to the degeneracy of pure Coulomb levels. Further, $T = 0, 1, \ldots, N/2$ and $T = 1/2, 3/2, \ldots, N/2$ for even and odd N, respectively. Therefore,

$$g_N = \sum_{T=0,1/2}^{N/2} g_N^T = \frac{(N+7)!}{7!N!},$$

i.e. we obtain the degeneracy of the energy levels for the $8D$ isotropic quantum oscillator.

7. CONCLUSIONS

Formulae (2) and (3) together with the ansatz (5) form the duality transrormation mapping of the $8D$ quantum oscillator into the charge-dyon system with the $SU(2)$-monopole. Let us stress the meaning we use for the term duality. Both equations (1) and (4) contain two quantities, ω and E. For equation (1) ω is the fixed parameter (coupling constant) and E is the quantity to be quantized (energy of the $8D$ oscillator). On the contrary, as it is easy to see from (5), for equation (4) E is a fixed parameter (Coulomb coupling constant) and ω is the quantity to be quantized (ω^2 is the energy of the final system). Thus, the $8D$ quantum oscillator and the charge-dyon bound system with the $SU(2)$-monopole are not identical, but dual to each other.

This type duality is valid not only for the $8D$, $4D$ and $2D$ oscillators, but also for oscillator-like systems with the potentials

$$V(u^2) = c_0 + c_1 u^2 + W(u^2),$$

where $W(u^2)$ has a polynomial form

$$W(u^2) = \sum_{n=2}^{\infty} c_n u^{2n}.$$

For such modified potentials, the ansatz (5) can be rewritten as

$$\epsilon = -\frac{c_1}{4}, \quad e^2 = \frac{E - c_0}{4}.$$

Thus, the value of the function $V(u^2)$ at $u^2 = 0$ contributes to the Coulomb coupling constant e^2. It is also easy to verify that the l.h.s. of equation (4) acquires the additional term $(-W(r)/4r)$.

Appendix

Consider the normalization of the wave function $\psi(x, \alpha_T, \beta_T, \gamma_T)$. A standard calculation shows that the radial wave function $R(r)$ normalized by the condition

$$\int_0^\infty r^4 [R_{n,\lambda}(r)]^2 dr = 1$$

has the form

$$R_{n_r\lambda}(r) = \frac{4}{r_0^{5/2}(n_r+\lambda+2)^3}\frac{1}{(2\lambda+3)!}\sqrt{\frac{(n_r+2\lambda+3)!}{(n_r)!}}\times$$

$$\times(2\kappa r)^{\lambda}e^{-\kappa r}F(-n_r;\, 2\lambda+4;\, 2\kappa r).$$

The full wave function

$$\psi = C^{\lambda}_{LTJ}R_{n_r\lambda}(r)Z_{\lambda LJ}(\theta)G^{JM}_{LTm'l'}(\alpha,\beta,\gamma;\,\alpha_T,\beta_T,\gamma_T)$$

is normalized by the condition

$$\int|\psi|^2 dv = 1,$$

where

$$dv = r^4\sin^3\theta dr d\theta d\Omega d\Omega_T$$

and

$$d\Omega = \frac{1}{8}\sin\beta d\beta d\alpha d\gamma, \quad d\Omega_T = \frac{1}{8}\sin\beta_T d\beta_T d\alpha_T d\gamma_T.$$

Using the formula

$${}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-y}{2}\right) =$$

$$= \frac{n!\Gamma(a+1)}{\Gamma(n+a+1)}P_n^{(a,b)}(y),$$

where $P_n^{(a,b)}(y)$ are the Jacobi polynomials, and taking into account that

$$\int_{-1}^{1}(1-y)^a(1+y)^b\{P_n^{(a,b)}(y)\}^2 dy = \frac{2^{a+b+1}}{2n+a+b+1}\times$$

$$\times\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{n!\Gamma(n+a+b+1)}\int D^{j_2^*}_{m_2m_2'}(\alpha,\beta,\gamma)\times$$

$$\times D^{j_1}_{m_1m_1'}(\alpha,\beta,\gamma)d\Omega = \frac{2\pi^2}{2j_1+1}\delta_{j_1j_2}\delta_{m_1m_2}\delta_{m_1'm_2'}$$

it is easy to obtain that

$$C^{\lambda}_{LJT} = \sqrt{\frac{(2L+1)(2T+1)(2\lambda+3)}{2^{2J+2L+5}\pi^4\Gamma(\lambda+J-L+2)}}\times$$

$$\times\sqrt{\frac{(\lambda-J-L)!\Gamma(\lambda+J+L+3)}{\Gamma(\lambda-J+L+2)}}.$$

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