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# HIDDEN SYMMETRY OF THE YANG—COULOMB MONOPOLE

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# 1 Introduction

As originally proposed by Yang [1], the Dirac monopole [2] can be generalized to the $SU(2)$ gauge group and such a generalization (Yang monopole) can be achieved only in the five-dimensional Euclidean space.

The simplest bound system connected to the Yang monopole is the Yang-Coulomb Monopole (YCM) which we define here as the system composed of the Yang monopole and a particle of the isospin coupled to the monopole by the $SU(2)$ and the Coulomb interaction.

It is of interest to ask what happens to the known $SO(6)$ hidden symmetry of the five-dimensional Coulomb system after $SU(2)$ generalization. In this note, we prove that $SU(2)$ leads to the $SO(6)$ group acting in a more general $\mathbb{R}^5 \otimes S^3$ space. We use this new symmetry for computation of the YCM energy spectrum by a pure algebraic method.

# 2 Notation and $\tau$ matrices

We keep the following notation: $j = 0,1,2,3,4$; $\mu = 1,2,3,4$; $a = 1,2,3$; $x_j$ are the Cartesian coordinates of the particle, $\hat{T}_a$ denote the $SU(2)$ gauge group generators; $\vec{A}^a = (0, A^a_\mu)$ is the triplet of Yang monopole's gauge potentials; $F^a_{ik}$ is the gauge field of the Yang monopole; $\sigma^a$ are the Pauli matrices, and $\tau^a$ are the 4x4 matrices

$$\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix}, \quad \tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}, \quad \tau^3 = \frac{1}{2}\begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$

$\tau^a$ matrices satify to relations

$$[\tau^a, \tau^b] = i\epsilon_{abc}\tau^c, \quad 4\tau^a_{\mu\lambda}\tau^b_{\lambda\nu} = \delta_{ab}\delta_{\mu\nu} + 2i\epsilon_{abc}\tau^c_{\mu\nu}$$

$$\epsilon_{abc}\tau^b_{\alpha\beta}\tau^c_{\mu\nu} = \frac{i}{2}\left(\delta_{\alpha\mu}\tau^a_{\nu\beta} - \delta_{\alpha\nu}\tau^a_{\mu\beta} + \delta_{\beta\nu}\tau^a_{\mu\alpha} - \delta_{\beta\mu}\tau^a_{\nu\alpha}\right)$$

and $r = (x_j x_j)^{1/2}$.

# 3 Yang monopole

Consider the formula

$$A^a_\mu = \frac{2i}{r(r + x_0)}\tau^a_{\mu\nu}x_\nu .$$

It is obvious that each term of the $\vec{A}^a$- triplet coincides with the gauge potential of the five-dimensional Dirac monopole with a unit topological charge and the line of singularity extended along the nonpositive part of the $x_0$-axis. The vectors $A^a_j$ are orthogonal to each other

$$A^a_j A^b_j = \frac{r - x_0}{r^2(r + x_0)}\delta_{ab}$$

and to the vector $x_j$ $(x_j A^a_j = 0)$.
By definition,

$$F^a_{ik} = \partial_i A^a_k - \partial_k A^a_i + \epsilon_{abc} A^b_i A^c_k$$

or, in a more explicit form,

$$F^a_{0\mu} = -\frac{2i}{r^3}\tau^a_{\mu\nu}x_\nu = -\frac{r+x_0}{r^2}A^a_\mu$$

$$F^a_{\mu\nu} = \frac{1}{r^2}\left(x_\nu A^a_\mu - x_\mu A^a_\nu - 2i\tau^a_{\mu\nu}\right).$$

The straighforward computation gives

$$F^a_{ik}F^b_{ik}\hat{T}_a\hat{T}_b = \frac{4}{r^4}\hat{T}^2 \tag{1}$$

where $\hat{T}^2 = \hat{T}_a\hat{T}_a$.

# 4 Yang SO(5) symmetry

The YCM is governed by the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{\pi}^2 + \frac{\hbar^2}{2mr^2}\hat{T}^2 - \frac{e^2}{r}$$

where $\hat{\pi}^2 = \hat{\pi}_j\hat{\pi}_j$,

$$\hat{\pi}_j = -i\hbar\frac{\partial}{\partial x_j} - \hbar A^a_j\hat{T}_a$$

and

$$[\hat{\pi}_i, x_k] = -i\hbar\delta_{ik}, \quad [\hat{\pi}_i, \hat{\pi}_k] = i\hbar^2 F^a_{ik}\hat{T}_a\,. \tag{2}$$

Let us consider the operator

$$\hat{L}_{ik} = \frac{1}{\hbar}(x_i\hat{\pi}_k - x_k\hat{\pi}_i) - r^2 F^a_{ik}\hat{T}_a\,.$$

It is easy to verify that

$$[\hat{L}_{ik}, x_j] = i\delta_{ij}x_k - i\delta_{kj}x_i\ . \tag{3}$$

For the commutator $[\hat{L}_{ik}, \pi_j]$ we have

$$[\hat{L}_{ik}, \hat{\pi}_j] = i\delta_{ij}\hat{\pi}_k - i\delta_{kj}\hat{\pi}_i + \hat{Q}_{ikj}$$

where

$$\hat{Q}_{ikj} = i\hbar\left(x_i F^a_{kj} - x_k F^a_{ij}\right)\hat{T}_a + [\hat{\pi}_j, r^2 F^a_{ik}\hat{T}_a] .$$

There are four possibilities for the indices $i, j, k$:

$$\begin{pmatrix} i \\ j \\ k \end{pmatrix} = \begin{pmatrix} \mu & \mu & 0 & 0 \\ \nu & \nu & \nu & \nu \\ \alpha & 0 & \alpha & 0 \end{pmatrix}$$

and, therefore, the direct calculation is required. After some algebra we obtain $\hat{Q}_{ikj} = 0$, and hence

$$[\hat{L}_{ik}, \hat{\pi}_j] = i\delta_{ij}\hat{\pi}_k - i\delta_{kj}\hat{\pi}_i \ . \qquad (4)$$

Now the commutation rule for the $SO(5)$ group generators

$$[\hat{L}_{ij}, \hat{L}_{mn}] = i\delta_{im}\hat{L}_{jn} - i\delta_{jm}\hat{L}_{in} - i\delta_{in}\hat{L}_{jm} + i\delta_{jn}\hat{L}_{im} \qquad (5)$$

can be derived from (3) and (4). Moreover, it follows from (3) and (4) that $\hat{L}_{ik}$ commutes with $\hat{H}$. This $SO(5)$ group was previously proposed by Yang [1] as the dynamical group of symmetry for the Hamiltonian $\hat{H}_Y - e^2/r$ including only a monopole–isospin interaction.

# 5 SO(6) symmetry of YCM

Let us consider the operator

$$\hat{M}_k = \frac{1}{2\sqrt{m}}\left(\hat{\pi}_i\hat{L}_{ik} - \hat{L}_{ik}\hat{\pi}_i + \frac{2me^2}{\hbar}\frac{x_k}{r}\right) \qquad (6)$$

by analogy with the Runge–Lenz vector. Long manipulation exercises yield $[\hat{H}, \hat{M}_k] = 0$, which means that $\hat{M}_k$ is the constant of motion. Now, from (3), (4) and (5) one can show

$$[\hat{L}_{ij}, \hat{M}_k] = i\delta_{ik}\hat{M}_j - i\delta_{jk}\hat{M}_i \ .$$

More complicated calculation leads to the formula

$$[\hat{M}_i, \hat{M}_k] = -2i\hat{H}\hat{L}_{ik} - \frac{i}{m}x_i x_k F^a_{mn} T_a \hat{\pi}_m \hat{\pi}_n - \frac{2\hbar^2}{m}\frac{x_i x_k}{r^4}\hat{T}^2 .$$

It is easily to verify from (1) and (2) that last two terms cancel each other and, therefore,

$$[\hat{M}_i, \hat{M}_k] = -2i\hat{H}\hat{L}_{ik} \ .$$

This commutator is identical with the corresponding commutator for the Coulomb problem. For $\hat{M}'_i = \left(-2\hat{H}\right)^{-1/2}\hat{M}_i$ one has

$$[\hat{M}'_i, \hat{M}'_k] = i\hat{L}_{ik} \ .$$

Now, introduce the 6x6 matrix

$$\hat D = \begin{pmatrix} \hat L_{ij} & -\hat M'_i \\ \hat M'_j & 0 \end{pmatrix} .$$

The components $\hat D_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3, 4, 5)$ give an $so(6)$ algebra

$$[\hat D_{\mu\nu}, \hat D_{\lambda\rho}] = i\delta_{\mu\lambda}\hat D_{\nu\rho} - i\delta_{\nu\lambda}\hat D_{\mu\rho} - i\delta_{\mu\rho}\hat D_{\nu\lambda} + i\delta_{\nu\rho}\hat D_{\mu\lambda} .$$

Since $[\hat H, \hat D_{\mu\nu}] = 0$, one concludes that **YCM** is provided by the $SO(6)$ group of hidden symmetry.

# 6 YCM energy spectrum

Having obtained the group of hidden symmetry one can calculate the energy eigenvalues by a pure algebraic method.

It is known [3] that the Casimir operators for $SO(6)$ are

$$\begin{aligned} \hat C_2 &= \frac{1}{2}\hat D_{\mu\nu}\hat D_{\mu\nu} \\ \hat C_3 &= \epsilon_{\mu\nu\rho\sigma\tau\lambda}\hat D_{\mu\nu}\hat D_{\rho\sigma}\hat D_{\tau\lambda} \\ \hat C_4 &= \frac{1}{2}\hat D_{\mu\nu}\hat D_{\nu\rho}\hat D_{\rho\tau}\hat D_{\tau\mu} . \end{aligned}$$

According to [3], the eigenvalues of these operators can be taken as

$$\begin{aligned} C_2 &= \mu_1(\mu_1+4) + \mu_2(\mu_2+2) + \mu_3^2 \\ C_3 &= 48(\mu_1+2)(\mu_2+1)\mu_3 \\ C_4 &= \mu_1^2(\mu_1+4)^2 + 6\mu_1(\mu_1+4) + \mu_2^2(\mu_2+2)^2 + \mu_3^4 - 2\mu_3^2 \end{aligned}$$

where $\mu_1$, $\mu_2$ and $\mu_3$ are the positive integer or half integer numbers and $\mu_1 \geq \mu_2 \geq \mu_3$.

The direct and very hard calculations lead to the representation

$$\hat C_2 = -\frac{e^4 m}{2\hbar^2 \hat H} + 2\hat T^2 - 4 \tag{7}$$

$$\hat C_3 = 48\left(-\frac{me^4}{2\hbar^2 \hat H}\right)^{1/2}\hat T^2 \tag{8}$$

$$\hat C_4 = \hat C_2^2 + 6\hat C_2 - 4\hat C_2\hat T^2 - 12\hat T^2 + 6\hat T^4 . \tag{9}$$

From the last equation we can obtain another expression for the eigenvalue $C_4$

$$C_4 = [C_2 - 2T(T+1)]^2 + 6[C_2 - 2T(T+1)] + 2T^2(T+1)^2$$

and conclude that

$$C_2 - 2T(T+1) = \mu_1(\mu_1 + 4) \tag{10}$$

$$\mu_2^2(\mu_2 + 2)^2 + \mu_3^4 - 2\mu_3^2 = 2T^2(T+1)^2 . \tag{11}$$

The energy levels of YCM can be obtained from (7) and (10)

$$\epsilon_N^T = -\frac{me^4}{2\hbar^2(\mu_1 + 2)^2} . \tag{12}$$

The substitution of the eigenvalues of $\hat{H}$ and $\hat{T}^2$ in the equation for $\hat{C}_3$ gives one more formula for $C_3$

$$C_3 = 48(\mu_2 + 2)T(T+1) .$$

Now we have two expressions for $C_3$ and the comparison leads to the relation

$$T(T+1) = (\mu_2 + 2)\mu_3 . \tag{13}$$

Comparing this with (11), we have the equation

$$\left(\mu_2^2 - \mu_3^2\right)\left[(\mu_2 + 2)^2 - \mu_3^2\right] = 0 .$$

Since $\mu_3 \le \mu_2$, one concludes that $\mu_3 = \mu_2$. Then, from (13) it follows that $\mu_2 = T$, which means that $\mu_1$ in (12) takes only values $\mu_1 = T, T+1, T+2, \ldots$


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Мардоян Л.Г., Сисакян А.Н., Тер-Антонян В.М. E2-98-46
Скрытая симметрия монополя Янга—Кулона

Рассмотрена связанная система, составленная из монополя Янга и изоспиновой частицы, скрепленных друг с другом $SU(2)$ и кулоновским взаимодействием. Найден обобщенный вектор Рунге—Ленца и установлена группа скрытой симметрии $SO(6)$. Показано также, что группа скрытой симметрии позволяет вычислить спектр системы алгебраическим путем.

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Mardoyan L.G., Sissakian A.N., Ter-Antonyan V.M. E2-98-46
Hidden Symmetry of the Yang—Coulomb Monopole

The bound system composed of the Yang monopole coupled to a particle of the isospin by the $SU(2)$ and Coulomb interaction is considered. The generalized Runge—Lenz vector and the $SO(6)$ group of hidden symmetry are established. It is also shown that the group of hidden symmetry makes it possible to calculate the spectrum of the system by a pure algebraic method.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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