Dynamical Chern-Simons term generation at finite density

A. N. Sissakian
Bogolubov Theoretical Laboratory, Joint Institute for Nuclear Research, Dubna, Moscow region 141980, Russia

O. Yu. Shevchenko[1] and S. B. Solganik[2]
Laboratory of Nuclear Problems, Joint Institute for Nuclear Research, Dubna, Moscow region 141980, Russia

Abstract

The Chern-Simons topological term dynamical generation in the effective action is obtained at arbitrary finite density. By using the proper time method and perturbation theory it is shown that $\mu^2 = m^2$ is the crucial point for Chern–Simons. So when $\mu^2 < m^2$ μ–influence disappears and we get the usual Chern-Simons term. On the other hand when $\mu^2 > m^2$ the Chern-Simons term vanishes because of non–zero density of background fermions. In particular for massless case parity anomaly is absent at any finite density. This result holds in any odd dimension as in abelian so as in nonabelian cases.

Since introducing the Chern-Simons (CS) topological term [1] and by now the great number of papers devoted to it appeared. Such interest is explained by variety of significant physical effects caused by CS secondary characteristic class. These are, for example, gauge particles mass appearance in quantum field theory, applications to condense matter physics such as the fractional quantum Hall effect and high T_c superconductivity, possibility of free of metric tensor theory construction and so on.

It was shown [2-4] in a conventional zero density gauge theory that the CS term is generated in the Eulier–Heisenberg effective action by quantum corrections. The main goal of this paper is to explore the parity anomalous CS term generation at finite density. In the excellent paper by Niemi [5] it was emphasized that the charge density at $\mu \neq 0$ becomes nontopological object, i.e contains as topological part so as nontopological one. The charge density at $\mu \neq 0$ (nontopological, neither parity odd nor parity even object)[3] in QED_3 at finite density was calculated and exploited in [6]. It must be emphasized that in [6] charge density contains as well parity odd part corresponding to CS term so as parity even part, which can't be covariantized and don't contribute to the mass of the gauge field. Here we are interested in effect of finite density influence on covariant parity odd form in action leading to the gauge field mass generation — CS topological term. Deep insight on this phenomena at small densities was done in [5, 7]. The result for CS term coefficient in QED_3 is $[\,\mathrm{sgn}(m-\mu) + \mathrm{sgn}(m+\mu)]$ (see [7], formulas (10.19)). However, to get this result it was heuristicaly supposed that at small densities index theorem could still be used and only odd in energy part of spectral density is responsible for

[1]e-mail: shevch@nusun.jinr.ru
[2]e-mail: solganik@thsun1.jinr.ru

[3]For abbreviation, speaking about parity invariance properties of local objects, we will keep in mind symmetries of the corresponding action parts.

parity nonconserving effect. Because of this in [7] it had been stressed that the result holds only for small μ. However, as we'll see below this result holds for any values of chemical potential. Thus, to obtain trustful result at any values of μ one have to use transparent and free of any restrictions on μ procedure, which would allow to perform calculations with arbitrary nonabelian background gauge fields.

Since the chemical potential term $\mu\bar{\psi}\gamma^0\psi$ is odd under charge conjugation we can expect that it would contribute to P and CP nonconserving quantity — CS term. As we will see, this expectation is completely justified.

The zero density approach usually is a good quantum field approximation when the chemical potential is small as compared with characteristic energy scale of physical processes. Nevertheless, for investigation of topological effects it is not the case. As we will see below, even a small density could lead to principal effects.

Introduction of a chemical potential μ in a theory corresponds to the presence of a nonvanishing background charge density. So, if $\mu > 0$, then the number of particles exceeds that of antiparticles and vice versa. It must be emphasized that the formal addition of a chemical potential looks like a simple gauge transformation with the gauge function μt. However, it doesn't only shift the time component of a vector potential but also gives corresponding prescription for handling Green's function poles. The correct introduction of a chemical potential redefines the ground state (Fermi energy), which leads to a new spinor propagator with the correct ϵ-prescription for poles. So, for the free spinor propagator we have (see, for example, [8, 9])

$$G(p;\mu) = \frac{\tilde{\not p} + m}{(\tilde{p_0} + i\epsilon\ \mathrm{sgn} p_0)^2 - \vec{p}^2 - m^2}, \tag{1}$$

where $\tilde{p} = (p_0 + \mu, \vec{p})$. Thus, when $\mu = 0$ one at once gets the usual ϵ-prescription because of the positivity of $p_0\ \mathrm{sgn} p_0$. In the presence of a background Yang–Mills field we consequently have for the Green function operator

$$\hat{G} = (\gamma\tilde{\pi} - m)\frac{1}{(\gamma\tilde{\pi})^2 - m^2 + i\epsilon(p_0 + \mu)\ \mathrm{sgn}(p_0)}, \tag{2}$$

where $\tilde{\pi_\nu} = \pi_\nu + \mu\delta_{\nu 0}$, $\pi_\nu = p_\nu - gA_\nu(x)$.

Let's first consider a (2+1) dimensional abelian case and choose the background field in the form

$$A^\mu = \frac{1}{2}x_\nu F^{\nu\mu}, \qquad F^{\nu\mu} = \mathrm{Const}.$$

To obtain the CS term in this case, it is necessary to consider the background current

$$\langle J^\mu \rangle = \frac{\delta S_{eff}}{\delta A_\mu}$$

rather than the effective action itself. This is because the CS term formally vanishes for such the choice of A^μ but its variation with respect to A^μ produces a nonvanishing current. So, consider

$$\langle J^\mu \rangle = -ig\ \mathrm{tr}\left[\gamma^\mu G(x, x')\right]_{x\to x'} \tag{3}$$

where

$$G(x, x') = \exp\left(-ig\int_{x'}^{x} d\zeta_\mu A^\mu(\zeta)\right)\langle x|\hat{G}|x'\rangle. \tag{4}$$

Let's rewrite Green function (2) in a more appropriate form

$$\hat{G} = (\gamma\tilde{\pi} - m)\Big[\frac{\theta((p_0+\mu)\ \mathrm{sgn}(p_0))}{(\gamma\tilde{\pi})^2 - m^2 + i\epsilon} + \frac{\theta(-(p_0+\mu)\ \mathrm{sgn}(p_0))}{(\gamma\tilde{\pi})^2 - m^2 - i\epsilon}\Big]. \tag{5}$$

Now, we use the well known integral representation of denominators

$$\frac{1}{\alpha \pm i0} = \mp i \int_0^\infty ds\ \mathrm{e}^{\pm i\alpha s},$$

which corresponds to introducing the "proper–time" s into the calculation of the Eulier–Heisenberg lagrangian by the Schwinger method [10]. We obtain

$$\begin{aligned}\hat{G} = (\gamma\tilde{\pi} - m)\Big[& - i\int_0^\infty ds \exp\left(is\left[(\gamma\tilde{\pi})^2 - m^2 + i\epsilon\right]\right)\theta((p_0+\mu)\ \mathrm{sgn}(p_0)) + \\ & + i\int_0^\infty ds \exp\left(-is\left[(\gamma\tilde{\pi})^2 - m^2 - i\epsilon\right]\right)\theta(-(p_0+\mu)\ \mathrm{sgn}(p_0))\Big]. \end{aligned} \tag{6}$$

For simplicity, we restrict ourselves only to the magnetic field case, where $A_0 = 0, [\tilde{\pi}_0, \tilde{\pi}_\mu] = 0$. Then we easily can factorize the time dependent part of Green function

$$G(x,x') = \int \frac{d^3p}{(2\pi)^3}\hat{G}\ \mathrm{e}^{ip(x-x')} = \int \frac{d^2p}{(2\pi)^2}\hat{G}_{\vec{x}}\ \mathrm{e}^{i\vec{p}(\vec{x}-\vec{x}')}\int\frac{dp_0}{2\pi}\hat{G}_{x_0}\ \mathrm{e}^{ip_0(x_0-x_0')}. \tag{7}$$

By using the obvious relation

$$(\gamma\tilde{\pi})^2 = (p_0+\mu)^2 - \vec{\pi}^2 + \frac{1}{2}g\sigma_{\mu\nu}F^{\mu\nu} \tag{8}$$

one gets

$$\begin{aligned} G(x,x')|_{x\to x'} = & -i\int\frac{dp_0}{2\pi}\frac{d^2p}{(2\pi)^2}(\gamma\tilde{\pi} - m)\int_0^\infty ds\Big[\mathrm{e}^{is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{-is\vec{\pi}^2}\ \mathrm{e}^{isg\sigma F/2} - \\ - & \ \theta(-(p_0+\mu)\ \mathrm{sgn}(p_0))\left(\mathrm{e}^{is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{-is\vec{\pi}^2}\ \mathrm{e}^{isg\sigma F/2} + \mathrm{e}^{-is(\tilde{p}_0^2-m^2)}\ \mathrm{e}^{is\vec{\pi}^2}\ \mathrm{e}^{-isg\sigma F/2}\right)\Big]. \end{aligned} \tag{9}$$

Here the first term corresponds to the usual μ–independent case and there are two additional μ–dependent terms. In the calculation of the current the following trace arises:

$$\mathrm{tr}\left[\gamma^\mu(\gamma\tilde{\pi} - m)\ \mathrm{e}^{isg\sigma F/2}\right] = 2\pi^\nu g^{\nu\mu}\cos(g|{}^*F|s) + 2\frac{\pi^\nu F^{\nu\mu}}{|{}^*F|}\sin(g|{}^*F|s) - 2im\frac{{}^*F^\mu}{|{}^*F|}\sin(g|{}^*F|s),$$

where ${}^*F^\mu = \varepsilon^{\mu\alpha\beta}F_{\alpha\beta}/2$ and $|{}^*F| = \sqrt{B^2 - E^2}$. Since we are interested in calculation of the parity odd part (CS term) it is enough to consider only terms proportional to the dual strength tensor ${}^*F^\mu$. On the other hand the term $2\pi^\nu g^{\nu\mu}\cos(g|{}^*F|s)$ at $\nu = 0$ (see expression for the trace, we take in mind that here there are only magnetic field) also gives nonzero contribution to the current $J^0_{c.s.}$ [6]

$$J^0_{\mathrm{even}} = \frac{|eB|}{2\pi}\left(\mathrm{Int}\left[\frac{\mu^2 - m^2}{2|eB|}\right] + \frac{1}{2}\right)\theta(|\mu| - |m|). \tag{10}$$

This part of current is parity invariant because under parity $B \to -B$. It is clear that this parity even object does contribute neither to the parity anomaly nor to the mass of the gauge field.

Moreover, this term has been obtained [6] in the pure magnetic background and scalar magnetic field occurs in the argument's denominator of the cumbersome function – integer part. So, the parity even term seems to be "noncovariantizable", i.e. it can't be converted in covariant form in effective action. For a pity, in papers [6] charge density consisting of both parity odd and parity even parts is dubbed CS, what leads to misunderstanding. The main goal of this paper is to explore the parity anomalous topological CS term in the effective action at finite density. So, just the term proportional to the dual strength tensor ${}^*F^\mu$ will be considered. The relevant part of the current reads

$$\begin{aligned} J^\mu_{CS} &= \frac{g}{2\pi}\int dp_0 \int \frac{d^2p}{(2\pi)^2}\int_0^\infty ds \frac{2im\,{}^*F^\mu}{|{}^*F|}\sin\left(g|{}^*F|s\right)\Big[\, \mathrm{e}^{is(\tilde{p}_0^2-m^2)}\,\mathrm{e}^{-is\vec{\pi}^2} - \\ &- \theta(-(p_0+\mu)\,\mathrm{sgn}(p_0))\left(\mathrm{e}^{is(\tilde{p}_0^2-m^2)}\,\mathrm{e}^{-is\vec{\pi}^2} - \mathrm{e}^{-is(\tilde{p}_0^2-m^2)}\,\mathrm{e}^{is\vec{\pi}^2}\right)\Big]. \end{aligned} \tag{11}$$

Evaluating two–momentum integral we derive

$$J^\mu_{CS} = \frac{g^2}{4\pi^2}m\,{}^*F^\mu \int_{-\infty}^{+\infty}dp_0\int_0^\infty ds\Big[\,\mathrm{e}^{is(\tilde{p}_0^2-m^2)} - \theta(-\tilde{p}_0\,\mathrm{sgn}(p_0))\left(\mathrm{e}^{is(\tilde{p}_0^2-m^2)} + \mathrm{e}^{-is(\tilde{p}_0^2-m^2)m}\right)\Big]. \tag{12}$$

Thus, we get besides the usual CS part [3], also the μ–dependent one. It is easy to calculate it by use of the formula

$$\int_0^\infty ds\,\mathrm{e}^{is(x^2-m^2)} = \pi\left(\delta(x^2-m^2) + \frac{i}{\pi}\mathcal{P}\frac{1}{x^2-m^2}\right)$$

and we get eventually

$$\begin{aligned} J^\mu_{CS} &= \frac{m}{|m|}\frac{g^2}{4\pi}{}^*F^\mu[1-\theta(-(m+\mu)\,\mathrm{sgn}(m)) - \theta(-(m-\mu)\,\mathrm{sgn}(m))] \\ &= \frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^2}{4\pi}{}^*F^\mu. \end{aligned} \tag{13}$$

Let's now discuss the non-abelian case. Then $A^\mu = T_a A^\mu_a$ in (2) and

$$\langle J^\mu_a\rangle = -ig\ \mathrm{tr}\left[\gamma^\mu T_a G(x,x')\right]_{x\to x'}.$$

It is well–known [3, 11] that there exist only two types of the constant background fields. The first is the "abelian" type (it is easy to see that the self–interaction $f^{abc}A^\mu_b A^\mu_c$ disappears under that choice of the background field)

$$A^\mu_a = \eta_a \frac{1}{2}x_\nu F^{\nu\mu}, \tag{14}$$

where η_a is an arbitrary constant vector in the color space, $F^{\nu\mu} = \mathrm{Const}$. The second is the pure "non–abelian" type

$$A^\mu = \mathrm{Const}. \tag{15}$$

Here the derivative terms (abelian part) vanish from the strength tensor and it contains only the self–interaction part $F^{\mu\nu}_a = gf^{abc}A^\mu_b A^\mu_c$. It is clear that to catch abelian part of the CS term we should consider the background field (14), whereas for the non–abelian (derivative noncontaining, cubic in A) part we have to use the case (15).

Calculations in the "abelian" case reduces to the previous analysis, except the trivial adding of the color indices in the formula (13):

$$J_a^\mu = \frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^2}{4\pi}{}^*F_a^\mu. \tag{16}$$

In the case (15) all calculations are similar. The only difference is that the origin of term $\sigma_{\mu\nu}F^{\mu\nu}$ in (8) is not the linearity A in x (as in abelian case) but the pure non–abelian $A^\mu=$ Const. Here term $\sigma_{\mu\nu}F^{\mu\nu}$ in (8) becomes quadratic in A and we have

$$J_a^\mu = \frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^3}{4\pi}\varepsilon^{\mu\alpha\beta}\ \mathrm{tr}\left[T_a A^\alpha A^\beta\right]. \tag{17}$$

Combining formulas (16) and (17) and integrating over field A_a^μ we obtain eventually

$$S_{eff}^{\mathrm{C.S}} = \frac{m}{|m|}\theta(m^2-\mu^2)\pi W[A], \tag{18}$$

where $W[A]$ is the CS term

$$W[A] = \frac{g^2}{8\pi^2}\int d^3x\varepsilon^{\mu\nu\alpha}\ \mathrm{tr}\left(F_{\mu\nu}A_\alpha - \frac{2}{3}gA_\mu A_\nu A_\alpha\right).$$

This result can be obtained also with an arbitrary initial field configuration by use of the perturbative expansion. Here we work at once in the nonabelian case. The effective action looks as

$$\begin{aligned} S_{eff} &= \frac{1}{2}\ \mathrm{tr}\int_x A_\mu(x)\int_p \mathrm{e}^{-ixp}A_\nu(p)\Pi^{\mu\nu}(p) \\ &+ \frac{1}{3}\ \mathrm{tr}\int_x A_\mu(x)\int_{p,r} \mathrm{e}^{-ix(p+r)}A_\nu(p)A_\alpha(r)\Pi^{\mu\nu\alpha}(p,r), \end{aligned} \tag{19}$$

where polarization operator and vertice have the standard form

$$\begin{aligned} \Pi^{\mu\nu}(p) &= g^2\int_k \mathrm{tr}\left[\gamma^\mu G(p+k;\mu)\gamma^\nu G(k;\mu)\right] \\ \Pi^{\mu\nu\alpha}(p,r) &= g^3\int_k \mathrm{tr}[\gamma^\mu G(p+r+k;\mu)\gamma^\nu G(r+k;\mu)\gamma^\alpha G(k;\mu)]. \end{aligned} \tag{20}$$

First consider the second order term. Signal for the mass generation (CS term) is $\Pi^{\mu\nu}(0)\neq 0$. Picking up a parity odd part we get

$$\Pi^{\mu\nu} = g^2\int_k(-i2me^{\mu\nu\alpha}p_\alpha)\frac{1}{(\tilde{k}^2-m^2+i\epsilon(k_0+\mu)\ \mathrm{sgn}(k_0))^2}. \tag{21}$$

After some simple algebra one obtains

$$\Pi^{\mu\nu} = -i\frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^2}{4\pi}e^{\mu\nu\alpha}p_\alpha. \tag{22}$$

In the same manner, handling the third order contribution one gets

$$\Pi^{\mu\nu\alpha} = -i\frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^3}{4\pi}e^{\mu\nu\alpha}. \tag{23}$$

Substituting vertices in the effective action we eventually get

$$S_{eff}^{C.S.} = \frac{m}{|m|}\theta(m^2 - \mu^2)\frac{g^2}{8\pi}\int d^3x e^{\mu\nu\alpha} \operatorname{tr}\left(A_\mu \partial_\nu A_\alpha - \frac{2}{3} g A_\mu A_\nu A_\alpha\right). \tag{24}$$

So, we obtain the same CS μ–dependent coefficient as in the previous method at once in the effective action.

Moreover, by use of the perturbative theory we also derive CS term at finite density in 5-dimensional nonabelian gauge theory. All calculations are similar to 3–dimensional case (details will be published elsewhere)

$$\begin{aligned} S_{eff} &= \frac{m}{|m|}\theta(m^2 - \mu^2)\frac{g^3}{48\pi^2}\int d^5x e^{\mu\nu\alpha\beta\gamma} \\ &\times \operatorname{tr}\Big(A_\mu \partial_\nu A_\alpha \partial_\beta A_\gamma + \frac{3}{2} g A_\mu A_\nu A_\alpha \partial_\beta A_\gamma + \frac{3}{5} g^2 A_\mu A_\nu A_\alpha A_\beta A_\gamma\Big). \end{aligned} \tag{25}$$

From the above direct calculations it is clearly seen that the chemical potential dependent coefficient is the same for all parity odd parts of action and doesn't depend on space dimension:

$$S_{eff}^{\mathrm{C.S}} = \frac{m}{|m|}\theta(m^2 - \mu^2)\pi W[A], \tag{26}$$

where $W[A]$ is the CS term in any odd dimension. Since only the lowest orders of perturbative series contribute to CS term at finite density (the same situation is well-known at zero density), the result obtained by using formally perturbative technique appears to be nonperturbative. Thus, the μ–dependent CS term coefficient reveals the amazing property of universality. Namely, it does depend on neither dimension of the theory nor abelian or nonabelian gauge theory is studied.

The arbitrariness of μ gives us the possibility to see Chern–Simons coefficient behavior at any masses. It is very interesting that $\mu^2 = m^2$ is the crucial point for Chern–Simons. Indeed, it is clearly seen from (26) that when $\mu^2 < m^2$ μ–influence disappears and we get the usual Chern-Simons term $I_{eff}^{\mathrm{C.S}} = \pi W[A]$. On the other hand when $\mu^2 > m^2$ the situation is absolutely different. One can see that here the Chern-Simons term disappears because of non–zero density of background fermions. We'd like to emphasize the important massless case $m = 0$ considered in [7]. Then even negligible density, which always take place in any physical processes, leads to vanishing of the parity anomaly.

In conclusion, let us stress again that we nowhere have used any restrictions on μ. Thus we not only confirm result in [7] for Chern–Simons in QED_3 at small density, but also expand it on arbitrary μ, nonabelian case and arbitrary odd dimension.

We would like to thank V.Fedyanin, V.Osipov and O.Veretin for helpful discussions.

References

[1] R. Jackiw, S. Templeton Phys.Rev. **D23**, 2291 (1981)

[2] A. J. Niemi and G. W. Semenoff Phys.Rev.Lett. **51**, 2077 (1983)

[3] A. N. Redlich Phys.Rev. **D29**, 2366 (1984)

[4] L. Avarez-Gaume, E. Witten Nucl.Phys. **B234**, 269 (1984)

[5] A. J. Niemi Nucl.Phys. **B251**[FS13] (1985) 155.

[6] J. D. Lykken, J. Sonnenschen and N. Weiss Phys.Rev **D42**, 2161 (1990); A. M. J. Schakel Phys.Rev. **D43**, 1428 (1991); V. Y. Zeitlin Mod.Phys.Lett. **A8**, 1821 (1993);

[7] A. J. Niemi and G. W. Semenoff Phys.Rep. **135** No.3 (1986) 99.

[8] E. V. Shuryak Phys.Rep. **61**, 73 (1980)

[9] A. Chodos, K. Everding and D. A. Owen Phys.Rev. **D42**, 2881 (1990)

[10] J. Schwinger Phys.Rev. **82**, 664 (1951)

[11] L. S. Brown, W. I. Weisberger Nucl.Phys. **B157**, 285 (1979)