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CONTRACTIONS OF LIE ALGEBRAS AND SEPARATION OF VARIABLES. TWO-DIMENSIONAL HYPERBOLOID

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The Inönü-Wigner contraction from the Lorentz group O(2, 1) to the Euclidean group E(2) is used to relate the separation of variables in the Laplace-Beltrami operators on the two corresponding homogeneous spaces. We consider the contractions on four levels: the Lie algebra, the commuting sets of second order operators in the enveloping algebra of o(2,1), the coordinate systems and some eigenfunctions of the Laplace-Beltrami operators.

1. Introduction

In this article we continue the investigation of the connection between contractions of Lie algebras and the separation of variables. In the first article[1] we restricted ourselves to the Inönü-Wigner contractions of the rotation algebra o(3) to the Euclidean algebra e(2). The two separable coordinates systems on the sphere $S_2 \sim O(3)/O(2)$ were related to the four separable systems on the plane $E_2 \sim E(2)/O(2)$. Here we consider the Inönü-Wigner contractions of the Lorentz algebra o(2,1) to the Euclidean algebra e(2). In this case the nine separable coordinates systems on the two-sheeted hyperboloid $L_2 \sim O(2,1)/O(2)$ are related to the four separable systems on the plane E_2. Our motivation for such an investigation and the results to be expected were discussed in detail in the previous article.[1]

2. Separable Coordinates on the Hyperboloid L_2

Consider the hyperboloid L_2: $u_0^2 - u_1^2 - u_2^2 = R^2$, $u_0 > 0$, where u_i $(i = 0, 1, 2)$ are Cartesian coordinates in the ambient space $E(2,1)$ and R is the radius of curvature of the two-sheeted hyperboloid L_2. The isometry group of L_2 is O(2,1). We choose a standard basis K_1, K_2, M_3 for the Lie algebra o(2,1):[2]

$$K_1 = -(u_0\partial_{u_2} + u_2\partial_{u_0}), \; K_2 = -(u_0\partial_{u_1} + u_1\partial_{u_0}), \; M_3 = (u_1\partial_{u_2} - u_2\partial_{u_1})$$

with commutation relations

$$[K_1, K_2] = -M_3, \quad [M_3, K_1] = K_2, \quad [K_2, M_3] = K_1. \tag{2.1}$$

The Laplace-Beltrami operator has the form:

$$\Delta_{LB} = \frac{1}{R^2}\left(K_1^2 + K_2^2 - M_3^2\right). \tag{2.2}$$

Following the general method[3–10] (that has in particular been applied to the hyperboloid[4] L_2) we look for separated eigenfunctions of the Laplace-Beltrami operator satisfying

$$R^2 \cdot \Delta_{LB}\Psi = l(l+1)\Psi, \quad I\Psi = \lambda^2\Psi; \quad \Psi_{l\lambda}(\zeta_1, \zeta_2) = \Xi_{l\lambda}(\zeta_1)\Phi_{l\lambda}(\zeta_2), \tag{2.3}$$

where l for the principal series of unitary irreducible representations has the form

$$l = -\frac{1}{2} + i\rho, \quad 0 < \rho < \infty \tag{2.4}$$

and I is a second order operator[4,11] in the enveloping algebra of o(2,1):

$$\begin{aligned} I = {} & aK_1^2 + b(K_1K_2 + K_2K_1) + cK_2^2 \\ & + d(K_1M_3 + M_3K_1) + e(K_2M_3 + M_3K_2) + fM_3^2, \end{aligned} \tag{2.5}$$

(I obviously commutes with the Laplace-Beltrami operator). We list all *coordinate systems* on the hyperboloid L_2 in which the Helmholtz equation (2.3) permits the separation of the variables[2,4,11,12] and corresponding integrals of motion I in Table 1. There are 9 such systems, all orthogonal, and they are in one to one correspondence with $O(2,1)$ conjugacy classes of operators I.

For the Euclidean plane E_2 we consider second order operators X in the enveloping algebra of the Euclidean algebra e(2).[1,4] We can take X into precisely one of the following operators by an $E(2)$ transformation:

$$X_S = L_3^2, \; X_C = p_1^2 - p_2^2, \; X_P = L_3p_1 + p_1L_3, \; X_E = L_3^2 + \frac{D^2}{2}(p_1^2 - p_2^2), \tag{2.6}$$

where L_3 is the angular momentum, $p_{1,2}$ the linear momenta which form the basis of e(2); $2D$ is the focal distance in the elliptic system of coordinates. Each of the operators (2.6) corresponds to a different separable coordinate system in the Helmholtz equation.[4] Thus X_C corresponds to Cartesian coordinates, X_S to polar ones, X_P to parabolic coordinates and X_E to elliptic ones.

3. The Contraction of the Lie Algebra

We shall use R^{-1} as the contraction parameter and consider contraction from o(2,1) to e(2). To realize the contraction explicitly, let us introduce the Beltrami coordinates on the hyperboloid L_2 putting

$$x_\mu = R\frac{u_\mu}{u_0} = R\frac{u_\mu}{\sqrt{R^2 + u_1^2 + u_2^2}}, \quad \mu = 1, 2. \tag{3.7}$$

Table 1. Coordinate Systems on the Two-Dimensional Hyperboloid

Coordinate System Integral of Motion I	Coordinates
I. Spherical $\tau > 0, \varphi \in [0, 2\pi)$ $I_S = M_3^2$	$u_0 = R\cosh\tau$ $u_1 = R\sinh\tau\cos\varphi$ $u_2 = R\sinh\tau\sin\varphi$
II. Equidistant $\tau_{1,2} \in \mathbb{R}$ $I_{EQ} = K_2^2$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2$ $u_2 = R\sinh\tau_1$
III. Horicyclic $\tilde{y} > 0, \tilde{x} \in \mathbb{R}$ $I_{HO} = (K_1 + M_3)^2$	$u_0 = R(\tilde{x}^2 + \tilde{y}^2 + 1)/2\tilde{y}$ $u_1 = R(\tilde{x}^2 + \tilde{y}^2 - 1)/2\tilde{y}$ $u_2 = R\tilde{x}/\tilde{y}$
IV. Elliptic[a)] $a_3 < a_2 < \rho_2 < a_1 < \rho_1$ $I_E = M_3^2 + \sinh^2 f K_2^2$	$u_0^2 = R^2(\rho_1 - a_3)(\rho_2 - a_3)/(a_1 - a_3)(a_2 - a_3)$ $u_1^2 = R^2(\rho_1 - a_2)(\rho_2 - a_2)/(a_1 - a_2)(a_2 - a_3)$ $u_2^2 = R^2(\rho_1 - a_1)(a_1 - \rho_2)/(a_1 - a_2)(a_1 - a_3)$
IV. Rotated Elliptic[b)] $\alpha \in (\mathrm{i}K', \mathrm{i}K' + 2K)$ $\beta \in [0, 4K')$ $I_E = \cosh 2f M_3^2 + \frac{1}{2}\sinh 2f\{K_1, M_3\}$	$u_0' = R\{\frac{1}{k}\mathrm{sn}(\alpha,k)\mathrm{dn}(\beta,k') + i\frac{k'}{k}\mathrm{cn}(\alpha,k)\mathrm{cn}(\beta,k')\}$ $u_1' = R\{\frac{k'}{k}\mathrm{sn}(\alpha,k)\mathrm{dn}(\beta,k') + \frac{i}{k}\mathrm{cn}(\alpha,k)\mathrm{cn}(\beta,k')\}$ $u_2' = iR\mathrm{dn}(\alpha,k)\mathrm{sn}(\beta,k')$
V. Hyperbolic[c)] $\rho_2 < a_3 < a_2 < a_1 < \rho_1$ $I_H = K_2^2 - \sin^2\alpha M_3^2$	$u_0^2 = R^2(\rho_1 - a_2)(a_2 - \rho_2)/(a_1 - a_2)(a_2 - a_3)$ $u_1^2 = R^2(\rho_1 - a_3)(a_3 - \rho_2)/(a_1 - a_3)(a_2 - a_3)$ $u_2^2 = R^2(\rho_1 - a_1)(a_1 - \rho_2)/(a_1 - a_2)(a_1 - a_3)$
VI. Semi-Hyperbolic $\mu_{1,2} > 0$ $I_{SH} = -\{K_1, M_3\}$	$u_0^2 = \frac{R^2}{4}\left\{\sqrt{(1 - i\mu_1)(1 + i\mu_2)} + \sqrt{(1 + i\mu_1)(1 - i\mu_2)}\right\}^2$ $u_1^2 = -\frac{R^2}{4}\left\{\sqrt{(1 - i\mu_1)(1 + i\mu_2)} - \sqrt{(1 + i\mu_1)(1 - i\mu_2)}\right\}^2$ $u_2^2 = R^2\mu_1\mu_2$
VII. Elliptic-Parabolic $a \in \mathbb{R}, \vartheta \in (-\pi/2, \pi/2)$ $I_{EP} = (K_1 + M_3)^2 + K_2^2$	$u_0 = R\frac{\cosh^2 a + \cos^2\vartheta}{2\cosh a\cos\vartheta}$ $u_1 = R\frac{\sinh^2 a - \sin^2\vartheta}{2\cosh a\cos\vartheta}$ $u_2 = R\tan\vartheta\tanh a$
VIII. Hyperbolic-Parabolic $b > 0, \vartheta \in (0, \pi)$ $I_{HP} = (K_1 + M_3)^2 - K_2^2$	$u_0 = R\frac{\cosh^2 b + \cos^2\vartheta}{2\sinh b\sin\vartheta}$ $u_1 = R\frac{\sinh^2 b - \sin^2\vartheta}{2\sinh b\sin\vartheta}$ $u_2 = R\cot\vartheta\coth b$
IX. Semi-Circular-Parabolic $\xi, \eta > 0$ $I_{SCP} = \{K_1, K_2\} + \{K_2, M_3\}$	$u_0 = R\frac{(\xi^2 + \eta^2)^2 + 4}{8\xi\eta}$ $u_1 = R\frac{(\xi^2 + \eta^2)^2 - 4}{8\xi\eta}$ $u_2 = R\frac{\eta^2 - \xi^2}{2\xi\eta}$

[a)]The parameter f is determined by the relation:

$$\sinh^2 f = (a_1 - a_2)/(a_2 - a_3) = k'^2/k^2 \ (k'^2 + k^2 = 1).$$

[b)]The rotated elliptic system u_i' is obtained from the elliptic one u_i by a hyperbolic rotation through the angle f about axis u_2.

[c)]Angle α is determined by the formula: $\sin^2\alpha = (a_2 - a_3)/(a_1 - a_3)$, where 2α is the angle between two focal lines.

The O(2,1) generators can be expressed as:

$$\begin{aligned} -\frac{K_1}{R} &\equiv \pi_2 = p_2 - \frac{1}{R^2}x_2(x_1p_1 + x_2p_2), \\ -\frac{K_2}{R} &\equiv \pi_1 = p_1 - \frac{1}{R^2}x_1(x_1p_1 + x_2p_2), \\ M_3 &= -L_3 = x_1p_2 - x_2p_1 = x_1\pi_2 - x_2\pi_1. \end{aligned} \tag{3.8}$$

The commutators of the o(2,1) algebra (2.1) in the new operators (3.8) take the form:

$$[\pi_1, \pi_2] = -\frac{L_3}{R^2}, \quad [L_3, \pi_1] = \pi_2, \quad [\pi_2, L_3] = \pi_1, \tag{3.9}$$

so, that for $R \to \infty$ the o(2,1) algebra contracts to the e(2) and the momenta π_μ to $p_\mu = \partial/\partial x_\mu$. The o(2,1) Laplace-Beltrami operator (2.2) contracts to the e(2) one:

$$\Delta_{LB} = \pi_1^2 + \pi_2^2 - \frac{M_3^2}{R^2} \to \Delta = (p_1^2 + p_2^2). \tag{3.10}$$

4. The Contractions of Coordinates from o(2,1) to e(2)

4.1. *Spherical Coordinates on L_2 to Polar Coordinates on E_2*

In the limit $R \to \infty$, $\tau \to 0$ putting $\tanh\tau \sim r/R$ we have:

$$I_S = M_3^2 \to X_S = L_3^2$$

and for Beltrami coordinates (3.7) we obtain:

$$x_1 = R\frac{u_1}{u_0} \to x = r\cos\varphi, \quad x_2 = R\frac{u_2}{u_0} \to y = r\sin\varphi.$$

4.2. *Equidistant Coordinates on L_2 to Cartesian on E_2*

For Beltrami coordinates (3.7) we have:

$$x_1 = R\tanh\tau_2, \quad x_2 = R\tanh\tau_1/\cosh\tau_2.$$

Taking the limit $R \to \infty$, $\tau_1, \tau_2 \to 0$ and putting $\sinh\tau_1 \sim y/R$, $\sinh\tau_2 \sim x/R$ we see that Beltrami coordinates go into Cartesian ones: $x_1 \to x$, $x_2 \to y$. For the integral of motion we get:

$$\frac{I_{EQ}}{R^2} = \pi_1^2 \to p_1^2 \sim X_C.$$

4.3. *Horicyclic Coordinates on L_2 to Cartesian on E_2*

For variables $\tilde{x}, \tilde{y}$ we obtain:

$$\tilde{x} = \frac{u_2}{u_0 - u_1}, \quad \tilde{y} = \frac{R}{u_0 - u_1}.$$

In the limit $R \to \infty$ we get: $\tilde{x} \to y/R$, $\tilde{y} \to 1 + x/R$ and the Beltrami coordinates go into Cartesian ones: $x_1 \to x$, $x_2 \to y$. For integral of motion we have:

$$\frac{I_{HO}}{R^2} = \pi_2^2 + \frac{M_3^2}{R^2} - \frac{1}{R}\{\pi_2, M_3\} \to p_2^2 \sim X_C.$$

4.4. *Elliptic Coordinates on L_2 to Elliptic Coordinates on E_2*

We put

$$\frac{R^2}{a_2 - a_3} = \frac{D^2}{a_1 - a_2}. \tag{4.11}$$

and in the limit $R^2 \sim (-a_3) \to \infty$ obtain:

$$I_E = M_3^2 + \frac{D^2}{R^2} K_2^2 \to L_3^2 + D^2 p_1^2 \sim X_E,$$

where $2D$ is the focal distance. Writing the coordinates as

$$\rho_1 = a_1 + (a_1 - a_2)\sinh^2\xi, \qquad \rho_2 = a_2 + (a_1 - a_2)\cos^2\eta$$

and using eq.(4.11) in the limit $R^2 \sim (-a_3) \to \infty$ we obtain the ordinary elliptic coordinates on the E_2 plane.[4,5]

4.5. *Elliptic Coordinates on L_2 to Cartesian on E_2*

We make the special choice $a_1 - a_2 = a_2 - a_3 \equiv a$. Then the variables $\xi_{1,2}$ are determined by the formula:

$$\xi_{1,2} = \frac{\rho_{1,2} - a_2}{a} = \frac{u_0^2 + u_2^2}{2R^2} \pm \frac{1}{2}\sqrt{\left(\frac{u_0^2 + u_2^2}{R^2}\right)^2 - 4\frac{u_1^2}{R^2}}.$$

Considering the limit $R \to \infty$ we obtain: $\xi_1 \to 1 + 2y^2/R^2$, $\xi_2 \to x^2/R^2$ and the Beltrami coordinates (3.7) take the Cartesian form: $x_1 \to x$, $x_2 \to y$. The operator I_E goes into the Cartesian one:

$$\frac{I_E}{R^2} = \frac{M_3^2}{R^2} + \pi_1^2 \to p_1^2 \sim X_C.$$

4.6. *Rotated Elliptic Coordinates on L_2 to Parabolic on E_2*

We choose $a_1 - a_2 = a_2 - a_3 \equiv a$. For all Jacobi elliptic functions modulus $k = k' = 1/\sqrt{2}$, then for rotated elliptic system (see Table 1) we obtain:

$$\mathrm{cn}\alpha = -\frac{i}{2}\sqrt{\left(1 + \frac{u_1'}{R\sqrt{2}} - \frac{u_0'}{R}\right)^2 + \frac{u_2'^2}{2R^2}} + \frac{i}{2}\sqrt{\left(1 - \frac{u_1'}{R\sqrt{2}} + \frac{u_0'}{R}\right)^2 + \frac{u_2'^2}{2R^2}}, \tag{4.12}$$

$$\mathrm{cn}\beta = \frac{1}{2}\sqrt{\left(1+\frac{u_1'}{R\sqrt{2}}-\frac{u_0'}{R}\right)^2+\frac{u_2'^2}{2R^2}}+\frac{1}{2}\sqrt{\left(1-\frac{u_1'}{R\sqrt{2}}+\frac{u_0'}{R}\right)^2+\frac{u_2'^2}{2R^2}}. \quad (4.13)$$

For large R from (4.12), (4.13) we get:

$$-i\mathrm{cn}\alpha \simeq 1-\frac{1}{2\sqrt{2}}\frac{u^2}{R}, \quad \mathrm{cn}\beta \simeq 1+\frac{1}{2\sqrt{2}}\frac{v^2}{R},$$

so, in the limit $R \to \infty$ we obtain the parabolic coordinates:

$$x_1 \to x = \frac{u^2-v^2}{2}, \quad x_2 \to y = uv.$$

In this case the integral of motion I_E transforms into the parabolic one:

$$\frac{I_E}{R\sqrt{2}} = \frac{3}{R\sqrt{2}}M_3^2 - \{\pi_2, M_3\} \to \{p_2, L_3\} \sim X_P.$$

4.7. *Hyperbolic Coordinates on L_2 to Cartesian Coordinates on E_2*

We start from the choice $a_1 - a_2 = a_2 - a_3 \equiv a$. For coordinates we put:

$$\frac{\rho_1 - a_2}{a} = \xi_1, \quad \frac{a_2-\rho_2}{a} = \xi_2,$$

then Beltrami coordinates can be written as:

$$x_1^2 = R^2\frac{(\xi_1+1)(\xi_2-1)}{2\xi_2\xi_1}, \quad x_2^2 = R^2\frac{(\xi_1-1)(\xi_2+1)}{2\xi_2\xi_1}.$$

Now considering the limit $R \to \infty$ we obtain: $\xi_1 \to 1 + y^2/R^2$, $\xi_2 \to 1 + x^2/R^2$ and Beltrami coordinates go into Cartesian ones: $x_1 \to x$, $x_2 \to y$. The integral of motion takes the form:

$$\frac{I_H}{R^2} = \pi_1^2 - \frac{1}{2R^2}M_3^2 \to p_1^2 \sim X_C.$$

4.8. *Semi-hyperbolic Coordinates on L_2 to Parabolic Coordinates on E_2*

The variables $\mu_{1,2}$ are determined by the formulae:

$$\mu_{1,2} = \sqrt{\frac{u_0^2u_1^2}{R^4}+\frac{u_2^2}{R^2}} \pm \frac{u_0u_1}{R^2}.$$

In the limit $R \to \infty$ the variables $\mu_{1,2}$ take the form: $\mu_1 \to u^2/R$, $\mu_2 \to v^2/R$ and the Beltrami coordinates contract to parabolic ones. The operator I_{SH} goes into the parabolic one X_P:

$$\frac{1}{R}I_{SH} = \{\pi_2, L_3\} \to \{p_2, L_3\} \sim X_P.$$

4.9. *Elliptic-parabolic Coordinates on L_2 to Parabolic Coordinates on E_2*

For variables ϑ and a we have:

$$\cos^2\vartheta = \frac{u_0 - \sqrt{u_0^2 - R^2}}{u_0 - u_1}, \quad \cosh^2 a = \frac{\sqrt{u_0^2 - R^2} + u_0}{u_0 - u_1}.$$

In the limit $R \to \infty$ we obtain: $\cos^2\vartheta \to 1 - v^2/R$, $\cosh^2 a \to 1 + u^2/R$ and hence the Beltrami coordinates x_1, x_2 go into the parabolic ones. For the integral of motion we get:

$$\frac{1}{R}\{I_{EP} + \hbar^2 R^2 \Delta_{LB}\} = \frac{2}{R}M_3^2 + \{\pi_2, M_3\} \to -\{p_2, L_3\} \sim X_P.$$

4.10. *Hyperbolic-parabolic Coordinates on L_2 to Cartesian ones on E_2*

Variables ϑ and b are determined by the formulae:

$$\cos^2\vartheta = \frac{u_0 - \sqrt{u_1^2 + R^2}}{u_0 - u_1}, \quad \cosh^2 b = \frac{u_0 + \sqrt{u_1^2 + R^2}}{u_0 - u_1}.$$

Then in the limit $R \to \infty$ we obtain: $\cos^2\vartheta \to y^2/2R^2$, $\cosh^2 b \to 2(1 + x/R)$ and Beltrami coordinates go into Cartesian ones. In this case the operator I_{HP} takes the form:

$$\frac{1}{R^2}I_{HP} = \pi_2^2 - \pi_1^2 + \frac{1}{R^2}M_3^2 + \frac{1}{R}\{\pi_2, M_3\} \to p_2^2 - p_1^2 = X_C.$$

4.11. *Semi-circular Parabolic Coordinates on L_2 to Cartesian Coordinates on E_2*

For variables η, ξ we have:

$$\eta^2 = \frac{\sqrt{R^2 + u_2^2} + u_2}{u_0 - u_1}, \quad \xi^2 = \frac{\sqrt{R^2 + u_2^2} - u_2}{u_0 - u_1}.$$

In the limit $R \to \infty$ the variables η, ξ take the form: $\eta^2 \to 1 + (x + y)/R$, $\xi^2 \to 1 + (x - y)/R$ and Beltrami coordinates go into Cartesian ones. The operator I_{SCP} may be written as:

$$\frac{1}{R^2}I_{SCP} = \{\pi_2, \pi_1\} + \frac{1}{R}\{\pi_2, M_3\} \to 2p_2p_1 \sim X_C.$$

5. Contraction of Basis Functions

Using the contraction properties of separable coordinates, we shall now consider the contraction limits for the two simplest eigenfunctions – pseudo-spherical and equidistant bases.

5.1. Spherical Basis on L_2 to Polar Basis on E_2

The normalized spherical eigenfunctions $\Psi^S_{\rho m}(\tau,\varphi)$ have the form:

$$\Psi^S_{\rho m}(\tau,\varphi) = \sqrt{\frac{\rho \sinh \pi\rho}{2\pi^2 R}} \mid \Gamma(\frac{1}{2} + i\rho + |m|) \mid \cdot P^{|m|}_{i\rho-1/2}(\cosh\tau)\exp(im\varphi),$$

where $\lambda \equiv m = 0, \pm 1, \pm 2, \ldots$. In the contraction limit $R \to \infty$ we put: $\tanh\tau \sim \tau \sim r/R$, $\rho \sim kR$. Using the asymptotic formula[13]

$$\lim_{|y|\to\infty} \mid \Gamma(x+iy) \mid \exp(\frac{\pi}{2} \mid y \mid) \mid y \mid^{\frac{1}{2}-x} = \sqrt{2\pi}$$

and rewriting the Legendre function in terms of the hypergeometric function, we obtain in the contraction limit $R \to \infty$:

$$\lim_{R\to\infty} F\left(\frac{1}{2} + |m| + i\rho, \frac{1}{2} + |m| - i\rho; 1 + |m|; -\sinh^2\frac{\tau}{2}\right) = \frac{2^{|m|}|m|!}{(kr)^{|m|}} J_{|m|}(kr).$$

So, the spherical functions in the contraction limit take the form:

$$\lim_{R\to\infty} \Psi^S_{\rho m}(\tau,\varphi) = \Phi^S_{km}(r,\varphi) = \sqrt{k}\cdot J_{|m|}(kr)\cdot\frac{e^{im\varphi}}{\sqrt{2\pi}},$$

where $J_\nu(z)$ is the Bessel function and the spherical basis contracts to the polar one.

5.2. Equidistant Basis on L_2 to Cartesian Basis on E_2

In the equidistant system the normalized eigenfunctions $\Psi^{EQ}_{\rho\lambda}(\tau_1,\tau_2)$ have the form:

$$\Psi^{EQ}_{\rho\lambda}(\tau_1,\tau_2) = \sqrt{\frac{\rho\sinh\pi\rho}{\cosh^2\pi\lambda + \sinh^2\pi\rho}}\cdot(\cosh\tau_1)^{-1/2} P^{i\rho}_{i\lambda-1/2}(-\tanh\tau_1)\cdot e^{i\lambda\tau_2}.$$

To perform the contraction we write the Legendre function in terms of the hypergeometric function

$$P^{i\rho}_{i\lambda-1/2}(-\tanh\tau_1) = \frac{\sqrt{\pi}2^{i\rho}(\cosh\tau_1)^{-i\rho}}{\Gamma\left(\frac{3}{4}-a\right)\Gamma\left(\frac{3}{4}-b\right)}\left\{ {}_2F_1\left(\frac{1}{4}+a, \frac{1}{4}+b; \frac{1}{2}; \tanh^2\tau_1\right)\right.$$
$$\left. +2\tanh\tau_1 \frac{\Gamma\left(\frac{3}{4}-a\right)\Gamma\left(\frac{3}{4}-b\right)}{\Gamma\left(\frac{1}{4}-a\right)\Gamma\left(\frac{1}{4}-b\right)} {}_2F_1\left(\frac{3}{4}-a, \frac{3}{4}-b; \frac{3}{2}; \tanh^2\tau_1\right)\right\},$$

where $a = i(\rho-\lambda)/2$, $b = i(\rho+\lambda)/2$. In the contraction limit $R\to\infty$ we put: $\rho \sim kR$, $\lambda \sim k_1 R$; $\tau_2 \sim x/R$, $\tau_1 \sim y/R$ where x, y are the Cartesian coordinates. Then we have asymptotic formulae:

$$\lim_{R\to\infty} {}_2F_1\left(\frac{1}{4}+a, \frac{1}{4}+b; \frac{1}{2}; \tanh^2\tau_1\right) = {}_0F_1\left(\frac{1}{2}; -\frac{y^2k_2^2}{4}\right) = \cos k_2 y,$$
$$\lim_{R\to\infty} {}_2F_1\left(\frac{3}{4}-a, \frac{3}{4}-b; \frac{3}{2}; \tanh^2\tau_1\right) = {}_0F_1\left(\frac{3}{2}; -\frac{y^2k_2^2}{4}\right) = \frac{1}{k_2 y}\sin k_2 y,$$

where $k_1^2 + k_2^2 = k^2$. In the contraction limit the equidistant basis goes into the Cartesian one:

$$\lim_{R\to\infty} \Psi_{\rho\lambda}^{EQ}(\tau_1, \tau_2) = \Phi_{k_1,k_2}(x, y) = \sqrt{\frac{k}{\pi k_2}} \exp(ik_1 x + ik_2 y).$$

6. Conclusions

In this paper we continue the investigation of some aspects of the theory of Lie group and Lie algebra contractions: the relation between separable coordinate systems in curved and flat spaces, related by the contraction of their isometry groups. We have considered the second simplest meaningful example, namely the original Inönü-Wigner contraction from a $O(2,1)$ to $E(2)$, as applied to the two-sheeted hyperboloid L_2 and Euclidean plane E_2.

We have followed through the contraction $R \to \infty$ on four levels: the Lie algebras as realized by vector fields and the Laplace-Beltrami operators in the two spaces, the second order operators in the enveloping algebras, characterizing separable systems, the separable coordinate systems themselves and two of the separated sets of eigenfunctions of the invariant operators. In particular, we have shown how *different* limiting procedures lead from nine separable systems on L_2, to four on the plane E_2.

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