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On the interbasis expansion for the Kaluza-Klein monopole system

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Abstract. We study the interbasis expansion of the wave-functions of the Kaluza-Klein monopole system in the parabolic coordinate system with respect to the spherical coordinate system, and vice versa. We show that the coefficients of the expansion are proportional to Clebsch-Gordan coefficients. We analyse the discrete and continuous spectrum as well, briefly discuss the feature that the (reduced) Kaluza-Klein monopole system is separable in three coordinate systems, and the fact that there are five functionally independent integrals of motion, respectively observables, a property which characterizes this system as super-integrable.

1 Introduction

In the framework of quantum mechanics magnetic monopoles have been first discussed by Dirac in his classical paper [1]. He described them as quantized singularities in the electromagnetic field, the quantization condition being

$$2ge = nc\hbar, \qquad (n \in \mathbb{N}) \tag{1}$$

(e – electric charge, g – magnetic charge, c – velocity of light), arising from the singlevaluedness requirement of the wave-function. The corresponding Schrödinger equation can be straightforwardly evaluated, and leads to a pure continuous spectrum of an electron moving in the field of a magnetic monopole. More general is the Dyon problem, where a Coulomb-interaction term $\propto eg/r$ is included, and bound states can appear. This problem has been discussed by several authors, see e.g. Barut et al. [2], Jackiw [3], and for discussions including spin, see e.g. D'Hoker and Vinet [4].

More elaborated monopole models have been developed since and monopole solutions seem to be inevitable in grand unified theories [5]. Important examples are the (Bogomolnyi-Prasad-Sommerfield) BPS monopoles, e.g. [6], which move along geodesics in a curved space, and Kaluza-Klein monopoles, e.g. [7–10], the latter emerging from the former by means of a static solution, i.e., large spatial separation, of the classical field equations of five-dimensional gravity (Taub-NUT limit, "Euclidean limit"). Then, the relevant metric for the (full) Kaluza-Klein monopole system is given by $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$) [7]

$$ds^2 = \frac{1}{\Lambda(r)} d\mathbf{x}^2 + \Lambda(r)(dx_5^2 + 4m\,\mathbf{A} \cdot d\mathbf{x})^2 \ , \quad \mathbf{A} = \frac{\pm 1 - z/r}{r^2 - z^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \ , \tag{2}$$

with $x_5 = 4m\psi$; the metric term Λ and the φ-component of the magnetic interaction in spherical coordinates (r, ϑ, φ) are given by

$$\Lambda(r) = \frac{1}{1 + \frac{4m}{r}} \ , \quad A_\varphi = 4m(\pm 1 - \cos\vartheta) \ , \quad A_r = A_\vartheta = 0 \ . \tag{3}$$

The additional angular variable ψ describes the relative phase. This form of the Kaluza-Klein monopole system is the simplest solution of the classical field equations. The singularity at the origin vanishes if the coordinate ψ is cyclic with period 4π [8, 9]. The quantity Λ in the metric (g_{ab}) represents the effects of gravity, and $\mathbf{A}$ is identified with the electromagnetic field interaction. $4m$ is the only parameter that characterizes the Kaluza-Klein monopole system, seen as a test particle in the Taub-NUT space, and the coupling $g = 4m < 0$ generates a discrete energy spectrum. Similarly as in the classical example of the $o(4)$, respectively $o(3,1)$ dynamical symmetry algebra in the Kepler problem, the total angular momentum operator $\mathbf{J}$ and a suitable rescaled Pauli-Runge-Lenz vector $\mathbf{K}$ close into an $o(4)$ or $o(3,1)$ algebra, depending on the sign of the energy, which can be extended to an $o(4,2)$ symmetry [2, 8, 10]. On the corresponding homogeneous spaces of the groups $O(4)$ and $O(3,1)$, i.e., the three-dimensional sphere and the three-dimensional hyperboloid, a convenient coordinate space representation may be chosen for perturbation investigations for the discrete spectrum, respectively scattering phenomena in particular channels, e.g. [11] for a review concerning coordinate systems in homogeneous spaces, and their corresponding path integral representations and solutions.

The dynamical symmetry allows for a complete algebraic description of the classical as well as the quantum motions for the Kaluza-Klein monopole system. By Barut et al. [2] it was found that quantum systems with an $O(4,2)$ symmetry can be related by the Kustaanheimo-Stiefel transformation [12] to a four-dimensional oscillator, a fact which has been extensively exploited in path integral evaluations concerning Kepler-Coulomb, e.g. [13–15], and Dyon problems [16–18], and references therein.

As already observed by Zwanziger [19] a monopole problem with Coulomb coupling constant $\propto \alpha = e_1 e_2 + g_1 g_2$, a magnetic interaction $\propto \mu = e_1 g_2 - e_2 g_1$, and an additional scalar potential $\propto \hbar^2 \mu^2 / 2Mr^2$, c.f. also [18], has the same dynamical symmetry as the Coulomb problem and admits separation of variables in spherical, parabolic and prolate spheroidal coordinates.

A thorough study of the classical and quantum properties of the BPS monopoles is due to Gibbons and Manton [7], and they have shown that these specific monopole problems admit due to their symmetry properties a solution in spherical and parabolic coordinates, i.e., the Taub-NUT limit leads surprisingly to a Coulomb-like Schrödinger equation which is exactly solvable in spherical as well as in parabolic coordinates. Prolate spheroidal coordinates have not been taken into account until now.

The metric (1.2) was used by Bernido [20] and Junker and Inomata [21] to establish a path integral solution of this problem in terms of polar coordinates. In Ref. [23] the corresponding path integral solution in parabolic coordinates was derived. The path integral solutions in Refs. [20–23] all made use of the Kustaanheimo-Stiefel transformation in order to obtain the propagator, its spectral expansion, respectively and the energy-dependent Green function.

In this paper we investigate the interbasis expansion of the (discrete and continuous) wave-functions in parabolic coordinates with respect to the spherical basis, and vice versa. We show that the coefficients in the interbasis expansion are proportional to Clebsch-Gordan coefficients.

The contents of the paper are organized as follows. In the following section we briefly define the Kaluza-Klein monopole system in spherical and parabolic coordinates. This includes the statements of the relevant line element ds^2, the metric (g_{ab}), the Hamiltonian $\underline{\mathrm{H}}$, and the constants of motion, respectively the observables, which are stated in the form of the total angular momentum **J** and the Pauli-Runge-Lenz vector **K**. We then just cite the relevant steps to obtain the properly normalized wave-functions, suitable for the interbasis expansion analysis. A simple separation Ansatz for the wave-functions yields ordinary differential equations in the variables (r, ϑ) and (ξ, η). We do not dwell into the corresponding path integral evaluations which make use of a space-time transformation within the path integral. For the properly normalized wave-functions in spherical coordinates we refer to [20, 21] for the discrete spectrum and to [24] for the continuous spectrum; in parabolic coordinates we rely on [23, 25].

In the third and fourth sections we present our principal results, i.e., we derive the interbasis expansions coefficients for the wave-functions of the Kaluza-Klein monopole systems for the parabolic basis expanded in terms of the spherical basis. Our investigation is done for the discrete as well as for the continuous basis, the latter often being neglected. The inverse expansion yields in the case of the continuous basis an integration with respect to the parabolic separation parameter β. The fifth section contains a short discussion of our results, and we briefly introduce the prolate spheroidal coordinates as the third separating coordinate system for the Kaluza-Klein monopole system.

2 Quantum mechanical solution

If the cyclic variable ψ is separated off, the observables in the (reduced) Kaluza-Klein monopole system are given by a suitably chosen angular momentum operator **J** and a Pauli-Runge-Lenz operator **K** which can be cast into the form, e.g. [8, 10, 26–28]

$$\left.\begin{aligned} \mathbf{J} &= \mathbf{x} \times \boldsymbol{\pi} - 4m\hbar q \frac{\mathbf{x}}{|\mathbf{x}|} \;, \\ \mathbf{K} &= \frac{1}{2M}(\boldsymbol{\pi} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\pi}) - 4m \frac{\mathbf{x}}{|\mathbf{x}|}\left(\underline{\mathrm{H}} - \frac{q^2\hbar^2}{M}\right) \;, \end{aligned}\right\} \tag{4}$$

where $\pi = \mathbf{p} - q\mathbf{A}$, $\mathbf{p} = -\mathrm{i}\hbar\nabla$. They satisfy the commutation relations

$$\left.\begin{aligned} [J_i, J_k] &= \mathrm{i}\hbar\varepsilon_{ikl}J_l \ , \\ [J_i, K_k] &= \mathrm{i}\hbar\varepsilon_{ikl}K_l \ , \\ [K_i, K_k] &= \mathrm{i}\hbar\left(\frac{q^2\hbar^2}{M} - 2\underline{\mathrm{H}}\right)\varepsilon_{ikl}J_l \ . \end{aligned}\right\} \tag{5}$$

Here, summation over repeated indices is understood, and ε_{ikl} is totally antisymmetric in the indices $i,k,l = 1,2,3$. On the fixed energy eigenspace $\underline{\mathrm{H}}\Psi = E\Psi$ one defines the rescaled Pauli-Lenz-Runge operator $\mathbf{M}$ by means of

$$\left.\begin{aligned} \mathbf{M} &= \left(\frac{\hbar^2 q^2}{M} - 2E\right)^{-1/2}\mathbf{K} \quad \text{for } E < \hbar^2 q^2/2M \ , \\ \mathbf{M} &= \mathbf{K} \quad \text{for } E = \hbar^2 q^2/2M \ , \\ \mathbf{M} &= \left(2E - \frac{\hbar^2 q^2}{M}\right)^{-1/2}\mathbf{K} \quad \text{for } E > \hbar^2 q^2/2M \ . \end{aligned}\right\} \tag{6}$$

The operators $\mathbf{M}$ and $\mathbf{J}$ close to an $\mathrm{o}(4)$ algebra for $E < \hbar^2 q^2/2M$, to an $\mathrm{o}(3,1)$ algebra for $E > \hbar^2 q^2/2M$, and to an $\mathrm{o}(3) \otimes \mathrm{I\!R}^3$ algebra in the case $E = \hbar^2 q^2/2M$ (e.g. [8]). The property of five functionally independent observables characterizes the (reduced) Kaluza-Klein monopole system as a super-integrable system [29, 30] in three-dimensional Euclidean space. A complete set of observables of the full system is given by (e.g. [10])

$$\{q, \underline{\mathrm{H}}, \mathbf{A}^2, A_3, \mathbf{B}^2, B_3\} \ , \tag{7}$$

where $\mathbf{A} = \frac{1}{2}(\mathbf{J} + \mathbf{M})$ and $\mathbf{B} = \frac{1}{2}(\mathbf{J} - \mathbf{M})$.

We consider the spherical and the parabolic coordinate system which separate the Kaluza-Klein monopole system. The *spherical coordinate system* is given by

$$\left.\begin{aligned} x &= r\sin\vartheta\cos\varphi \ , \quad & r &> 0 \ , \\ y &= r\sin\vartheta\sin\varphi \ , \quad & 0 &< \vartheta < \pi \ , \\ z &= r\cos\vartheta \ , \quad & 0 &\leq \varphi < 2\pi \ . \end{aligned}\right\} \tag{8}$$

The *parabolic coordinate system* has the form

$$\left.\begin{aligned} x &= \xi\eta\cos\varphi \ , \quad & \xi,\eta &> 0 \ , \\ y &= \xi\eta\sin\varphi \ , \quad & 0 \leq \varphi &< 2\pi \ , \\ z &= \frac{1}{2}(\xi^2 - \eta^2) \ . \end{aligned}\right\} \tag{9}$$

In terms of these coordinates the line element ds^2 takes on the form ($dx_5 = 4m d\psi, \psi \in [0, 4\pi)$)

$$ds^2 = \frac{1}{\Lambda(r)} d\mathbf{x}^2 + \Lambda(r)(dx_5^2 + 4m\,\mathbf{A}\cdot d\mathbf{x})^2 \tag{10}$$

Polar Coordinates :

$$= \frac{1}{\Lambda(r)}\Big(dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\Big) + (4m)^2\Lambda(r)\Big(d\psi + (\pm 1 - \cos\vartheta)d\varphi\Big)^2 \tag{11}$$

Parabolic Coordinates

$$= \frac{1}{\Lambda(r)}\Big((\xi^2+\eta^2)(d\xi^2 + d\eta^2) + \xi^2\eta^2 d\varphi^2\Big) + (4m)^2\Lambda(r)\left[d\psi + \left(\pm 1 - \frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)d\varphi\right]^2 . \tag{12}$$

In the following we only take into account the "+"-sign which is sufficient for our purposes. The metric-tensor in *polar coordinates* has the form

$$(g_{ab}) = \frac{1}{\Lambda(r)}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\vartheta + (4m\Lambda)^2(1-\cos\vartheta)^2 & (4m\Lambda)^2(1-\cos\vartheta) \\ 0 & 0 & (4m\Lambda)^2(1-\cos\vartheta) & (4m\Lambda)^2 \end{pmatrix}, \tag{13}$$

with

$$g = \det(g_{ab}) = \left(\frac{4mr^2\sin\vartheta}{\Lambda(r)}\right)^2 , \tag{14}$$

and its inverse (g^{ab}) is given by

$$(g^{ab}) = \Lambda(r)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\vartheta} & -\frac{1-\cos\vartheta}{r^2\sin^2\vartheta} \\ 0 & 0 & -\frac{1-\cos\vartheta}{r^2\sin^2\vartheta} & \frac{1}{(4m\Lambda)^2} + \frac{(1-\cos\vartheta)^2}{r^2\sin^2\vartheta} \end{pmatrix} . \tag{15}$$

In *parabolic coordinates* we have

$$
(g^{ab}) = \frac{1}{\Lambda(r)}
\times \begin{pmatrix}
\xi^2+\eta^2 & 0 & 0 & 0 \\
0 & \xi^2+\eta^2 & 0 & 0 \\
0 & 0 & \xi^2\eta^2 + (4m\Lambda)^2\left(1+\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)^2 & (4m\Lambda)^2\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right) \\
0 & 0 & (4m\Lambda)^2\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right) & (4m\Lambda)^2
\end{pmatrix},
\tag{16}
$$

with

$$
\left.\begin{aligned}
g &= \det(g_{ab}) = (\xi^2+\eta^2)^2\left(\frac{4m\xi\eta}{\Lambda(r)}\right)^2 , \\
\Lambda(r) &= \left(1+\frac{8m}{\xi^2+\eta^2}\right)^{-1} .
\end{aligned}\right\}
\tag{17}
$$

The inverse (g^{ab}) is given by

$$
(g^{ab}) = \Lambda(r)
\times \begin{pmatrix}
\frac{1}{\xi^2+\eta^2} & 0 & 0 & 0 \\
0 & \frac{1}{\xi^2+\eta^2} & 0 & 0 \\
0 & 0 & \frac{1}{\xi^2\eta^2} & -\frac{1}{\xi^2\eta^2}\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right) \\
0 & 0 & -\frac{1}{\xi^2\eta^2}\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right) & \frac{1}{(4m\Lambda)^2}+\frac{1}{\xi^2\eta^2}\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)^2
\end{pmatrix}.
\tag{18}
$$

Therefore we obtain for the Hamiltonian operator $\underline{\mathrm{H}}$ in the two coordinate systems

Polar Coordinates :

$$
\begin{aligned}
\underline{\mathrm{H}} = -\frac{\hbar^2}{2M}\Lambda(r)\Bigg[&\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\vartheta^2}+\cot\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right) \\
&+\left(\frac{1}{(4m\Lambda)^2}+\frac{(1-\cos\vartheta)^2}{r^2\sin^2\vartheta}\right)\frac{\partial^2}{\partial\psi^2}-\frac{2}{r^2}\frac{1-\cos\vartheta}{\sin^2\vartheta}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\psi}\Bigg]
\end{aligned}
\tag{19}
$$

Parabolic Coordinates :

$$= -\frac{\hbar^2}{2M}\Lambda(r)\left\{\frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)+\frac{1}{\xi^2\eta^2}\frac{\partial^2}{\partial\varphi^2}\right.$$
$$\left.+\left[\frac{1}{(4m\Lambda)^2}+\frac{1}{\xi^2\eta^2}\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)^2\right]\frac{\partial^2}{\partial\psi^2}-\frac{2}{\xi^2\eta^2}\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\psi}\right\}. \tag{20}$$

In the spherical coordinate system the time-independent Schrödinger equation

$$\underline{H}\Psi(r,\vartheta,\varphi,\psi) = E\Psi(r,\vartheta,\varphi,\psi) \tag{21}$$

is solved by making the Ansatz for the wave-functions according to

$$\Psi(r,\vartheta,\varphi,\psi) = R(r)Z(\vartheta)\frac{e^{i(\nu\varphi+k\psi)}}{4\pi\sqrt{2|m|}} \ . \tag{22}$$

Observing

$$\frac{1}{4\pi}\int_0^{4\pi} e^{-i(\nu-q/\hbar)\psi}d\psi = \delta_{\nu,q/\hbar} \ , \tag{23}$$

which yields the quantization condition of the monopole charge $q = s\hbar$, $2s = 0, \pm 1, \pm 2, \ldots$. Note that in the Dirac monopole quantization we have $\beta q/\hbar = n/2$ with $\beta = 1, n \in \mathbb{N}$, and in the Schwinger quantization condition $\beta = 1/2$, c.f.e.g. [22], and references therein. Therefore $\nu \in \mathbb{Z}$, $q \equiv k/4m \equiv s/4m$, $2k = 2s = 0, \pm 1, \pm 2, \ldots$, and the operator $\hat{q}$, conjugate to $\hat{\psi}$, corresponding to the quantum number q is conserved and identified with the relative electric charge [7, 10, 28].

The functions $R(r)$ and $Z(\vartheta)$ are normalized, and the particular form of Ψ guarantees that the wave-functions are normalized to unity with respect to the scalar product

$$(f,g) = \int_0^\infty \frac{r^2 dr}{\Lambda(r)}\int_0^\pi \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi \int_0^{4\pi} 4|m|\, d\psi f^* g \ . \tag{24}$$

Explicitly, this gives for the angular and radial variables the ordinary differential equations

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\sin\vartheta\frac{dZ}{d\vartheta} + \left[J(J+1) - \frac{\nu^2}{4\sin^2\frac{\vartheta}{2}} - \frac{(\nu-2k)^2}{4\cos^2\frac{\vartheta}{2}}\right]Z = 0 \ , \tag{25}$$

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} + \left[\left(\frac{2ME}{\hbar^2} - \frac{k^2}{16m^2}\right) + \frac{1}{r}\left(\frac{8mME}{\hbar^2} - \frac{k^2}{2m}\right) - \frac{J(J+1)}{r^2}\right]R = 0 \ , \tag{26}$$

where J is the spherical quantum number. We see that a discrete spectrum can only occur for

$$g = 4m < 0, \qquad E < \hbar k^2/32m^2 M \ . \tag{27}$$

The solution of these differential equations, respectively the wave-functions of the spectral expansion of the corresponding path integral solution, are in terms of Jacobi polynomials in $\cos\vartheta$, and Laguerre polynomials for the discrete and Whittaker functions for the continuous spectrum in the radial variable. Thus we get for $m < 0$ the bound state wave-functions $\Psi_{NJ\nu k}$ in spherical coordinates [7, 20, 21]

$$\Psi_{NJ\nu k}(r,\vartheta,\varphi,\psi) = \frac{e^{i\nu\varphi + ik\psi}}{4\pi\sqrt{2|m|}} R_{NJ}(r)\, Z_{J\nu k}(\vartheta) \ , \tag{28}$$

where the radial $R_{NJ}(r)$ wave-functions have the form

$$R_{NJ}(r) = \left[\frac{(N-J-1)!}{(N+J)!}\,\frac{4}{a^3N^3\sqrt{N^2-s^2}}\right]^{1/2} \left(\frac{2r}{aN}\right)^J \exp\left(-\frac{r}{aN}\right) L_{N-J-1}^{(2J+1)}\left(\frac{2r}{aN}\right) \ , \tag{29}$$

and the functions $Z_{J\nu k}(\vartheta)$ are given by

$$\begin{aligned} Z_{J\nu k}(\vartheta) &= \left[\frac{(2J+1)[(J-\frac{1}{2}(|\nu|+|\nu-2k|)]![(J+\frac{1}{2}(|\nu|+|\nu-2k|)]!}{2[J+\frac{1}{2}(|\nu|-|\nu-2k|)]![J-\frac{1}{2}(|\nu|-|\nu-2k|)]!}\right]^{1/2} \\ &\quad \times \left(\sin\frac{\vartheta}{2}\right)^{|\nu|} \left(\cos\frac{\vartheta}{2}\right)^{|\nu-2k|} P_{J-\frac{1}{2}(|\nu|+|\nu-2k|)}^{(|\nu|,|\nu-2k|)}(\cos\vartheta) \ . \end{aligned} \tag{30}$$

The bound state energy levels are

$$E_N = \frac{\hbar^2}{(4m)^2 M}\sqrt{N^2-s^2}\left(\pm N - \sqrt{N^2-s^2}\right) \ , \qquad N = |s|+1, |s|+2, \cdots \ . \tag{31}$$

Here $N = n_r + l + \frac{|\nu|+|\nu-2k|}{2} + 1$ is the principal quantum number, l the orbital, and n_r the radial quantum numbers, which have for fixed N and s, the values $0, 1, \cdots N-|s|-1$, respectively. The quantum numbers J and l are related through $J = l + \frac{|\nu|+|\nu-2k|}{2}$. Note that $a = 1\Big/\left(N\sqrt{q^2 - 2E_N M/\hbar^2}\right) = 4|m|\Big/\left[N(N-\sqrt{N^2-s^2})\right]$ is a "Bohr-radius".

Gibbons and Manton [7] have argued that the energy levels with the "−"- sign are artefacts of the asymptotic approximation. The $L_n^{(\alpha)}(x)$ are Laguerre polynomials [31, p. 1037], and the $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials [31, p. 1035]. The levels with the "+"- sign give for $N \gg |s|$

$$E_N \simeq \frac{\hbar^2}{(4m)^2 M}\left[\frac{s^2}{2} - \frac{s^2}{8N^2} + O\left(\frac{1}{N^4}\right)\right] , \tag{32}$$

which exhibits a Coulomb-like behaviour.

The continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2M}(p^2 + q^2) , \tag{33}$$

with largest lower bound $E_0 = \hbar^2 q^2/2M = \hbar^2 k^2/32m^2 M$, and the continuous wave-functions $\Psi_{pJ\nu k}$ are

$$\Psi_{pJ\nu k}(r, \vartheta, \varphi, \psi) = \frac{e^{i\nu\varphi + ik\psi}}{4\pi\sqrt{2|m|}} R_{pJ}(r)\, Z_{J\nu k}(\vartheta) , \tag{34}$$

with the radial wave-functions $R_{pJ}(r)$ given by

$$R_{pJ}(r) = \frac{|\Gamma[J + 1 - 2i|m|(p^2 - q^2)/p]|}{(2J+1)!\sqrt{2\pi}\, r} \exp\left[\frac{\pi|m|}{p}(p^2 - q^2)\right] M_{2i|m|(p^2-q^2)/p, J+\frac{1}{2}}(-2ipr) . \tag{35}$$

The $M_{\kappa,\lambda}(z) = e^{-x/2} x^{\lambda+1/2} {}_1F_1(\frac{1}{2} - \kappa + \lambda; 2\lambda + 1; x)$ are Whittaker functions [31, p. 1059]. According to Meixner [32] and Mukunda [33] the functions (29, 35) are orthogonal and form a complete set, i.e.,

$$\int_0^\infty \frac{r^2 dr}{\Lambda(r)} R_{NJ}(r) R^*_{N'J}(r) = \delta_{NN'} , \int_0^\infty \frac{r^2 dr}{\Lambda(r)} R_{pJ}(r) R^*_{p'J}(r) = \delta(p - p') , \tag{36}$$

and

$$\sum_{N=1}^{\infty} R_{NJ}(r') R_{NJ}(r'') + \int_0^\infty R^*_{pJ}(r') R_{pJ}(r'') dp = \frac{1}{r^2}\delta(r'' - r') . \tag{37}$$

In parabolic coordinates one makes the Ansatz

$$\Psi(\xi, \eta, \varphi, \psi) = f_1(\xi) f_2(\eta) \frac{e^{i(\nu\varphi + k\psi)}}{4\pi\sqrt{2|m|}} . \tag{38}$$

The wave-functions Ψ are normalized according to the scalar product

$$(f, g) = \int_0^\infty d\xi \int_0^\infty d\eta (\xi^2 + \eta^2) \frac{\xi\eta}{\Lambda(r)} \int_0^{2\pi} d\varphi \int_0^{4\pi} 4|m| d\psi f^* g . \tag{39}$$

This gives the ordinary differential equations

$$\frac{d^2f_1}{d\xi^2}+\frac{1}{\xi}\frac{df_1}{d\xi}+\left[\left(\frac{2ME}{\hbar}-\frac{k^2}{16m^2}\right)\xi^2-\frac{(\nu-2k)^2}{\xi^2}+\beta_1\right]f_1=0\ , \tag{40}$$

$$\frac{d^2f_2}{d\eta^2}+\frac{1}{\eta}\frac{df_2}{d\eta}+\left[\left(\frac{2ME}{\hbar}-\frac{k^2}{16m^2}\right)\eta^2-\frac{\nu^2}{\eta^2}+\beta_2\right]f_2=0\ , \tag{41}$$

where $\beta_{1,2}$ are related through

$$\beta_1+\beta_2=\frac{16mME}{\hbar^2}-\frac{k^2}{m}\ . \tag{42}$$

The solutions of these ordinary differential equations, respectively the solution for the corresponding path integral formulation, are for the bound state wave-functions Laguerre polynomials in ξ and η, respectively. We obtain [7, 23] for the bound state wave-functions ($n_1,n_2\in\mathbb{N}; N=n_1+n_2+\frac{1}{2}(|\nu|+|\nu-2k|)+1$)

$$\Psi_{n_1n_2\nu k}(\xi,\eta,\varphi,\psi)=\frac{e^{i\nu\varphi+ik\psi}}{4\pi\sqrt{2|m|}}\left[\frac{2}{a^3N^3\sqrt{N^2-s^2}}\frac{n_1!n_2!}{(n_1+|\nu-2k|)!(n_2+|\nu|)!}\right]^{1/2}$$
$$\times\left(\frac{\xi^2}{aN}\right)^{\frac{|\nu-2k|}{2}}\left(\frac{\eta^2}{aN}\right)^{\frac{|\nu|}{2}}\exp\left(-\frac{\xi^2+\eta^2}{2aN}\right)L_{n_1}^{(|\nu-2k|)}\left(\frac{\xi^2}{aN}\right)L_{n_2}^{(|\nu|)}\left(\frac{\eta^2}{aN}\right)\ , \tag{43}$$

with the energy spectrum (31). The continuous states $\Psi_{p\beta\nu k}$ are given by [23]

$$\Psi_{p\beta\nu k}(\xi,\eta,\varphi,\psi)=\frac{e^{i\nu\varphi+ik\psi}}{4\pi\sqrt{2|m|}}\exp\left[\frac{\pi|m|}{p}(p^2-q^2)\right]$$
$$\times\frac{|\Gamma(\frac{1}{2}+|\frac{\nu}{2}-k|-i\beta_1)\Gamma(\frac{1+|\nu|}{2}-i\beta_2)|}{\sqrt{2\pi^2p}\,\xi\eta|\nu-2k|!\,|\nu|!}M_{i\beta_1,\frac{|\nu-2k|}{2}}(-ip\xi^2)M_{i\beta_2,\frac{|\nu|}{2}}(-ip\eta^2)\ , \tag{44}$$

where $\beta_{1,2}=\frac{1}{4}\left[4|m|(p-q^2/p)\pm 2\beta/p\right]$, β are the parabolic separation parameter, and E_p as in (33). The wave-functions (43, 44) are orthogonal (compare e.g. [23])

$$\int_0^\infty d\xi\int_0^\infty(\xi^2+\eta)^2\frac{\xi\eta d\xi d\eta}{\Lambda(r)}\int_0^{2\pi}d\varphi\int_0^{4\pi}4|m|d\psi\,\Psi_{n_1n_2\nu k}(\xi,\eta,\varphi,\psi)$$
$$\times\Psi^*_{n_1'n_2'\nu'k'}(\xi,\eta,\varphi,\psi)=\delta_{n_1,n_1'}\delta_{n_2,n_2'}\delta_{\nu,\nu'}\delta_{k,k'}\ , \tag{45}$$

$$\int_0^\infty d\xi\int_0^\infty(\xi^2+\eta)^2\frac{\xi\eta d\xi d\eta}{\Lambda(r)}\int_0^{2\pi}d\varphi\int_0^{4\pi}4|m|d\psi\,\Psi_{p\beta\nu k}(\xi,\eta,\varphi,\psi)$$
$$\Psi^*_{p'\beta'\nu'k'}(\xi,\eta,\varphi,\psi)=\delta(p-p')\delta(\beta-\beta')\delta_{\nu,\nu'}\delta_{k,k'}\ , \tag{46}$$

and form a complete set

$$\sum_{2k\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}}\left\{\sum_{n_1,n_2\in\mathbb{N}_0}\Psi_{n_1n_2\nu k}(\xi,\eta,\varphi,\psi)\Psi^*_{n_1n_2\nu k}(\xi',\eta',\varphi',\psi')\right.$$
$$\left.+\int_0^\infty dp\int_{\mathbb{R}}d\beta\,\Psi_{p\beta\nu k}(\xi,\eta,\varphi,\psi)\Psi^*_{p\beta\nu k}(\xi',\eta',\varphi',\psi')\right\}$$
$$=\frac{\Lambda(r)}{4|m|\xi\eta(\xi^2+\eta^2)}\delta(\varphi-\varphi')\delta(\psi-\psi')\delta(\xi-\xi')\delta(\eta-\eta')\ . \tag{47}$$

3 Interbasis expansion for the discrete basis

Let us consider the interbasis expansion of the parabolic bound state wave-functions (43) with respect to the spherical wave-functions (28), i.e.,

$$\Psi_{n_1n_2\nu k}(\xi,\eta,\varphi,\psi)=\sum_{l=0}^{n_1+n_2}W^l_{n_1n_2}\Psi_{n_rl\nu k}(r,\vartheta,\varphi,\psi)\ , \tag{48}$$

where $n_1+n_2=n_r+l$; we have included the dependence of ψ, and have re-inserted the angular quantum number l and the radial quantum number n_r in the principal quantum number $N=n_r+l+1\equiv n_r+l+\frac{1}{2}(|\nu|+|\nu-2k|)+1$. The parabolic variables can be expressed in terms of the spherical variables by means of

$$\xi^2=r+z=r(1+\cos\vartheta)\ ,\qquad \eta^2=r-z=r(1-\cos\vartheta)\ . \tag{49}$$

We consider (48) in the limit $r\to\infty$. From the property of the Laguerre polynomials $L_n^{(\alpha)}(x)\to(-1)^nx^2/n!$, as $x\to\infty$, we see that the dependence on r cancels on both sides of (48).

Using the orthogonality condition of the angular wave-functions (30) we find the following expression for the interbasis coefficients $W^l_{n_1n_2}$

$$W^l_{n_1n_2}=\frac{(-1)^l}{2^{n_1+n_2+|\nu|+|\nu-2k|}}\sqrt{\frac{l!(l+|\nu|+|\nu-2k|)!}{(l+|\nu|)!(l+|\nu-2k|)!}}$$
$$\times\sqrt{\frac{[l+\frac{1}{2}(|\nu|+|\nu-2k|+1)]n_r!(n_r+2l+|\nu|+|\nu-2k|+1)!}{n_1!n_2!(n_1+|\nu-2k|)!(n_2+|\nu|)!}}\,I^l_{n_1n_2}\ , \tag{50}$$

with the quantity $I^l_{n_1n_2}$ given by

$$I^l_{n_1n_2}=\int_{-1}^1dx\,(1+x)^{n_1+|\nu-2k|}(1-x)^{n_2+|\nu|}P_l^{(|\nu|,|\nu-2k|)}(x)\ . \tag{51}$$

The integral $I^l_{n_1n_2}$ can be evaluated by means of [31, p. 841]

$$\int_{-1}^{1} dx(1+x)^{\sigma}(1-x)^{\rho}P_n^{(\alpha,\beta)} = \frac{2^{\rho+\sigma+1}\Gamma(1+\rho)\Gamma(1+\sigma)\Gamma(n+1+\alpha)}{n!\Gamma(\rho+\sigma+2)\Gamma(1+\alpha)}$$
$$\times {}_3F_2\left(\begin{matrix} -n & \alpha+\beta+n+1 & \rho+1 \\ \alpha+1 & \rho+\sigma+2 & \end{matrix}\Bigg| 1\right) . \tag{52}$$

However, using the Rodriguez formula for the Jacobi polynomials [31, p. 1035]

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \tag{53}$$

and comparing with the integral representation of the Clebsch-Gordan coefficients [34, p. 243]

$$C_{j_1m_1,j_2m_2}^{jm} = \delta_{m_1+m_2,m}\frac{(-1)^{j_1+m_2-j}}{2^{j_1+j_2+j+1}}$$
$$\times \sqrt{\frac{(2j+1)(j+m)!(j_1+j_2+j+1)!(j_1+j_2-j)!}{(j_2-j_1+j)!(j_1-j_2+j)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!(j-m)!}}$$
$$\times \int_{-1}^{1} dx\,(1-x)^{j_1-m_1}(1+x)^{j_2-m_2}\frac{\mathrm{d}^{j-m}}{\mathrm{d}x^{j-m}}\left[(1-x)^{j_2-j_1+j}(1+x)^{j_1-j_2+c}\right] , \tag{54}$$

yields that the coefficients $W_{n_1n_2}^{l}$ of the interbasis expansion (3.1) are given in terms of the Clebsch-Gordan coefficients $C_{j_1m_1,j_2m_2}^{jm}$, i.e.,

$$W_{n_1n_2}^{l} = (-1)^{n_2}C_{j_1m_1,j_2m_2}^{jm} , \tag{55}$$

where

$$\left.\begin{aligned} j_1 &= \frac{n_1+n_2+|\nu-2k|}{2} , j_2 = \frac{n_1+n_2+|\nu|}{2} , j = l+\frac{|\nu|+|\nu-2k|}{2} , \\ m_1 &= \frac{n_1-n_2+|\nu-2k|}{2} , m_2 = \frac{n_2-n_1+|\nu|}{2} , m = \frac{|\nu|+|\nu-2k|}{2} . \end{aligned}\right\} \tag{56}$$

Using the orthogonality condition of the Clebsch-Gordan coefficients, we can invert (48) yielding

$$\Psi_{n_rl\nu k}(r,\vartheta,\varphi,\psi) = \sum_{n_1=0}^{n_r+l} W_{n_1n_2}^{l}\Psi_{n_1n_2\nu k}(\xi,\eta,\varphi,\psi) , \tag{57}$$

which represents the expansion of the spherical basis with respect to the parabolic basis.

4 Interbasis expansion for the continuous basis

4.1 Interbasis expansion of the parabolic basis with respect to the spherical basis

Let us consider the interbasis expansion of the parabolic scattering basis with respect to the spherical basis, i.e.,

$$\Psi_{p\beta\nu k}(\xi,\eta,\varphi,\psi)=\sum_{l=0}^{\infty} W_{l\nu k}^{p\beta}\Psi_{pJ\nu k}(r,\vartheta,\varphi,\psi) \quad , \tag{58}$$

with the spherical wave-functions $\Psi_{pl\nu k}$ (34) and the parabolic wave-functions $\Psi_{p\beta\nu k}$ (44). We now consider the following expression

$$\begin{aligned} &W_{l\nu k}^{p\beta}\cdot {}_1F_1(a+b+l;2l+|\nu|+|\nu-2k|+2;-2\mathrm{i}pr)\\ &=\frac{(-\mathrm{i})^l}{2^{l+|\nu|+|\nu-2k|}}\frac{|\Gamma(a)\Gamma(b)|}{|\Gamma(l+a+b)|}\frac{(2l+(|\nu|+|\nu-2k|)+1)!}{(|\nu|)!(|\nu-2k|)!}\\ &\quad\times\sqrt{\frac{(2l+|\nu|+|\nu-2k|+1)l!(l+|\nu|-|\nu-2k|)!}{16\pi p\,(l+|\nu|)!(l+|\nu-2k|)!}}\\ &\quad\times\sum_{s,t=0}^{\infty}\frac{(a)_s(b)_t}{(1+|\nu-2k|)_s(1+|\nu|)_t}\frac{(-\mathrm{i}pr)^{s+t-l}}{s!t!}\Theta_{st} \quad , \end{aligned} \tag{59}$$

where $a=\frac{1}{2}(1+|\nu-2k|)-\mathrm{i}|m|(p-q^2/p)-\mathrm{i}\beta/p$, $b=\frac{1}{2}(1+|\nu|)-\mathrm{i}|m|(p-q^2/p)$ $+\mathrm{i}\beta/p$ and $(z)_s$ denotes Pochhammer's symbol [35]. The quantity Θ_{st} is given by

$$\Theta_{st}=\int_{-1}^{1}(1+x)^{|\nu-2k|+s}(1-x)^{|\nu|+t}P_l^{(|\nu|,|\nu-2k|)}(x)dx \quad . \tag{60}$$

Using the Rodriguez formula (53) one shows that Θ_{st} is equal to zero for $s+t\geq l$. Therefore, we can consider the limit $r\longrightarrow 0$ on both sides of (59) and obtain

$$\begin{aligned} W_{l\nu k}^{p\beta}&=\frac{(-\mathrm{i})^l}{2^{l+|\nu|+|\nu-2k|}}\frac{|\Gamma(a)\Gamma(b)|}{|\Gamma(l+a+b)|}\frac{(2l+|\nu|+|\nu-2k|+1)!}{|\nu|!|\nu-2k|!}\\ &\quad\times\sqrt{\frac{(2l+|\nu|+|\nu-2k|+1)l!(l+|\nu|-|\nu-2k|)!}{16\pi p\,(l+|\nu|)!(l+|\nu-2k|)!}}\\ &\quad\times\sum_{s=0}^{l}\frac{(a)_s(b)_{l-s}}{(1+|\nu-2k|)_s(1+|\nu|)_{l-s}}\frac{\Theta_{s,l-s}}{s!(l-s)!} \quad . \end{aligned} \tag{61}$$

Using the integral (53) we can evaluate $\Theta_{s,l-s}$, and therefore we obtain for the interbasis coefficients $W_{l\nu k}^{p\beta}$

$$W_{l\nu k}^{p\beta} = \frac{(-\mathrm{i})^l}{|\nu - 2k|!} \frac{|\Gamma(a)\Gamma(b)|\Gamma(l+b)}{|\Gamma(l+a+b)|\Gamma(b)}$$
$$\times \sqrt{\frac{(2l + |\nu| + |\nu - 2k| + 1)(l + |\nu - 2k|)!(l + |\nu| + |\nu - 2k|)!}{4\pi p\, l!(l + |\nu|)!}}$$
$$\times {}_3F_2\left(\begin{matrix} a & l + |\nu| + |\nu - 2k| & -l \\ 1 + |\nu - 2k| & 1 - l - b & \end{matrix} \middle| 1 \right) . \tag{62}$$

Using a symmetry transformation for ${}_3F_2(1)$ according to [36]

$${}_3F_2\left(\begin{matrix} -n & b & c \\ d & e & \end{matrix} \middle| 1 \right) = \frac{(d-b)_n}{(a)_n} {}_3F_2\left(\begin{matrix} -n & b & e-c \\ e & b-d-n+1 & \end{matrix} \middle| 1 \right) , \tag{63}$$

yields

$$W_{l\nu k}^{p\beta} = \frac{(-\mathrm{i})^l}{|\nu - 2k|!} \frac{\Gamma(l+a+b)|\Gamma(a)\Gamma(b)|}{\Gamma(a+b)|\Gamma(l+a+b)|}$$
$$\times \sqrt{\frac{(2l + |\nu| + |\nu - 2k| + 1)(l + |\nu - 2k|)!(l + |\nu| + |\nu - 2k|)!}{4\pi p\, l!(l + |\nu|)!}}$$
$$\times {}_3F_2\left(\begin{matrix} a & -l & l + |\nu| + |\nu - 2k| \\ 1 + |\nu - 2k| & a+b & \end{matrix} \middle| 1 \right) . \tag{64}$$

Comparing (64) with the corresponding formula for the analytic continuation of the ordinary SU(2) Clebsch-Gordan coefficients [34, 37]

$$C_{j_1m_1,j_2m_2}^{jm} = (-1)^{j_1-m_1} \delta_{m,m_1+m_2} \frac{(j_1+j_2-m)!}{(j_2-j_1+m)!}$$
$$\times \sqrt{\frac{(2j+1)(j_2-j_1+j)!(j_1+m_1)!(j_2+m_2)!(j+m)!}{(j_1-m_1)!(j_2-m_2)!(j-m)!(j_1+j_2-j)!(j_1-j_2+j)!(j_1+j_2+j+1)!}}$$
$$\times {}_3F_2\left(\begin{matrix} -j_1+m_1 & -j+m & j+m+1 \\ -j_1-j_2+m & j_2-j_1+m & \end{matrix} \middle| 1 \right) , \tag{65}$$

we finally obtain for the coefficients $W_{l\nu k}^{p\beta}$ of the interbasis expansion

$$W_{l\nu k}^{p\beta} = (-1)^{l+a} \sqrt{\frac{\Gamma(a)\Gamma(b)\Gamma(1-a)\Gamma(1-b)}{4\pi p\, \Gamma(a+b)\Gamma(1-a-b)}}\, C_{j_1m_1,j_2m_2}^{jm} , \tag{66}$$

where

$$\left.\begin{aligned} j_1 &= \frac{|\nu| - a - b}{2} , \quad j_2 = \frac{|\nu - 2k| - a - b}{2} , \quad j = l + \frac{|\nu| + |\nu - 2k|}{2} , \\ m_1 &= \frac{|\nu - 2k| - a + b}{2} , \quad m_2 = \frac{|\nu| + a - b}{2} , \quad m = \frac{|\nu| + |\nu - 2k|}{2} . \end{aligned}\right\} \tag{67}$$

Thus we have established the interbasis expansion of the continuous basis in parabolic coordinates in terms of the spherical basis.

4.2 Interbasis expansion of the spherical basis with respect to the parabolic basis

Because the parabolic parameter β may have in general a complex value, we must clarify the range of integration for β in $\mathbb{C}$ for the inverse interbasis expansion. Let us consider the integral

$$Q_{ll'} = \int_{\mathbb{R}} W^{p\beta *}_{l'\nu k} W^{p\beta}_{l\nu k} d\beta \ . \tag{68}$$

Using the form for the interbasis coefficient $W^{p\beta}_{l\nu k}$ in terms of the ${}_3F_2$ function (64) we have

$$\begin{aligned} Q_{ll'} =& \frac{\mathrm{i}^{l-l'}}{4\pi p} \frac{E^{\nu,k}_{ll'}}{(|\nu-2k|!)^2} \sum_{s=0}^{l'} \frac{(-l')_s(l'+1+|\nu|+|\nu-2k|)_s}{(1+|\nu-2k|)_s \Gamma(a+b+s)\, s!} \\ &\times \sum_{t=0}^{l} \frac{(-l)_t(l+1+|\nu|+|\nu-2k|)_t}{(1+|\nu-2k|)_t \Gamma(2+|\nu|+|\nu-2k|-a-b+t) t!} A_{k\nu} \ , \end{aligned} \tag{69}$$

where the quantity $E^{\nu,k}_{ll'}$ is given by

$$\begin{aligned} E^{\nu,k}_{ll'} =& \sqrt{(2l+|\nu|+|\nu-2k|+1)(2l'+|\nu|+|\nu-2k|+1)} \\ &\times \sqrt{\frac{\Gamma(l+a+b)\Gamma(l'+|\nu|+|\nu-2k|-a-b+2)}{\Gamma(l'+a+b)\Gamma(l+|\nu|+|\nu-2k|-a-b+2)}} \\ &\times \sqrt{\frac{(l+|\nu-2k|)!(l+|\nu|+|\nu-2k|)!(l'+|\nu|)!l'!}{(l'+|n-2k|)!(l'+|\nu|+|\nu-2k|)!(l+|\nu|)!l!}} \ , \end{aligned} \tag{70}$$

and the quantity $A_{k\nu}$ has the form $(z = \mathrm{i}\beta/2p)$

$$\begin{aligned} A_{k\nu} =& (-2\mathrm{i}p) \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma\left(\frac{1+|\nu|}{2} - \frac{\mathrm{i}|m|(p^2-q^2)}{p} + z\right) \Gamma\left(\frac{1+|\nu|}{2} + \frac{\mathrm{i}|m|(p^2-q^2)}{p} - z\right) \\ &\times \Gamma\left(\frac{1+|\nu-2k|}{2} - \frac{\mathrm{i}|m|(p^2-q^2)}{p} + s - z\right) \\ &\times \Gamma\left(\frac{1+|\nu-2k|}{2} + \frac{\mathrm{i}|m|(p^2-q^2)}{p} + t + z\right) dz \ . \end{aligned} \tag{71}$$

According to Barnes' Lemma [38, §1.19(8)]

$$\frac{1}{2i\pi}\int_{-i\infty}^{i\infty}\Gamma(\alpha+z)\Gamma(\beta+z)\Gamma(\gamma-z)\Gamma(\delta-z)\,dz$$
$$=\frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}\ , \tag{72}$$

if the path of integration is indented so that the poles of $\Gamma(\gamma-z)\Gamma(\delta-z)$ lie to the right, and the poles of the expression $\Gamma(\alpha+z)\Gamma(\beta+z)$ to the left of it, and it is supposed that $\alpha,\beta,\gamma,\delta$ are such that no pole of the first set coincides with any pole of the second set. In our case the Lemma applies and we obtain

$$Q_{ll'} = \mathrm{i}^{l-l'}\,E_{ll'}^{\nu,k}\,\frac{|\nu|!}{|\nu-2k|!}\sum_{s=0}^{l'}\frac{(-l')_s(l'+1+|\nu|+|\nu-2k|)_s}{\Gamma(2+|\nu|+|\nu-2k|+s)\,s!}$$
$$\times\sum_{t=0}^{l}\frac{(-l)_t(l+1+|\nu|+|\nu-2k|)_t(1+|\nu-2k|+s)_t}{(1+|\nu-2k|)_t(2+|\nu|+|\nu-2k|+s)_t\,t!}\ . \tag{73}$$

Using now the Saalschütz Theorem [38, §4.4(3)]

$$\sum_{p}^{n}\frac{(a)_p(b)_p(-n)_p}{(c)_p(1+a+b-c-n)_p\,p!}=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}\ , \tag{74}$$

we get

$$Q_{ll'}=(-\mathrm{i})^{l+l'}\,E_{ll'}^{\nu,k}\,\frac{(l+|\nu|)!}{(l+|\nu-2k|)!}\sum_{s=0}^{l'}\frac{(-l')_s(l'+1+|\nu|+|\nu-2k|)_s}{(l+|\nu|+|\nu-2k|+s+1)!\,\Gamma(1-l+s)}\ . \tag{75}$$

Because $s_{\max}=l'$ for the case $l'<l$, we have for all s $1-l+s\leq 0$, and it follows that $Q_{ll'}$ equals zero. We get the same result in the case $l'>l$ because the equation for $Q_{ll'}$ is symmetric with respect to $l\to l'$. Thus, for $l=l'$ we consider only the last term with $s=l'$, and it follows in (75)

$$Q_{ll'}=\int_{\mathbb{R}}W_{l'\nu k}^{p\beta *}\,W_{l\nu k}^{p\beta}d\beta=\delta_{ll'}\ . \tag{76}$$

Thus we have for the inverse interbasis expansion of the spherical basis with respect to the parabolic basis

$$\Psi_{pl\nu k}(r,\vartheta,\varphi,\psi)=\int_{\mathbb{R}}d\beta\,W_{l\nu k}^{p\beta *}\Psi_{p\beta\nu k}(\xi,\eta,\varphi,\psi)\ , \tag{77}$$

and the β-integration is taken along the real axis.

5 Summary

In this paper we have derived the interbasis coefficients of the wave-functions of the Kaluza-Klein monopole system which relate the spherical and parabolic bases with each other. We have found that the coefficients for the discrete as well as for the continuous bases are proportional to Clebsch-Gordan coefficients; in the case of the discrete basis we have found the difference is but a phase factor. An extension of our results to the Dyon problem with an additional scalar potential $\propto \hbar^2\mu^2/2Mr^2$ according to [18, 19] is straightforward and omitted.

We have not dwelled into the corresponding analysis of the Kaluza-Klein monopole in the third separating coordinate system. Let us consider the *prolate spheroidal coordinate system*, which is given by

$$\left.\begin{aligned} x &= d\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi &&= d\sinh\mu\sin\nu\cos\varphi\ ,\\ y &= d\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi &&= d\sinh\mu\sin\nu\sin\varphi\ ,\\ z &= d\xi\eta &&= d\cosh\mu\cos\nu \end{aligned}\right\} \tag{78}$$

$\xi > 1, |\eta| < 1$, $\mu > 0, 0 < \nu < \pi$, $\varphi \in [0, 2\pi)$, and $R = 2d$ is the interfocus distance. For convenience we can also introduce the alternative representation of the coordinates in terms of trigonometric and hyperbolic functions via $\xi = \cosh\mu$, $\eta = \cos\nu$. Replacing $z \mapsto z = d(\cosh\mu\cos\nu + 1)$ gives the *prolate spheroidal II coordinate system* (c.f. [30] and references therein), which actually separates the Kaluza-Klein monopole system and the usual Kepler-Coulomb problem as well. Note that the coordinates (78) separate the two-center Coulomb problem [39]. The property that the Kaluza-Klein monopole system separates in a third coordinate system is connected with the fact that we have the five observables (8), and the observable corresponding to the spheroidal system is a combination of the observables in the spherical and parabolic coordinate systems.

In a forthcoming contribution we will analyse this case of the prolate spheroidal basis, where matters are much more involved and the wave-functions can only be defined recursively. The separating procedure in prolate spheroidal coordinates is similar as in the case of the Hartmann potential (c.f. [30] and references therein). As it turns out, the interbasis expansion approach is most convenient in this case because no solution in usually known higher transcendental functions is possible, similarly as for the pure Kepler-Coulomb problem [40]. After obtaining recurrence relations for the interbasis coefficients, they can serve as a starting point for an algebraic perturbation description [41–43] for the construction of the wave-functions. Also, interbasis expansions relating parametric bases with, e.g., a spherical basis enables one to derive path integral representations in parametric coordinate systems [11]. However, this will be discussed elsewhere.

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