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A.N.Sissakian
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METHOD OF VARIATIONAL PERTURBATION THEORY IN QCD

A.N. Sissakian, I.L. Solovtsov, O.P. Solovtsova

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, 141980 Dubna, Moscow region, Russia

In recent years great efforts have been made to develop methods which somehow make it possible to go beyond perturbation theory. For this purpose here we consider the method of variational perturbation theory (VPT). Within this approach the original action is rewritten using some variational addition and an expansion in the effective interaction is made. Therefore, in contrast to many variational methods, in the VPT the quantity under study from the very beginning is written in the form of a series which makes it possible to calculate the needed corrections. The VPT method thereby allows for the possibility of determining the degree to which the principal contribution found variationally using some optimization principle adequately reflects the problem in question and of determining the region of applicability of the results obtained. The possibility of performing calculations using this approach is based on the fact that the VPT method, like standard perturbation theory, uses only Gaussian functional quadratures. Here, of course, the VPT series possesses a different structure and, in addition, the Feynman rules are modified at the level of the propagators and vertices. The form of the diagrams does not change, which is very important technically. The auxiliary parameters arising in the VPT expansion allow the convergence properties of the series to be controlled. The scalar models have been considered in Refs. [1-3]. In this paper we will concentrate on applications of the VPT method to quantum chromodynamics and give a brief review of the results obtained.

In the case of QCD it is not immediately obvious how to introduce a useful variable split between bare and interaction Lagrangians which respects gauge invariance. A solution was found in Refs. [4,5] using as an intermediate step the device of an auxiliary χ–field. Let us write the QCD action functional in the form

$$S(A,q,\varphi) = S_2(A) + S_2(q) + S_2(\varphi) + g\,S_3(A,q,\varphi) + g^2\,S_4(A)\,, \tag{1}$$

where $S_2(A)$, $S_2(q)$ $S_2(\varphi)$ are free action functionals of the gluon, quark, and ghost fields, respectively; the term $S_2(A)$ also contains a term fixing the covariant α_G–gauge. The term $S_3(A,q,\varphi)$ describes the Yukawa interaction of gluons, gluons with quarks, and gluons with ghosts

$$S_3(A,q,\varphi) = S_3(A) + S_3(A,q) + S_3(A,\varphi)\,. \tag{2}$$

The terms $S_3(A)$, $S_3(A,q)$ and $S_3(A,\varphi)$ generate, respectively, three–line vertices (AAA), $(\bar{q}Aq)$ and $(\varphi A\varphi)$; whereas the term $S_4(A)$ in (1), four–gluon vertices $(AAAA)$. We will transform the latter term by introducing auxiliary fields $\chi^a_{\mu\nu}$ [4]. Upon the χ–transformation, the diagrams of the Green functions will consist only of diagrams of the Yukawa type. In addition to the usual three–line vertices of QCD, vertices of the type $A\chi A$ will appear. Thus, a certain Green function of QCD can be represented in the following functional integral form

$$G(\cdots) = \int D\chi\, D_{\rm QCD}\,(\cdots) \times \exp\Big\{ i \Big[S(A,\chi) + S_2(q) + S_2(\varphi) + S_2(\chi) + g\, S_3(A,q,\varphi) \Big] \Big\}, \qquad (3)$$

where

$$S(A,\chi) = \frac{1}{2} \int dx dy\, A^a_\mu(x) \Big[D^{-1}(x,y|\chi) \Big]^{ab}_{\mu\nu} A^b_\nu(y) \qquad (4)$$

with the gluon propagator $D(x,y|\chi)$ in the χ–field

$$\Big[D^{-1}(x,y|\chi) \Big]^{ab}_{\mu\nu} = \big[(-g_{\mu\nu}\, \partial^2 + \partial_\mu \partial_\nu)\, \delta^{ab} + g\sqrt{2}\, f^{abc}\, \chi^c_{\mu\nu} + \text{gauge terms} \big]\, \delta(x-y)\,. \qquad (5)$$

and the term $(\cdots)$ is a set of ν gluon, quark and ghost fields. Integration measure $D_{\rm QCD}$ in (3) defines standard integrations over gluon, quark, and ghost fields.

Following the ideas of the VPT method, we introduce auxiliary parameters ζ and ξ and rewrite the action in (3) in the form

$$S(A,q,\varphi,\chi) = S_0'(A,q,\varphi,\chi) + S_I'(A,q,\varphi,\chi)\,, \qquad (6)$$

where

$$S_0'(A,q,\varphi,\chi) = \zeta^{-1} [S(A,\chi) + S_2(q) + S_2(\varphi)] + \xi^{-1}\, S_2(\chi)\,, \qquad (7)$$

$$S_I'(A,q,\varphi,\chi) = g\, S_3(A,q,\varphi) - (\zeta^{-1}-1)\, [S(A,\chi) + S_2(q) + S_2(\varphi)] - (\xi^{-1}-1)\, S_2(\chi)\,. \qquad (8)$$

The exact value of the quantity under consideration, for instance, the Green function does not depend on the parameters ζ and ξ. However, the approximation of that quantity with a finite number of terms of the VPT series, that results from the expansion in powers of the action $S_I'(A,q,\varphi,\chi)$, does depend on those parameters. We can employ the freedom in the choice of the parameters ζ and ξ for our aim, construction of a new small parameter of the expansion.

It is more convenient to rewrite $S_0'(A,q,\varphi,\chi)$ in (7) by replacing ζ^{-1} to $[1+\kappa\,(\zeta^{-1}-1)]$ and ξ^{-1} to $[1+\kappa\,(\xi^{-1}-1)]$ and putting $\kappa = 1$ at the end of calculations. In this case, any power of the expression $(\zeta^{-1}-1)\,[S(A,\chi) + S_2(q)\; S_2(\varphi)] + (\xi^{-1}-1)\, S_2(\chi),$, appearing in the factor of the exponential upon expanding the Green function in powers of (8), can be obtained by differentiating with respect to the parameter κ as many times as required. Then, the integrand in the factor of the exponential will contain only the powers of the action $g\, S_3(A,q,\varphi)$ that generate the QCD Yukawa diagrams with modified propagators defined by appropriate quadratic forms in the new "free" action S_0'. The VPT series for the Green function is given by

$$G(\cdots) = \sum_n \sum_{k=0}^{n} \frac{1}{(n-k)!} \Big(-\frac{\partial}{\partial\kappa}\Big)^{n-k} \frac{i^k}{k!} \times \int D\chi\, D_{\rm QCD}(\cdots)\, \big[g\, S_3(A,q,\varphi \big]^k \exp\big[i\, S_0'(A,q,\varphi,\chi) \big] \qquad (9)$$

with the above replacement in $S_0'(A,q,\varphi,\chi)$. Further, it is convenient to rescale the fields

$$(A,q,\varphi) \Rightarrow \frac{(A,q,\varphi)}{\sqrt{1+\kappa(\zeta^{-1}-1)}}\quad,\quad \chi \Rightarrow \frac{\chi}{\sqrt{1+\kappa(\xi^{-1}-1)}}\,. \tag{10}$$

As a result, the propagators acquire the standard form and only the diagram vertices get modified. Integrating then over the field χ we obtain for the Green function

$$\begin{aligned} G(\cdots) &= \sum_n \sum_{k=0}^{n} \frac{1}{(n-k)!}\Big(-\frac{\partial}{\partial\kappa}\Big)^{n-k}\frac{i^k}{k!}\,\frac{1}{[1+\kappa(\zeta^{-1}-1)]^{\nu/2}} \\ &\times \int D_{\rm QCD}(\cdots)\,[g_3\,S_3(A,q,\varphi)]^k \exp\{\,i\,[\,S_0(A,q,\varphi) + g_4^2\,S_4(A)\,]\,\}\,. \end{aligned} \tag{11}$$

Here $S_0(A,q,\varphi)$ does no longer contain the term describing the field χ and represents a usual functional of the QCD free action, whereas g_3 and g_4 in the Yukawa and four-gluon vertices are defined as follows:

$$g_3 = \frac{g}{[1+\kappa(\zeta^{-1}-1)]^{3/2}}\,,\qquad g_4 = \frac{g}{[1+\kappa(\xi^{-1}-1)]^{1/2}}\,. \tag{12}$$

Analysis of the structure of the VPT series shows [4] that we will succeed in constructing the small expansion parameter if we put $\xi=\zeta^3$ and if the parameter ζ is connected with the coupling constant by the equation

$$\lambda = \frac{g^2}{(4\pi)^2} = \frac{1}{C}\,\frac{a^2}{(1-a)^3}\,,\qquad a = 1-\zeta\,, \tag{13}$$

where C is a positive constant. As follows from (13), at any values of the coupling constant g, the new expansion parameter a obeys the inequality $0\le a<1$. It is interesting that the connection between the initial coupling constant g and the expansion parameter a, given by Eq. (13), is the same as for the anharmonic oscillator [6].

In a genuine field theory such as QCD the coupling constant run in a way determined by the renormalization group method. A momentum dependence of our expansion parameter $a=a(Q^2)$ is given by the following transcendental equation [5,7]

$$Q^2 = Q_0^2\,\exp\Big\{\frac{C}{2b_0}\Big[f(a) - f(a_0)\Big]\Big\}\,, \tag{14}$$

where

$$f(a) = \frac{2}{a^2} - \frac{6}{a} - 48\ln a - \frac{18}{11}\frac{1}{1-a} + \frac{624}{121}\ln(1-a) + \frac{5184}{121}\ln(1+\frac{9}{2}a) \tag{15}$$

and $b_0 = 11 - \frac{2}{3}N_f$, N_f is number of flavours.

The parameters C has been determined from the condition that the renormalization group β-function at large enough values of the coupling constant behaves as $\beta(\lambda)\simeq -\lambda$. This behaviour corresponds to the singular infrared behaviour of the invariant charge $\alpha_S(Q^2)\sim Q^{-2}$ and ensures the linear growth of the nonrelativistic static quark-antiquark potential at large distances. Thus, we fix parameters C on the basis of data of hadron spectroscopy, which gives $C=4.1$ [5]. To determine all parameters, we use the condition

$\lambda_{\rm eff}(Q_0) = \lambda_0$ at some normalization point Q_0 with an experimental value of the coupling constant λ_0.

In Ref. [8], the process of e^+e^- annihilation into hadrons at low energies has been analyzed in the framework of the given approach. We applied the smearing method [9] to compare the obtained theoretical prediction for $R_{e^+e^-}$–ratio with experimental data and found a good agreement of the first order of our approximation down to lowest energies.

Recently, the problem of "freezing" the QCD coupling constant at low energies has arisen. This freezing is required in many models based on QCD (for detailed discussion, see [10] and references therein). As the "experimental" value, for comparison it is convenient to use the integral independent of the fit of data [10]

$$\int_0^{1\,\rm GeV} dQ \, \frac{\alpha_s^{\rm eff}(Q)}{\pi} \simeq 0.2\,{\rm GeV}\,. \tag{16}$$

In our case, this integral equals 0.237 GeV.

The τ decay process with hadronic final states represents an important test of quantum chromodynamics. Due to the inclusive character of the process, the ratio R_τ is a very convenient quantity both for a theoretical investigation and for the definition of the QCD coupling constant $\alpha_s(M_\tau^2)$. A detailed theoretical analysis of this problem has been given in Ref. [11] (see also Refs. [12-15], in which different aspects of the problem are discussed).

The starting point of the theoretical analysis is the expression

$$R_\tau = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right)\tilde{R}(s)\,, \tag{17}$$

where

$$\begin{aligned}\tilde{R}(s) &= \frac{\rm N}{2\pi {\rm i}}\left[\Pi(s+{\rm i}\epsilon) - \Pi(s-{\rm i}\epsilon)\right]\,,\\ \Pi(s) &= \sum_{q=d,s}|V_{uq}|^2\left(\Pi_{uq,V}(s) + \Pi_{uq,A}(s)\right)\,.\end{aligned} \tag{18}$$

The normalization factor N is defined so that in zeroth order perturbation theory $\tilde{R}^{(0)}_{\rm pert} = 3$. In the framework of standard perturbation theory the integral (17) cannot be evaluated directly since the integration region in (17) includes small values of momentum for which perturbation theory is invalid[1]. Instead of Eq. (17), the expression for R_τ may be rewritten, using Cauchy's theorem, as a contour integral in the complex s-plane with the contour running clockwise around the circle $|s| = M_\tau^2$. It seems that this trick allows one to avoid the problem of calculating the nonperturbative contribution, which is needed if one uses Eq. (17). However, the application of Cauchy's theorem is based on specific analytic properties of Π(s) or the Adler D function

$$D(q^2) = q^2\left(-\frac{d}{d\,q^2}\right){\rm N}\,\Pi(q^2)\,. \tag{19}$$

The function $D(q^2)$ is an analytic function in the complex q^2–plane with a cut along the positive real axis. It is clear that the approximation of the D-function by perturbation

[1] In Ref.[16], the integral (17) has been calculated within the method of optimized perturbative series [17].

theory breaks these analytic properties. For example, the one-loop approximation for the QCD running coupling constant has a singularity at $Q^2 = \Lambda^2_{QCD}$, the existence of which prevents the application of Cauchy's theorem. Moreover, to define the running coupling constant in the timelike domain, one usually uses the dispersion relation for the D function derived on the basis of the above-mentioned analytic properties. In the framework of perturbation theory, this method gives the so-called π^2–term contribution which plays an important role in the analysis of various processes [18-22]. However, the same problem arises: the perturbative approximation breaks the analytic properties of $\lambda^{\rm eff}(q^2)$ which are required to write the dispersion relation. In addition, there is the problem of taking into account of threshold effects. As follows from Eq. (17), the initial expression for R_τ "knows" about the quark thresholds. But all the threshold information is lost if one rewrites this equation as a contour integral and uses a fixed number of flavours for the calculation of $\tilde{R}(s)$ on this contour.

Here we will concentrate on both aspects of the problem. In the framework of our approach there exists a well-defined procedure for defining the running coupling in the timelike domain which does not conflict with the dispersion relation [23,24]. We will use the following definitions: $\lambda^{\rm eff} = \alpha_{\rm QCD}/(4\pi)$ is the initial effective coupling constant in the t–channel (spacelike region) and $\lambda^{\rm eff}_s$ is the effective coupling constant in the s–channel (timelike region). From the dispersion relation for the D–function we obtain

$$\lambda^{\rm eff}(q^2) = -q^2 \int_0^\infty \frac{d\,s}{(s-q^2)^2}\,\lambda^{\rm eff}_s(s)\,. \tag{20}$$

Thus, the initial running coupling constant $\lambda^{\rm eff}(q^2)$ is an analytic function in the complex q^2–plane with a cut along the positive real axis. This function does not exist for real positive q^2, so the definition of the running coupling constant in the timelike domain is a crucial question. Here we use the standard definition of $\lambda^{\rm eff}_s(s)$ in the s–channel based on the dispersion relation for the Adler D–function. In this case, parametrization of timelike quantities, for example $R_{e^+e^-}(s)$ or $\tilde{R}(s)$, by the function $\lambda^{\rm eff}_s(s)$ is similar to parametrization of spacelike processes by the function $\lambda^{\rm eff}(q^2)$.

The inverse relation of Eq. (20), given the analytic properties of $\lambda^{\rm eff}(q^2)$, is of the form

$$\lambda^{\rm eff}_s(s) = -\frac{1}{2\pi {\rm i}} \int_{s-{\rm i}\,\epsilon}^{s+{\rm i}\,\epsilon} \frac{d\,q^2}{q^2}\,\lambda^{\rm eff}(q^2)\,, \tag{21}$$

where the contour goes from the point $q^2 = s - {\rm i}\epsilon$ to the point $q^2 = s + {\rm i}\epsilon$ and lies in the region where $\lambda^{\rm eff}(q^2)$ is an analytic function of q^2. Equation (21) defines the running coupling constant in the timelike region which we must use to calculate $\tilde{R}(s)$ in Eq. (17).

To write Eq. (21), it was important that the function $\lambda^{\rm eff}(q^2)$ had the above-mentioned analytic properties. For example, to use the one-loop approximation, one needs to modify its infrared behaviour at $Q^2 = \Lambda^2$ in an *ad hoc* manner so that the singularity at $Q^2 = \Lambda^2$ is absent in the new expression for $\lambda(Q^2)$. A self-consistent formulation of the analytic continuation problem is, however, possible within the scope of a systematic non-perturbative approach. Within this approach we can maintain the mentioned above analytic properties [23-25]. Taking the experimental value $R_\tau = 3.552$ [26] as an input, we obtain $\alpha_s(M_\tau^2) = 0.37$ and $\alpha(M_\tau^2) = 0.40$. The values of the coupling constant in the s- and t–channels are clearly different from each other; the ratio is $\alpha_s(M_\tau^2)/\alpha(M_\tau^2) = 0.92$.

The experimentally measurable quantity R_τ can be parametrized both by the function $\alpha_s(s)$ defined in the time–like region and entering into the initial expression for R_τ and by the running coupling constant $\alpha(q^2)$ used in the contour integral. The perturbative expansion does not allow one to perform the integration in Eq.(17) directly because it involves a non-perturbative region. Instead, one usually uses the perturbative formula to evaluate the contour integral. However, we believe this to be inconsistent because the analytic properties which are required to write down the Cauchy integral are not respected by the perturbative formula. The method proposed allows one to evaluate both the initial integral for R_τ and the expression obtained by the use of Cauchy's theorem. Of course, as it should be, they are equal. We have also demonstrated that the distinction between the functions $\alpha_s(s)$ and $\alpha(q^2)$ is not simply a matter of the standard π^2 terms, which may be important for understanding certain discrepancies [27] arising in the determination of the QCD coupling constant from various experiments.

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References

[1] A.N. Sissakian and I.L. Solovtsov, Phys. Part. Nucl. **25** (1994) 478.

[2] A.N. Sissakian, I.L. Solovtsov, and O.Yu. Shevchenko, in Proc. Int. Workshop *Symmetry Methods in Physics*, in Memory of Professor Ya.A. Smorodinsky, ed. by A.N. Sissakian, G.S. Pogosyan, and S.I. Vinitsky, Dubna, 1994, v.2, p.494.

[3] A.N. Sissakian, I.L. Solovtsov, and O.P. Solovtsova, Phys. Lett. **B321** (1994) 381.

[4] I.L. Solovtsov, Phys. Lett. **B327** (1994) 335.

[5] I.L. Solovtsov, Phys. Lett. **B340** (1994) 245.

[6] R. Seznec and J. Zinn-Justin, J. Math. Phys. **20** (1979) 1398.

[7] A.N. Sissakian, I.L. Solovtsov, and O.P. Solovtsova, Mod. Phys. Lett. **A9** (1994) 2437.

[8] I.L. Solovtsov and O.P. Solovtsova, Phys. Lett. **B344** (1995) 377.

[9] E.C. Poggio, H.R. Quinn, and S. Weinberg, Phys. Rev. **D13** (1976) 1958.

[10] A.C. Mattingly and P.M. Stevenson, Phys. Rev. Lett. **69** (1992) 1320; Phys. Rev. **D49** (1994) 437.

[11] E. Braaten, S. Narison, and A. Pich, Nucl. Phys. **B373** (1992) 581.

[12] A. Pich, Nucl. Phys. B (Proc. Suppl.) **39B**, C (1995) 326.

[13] E. Braaten, Phys. Rev. Lett. **60** (1988) 1606; Phys. Rev. **D39** (1989) 1458;
S. Narison, Preprint CERN-TH.7188/94, PM 94/08;
F. LeDiberder and A. Pich, Phys. Lett. **B286** (1992) 147.

[14] G. Altarelli, Preprints CERN-TH.7493/94, CERN-TH-95/196;
G. Altarelli, P. Nason, and G Ridolfi, Preprint CERN-TH.7537/94 [hep-ph/9501240], GeF-TH-1/95, IFUM 489/FT.

[15] T.N. Truong, Phys. Rev. **D47** (1993) 3999;
K.G. Chetyrkin and A. Kwiatkowski, Z. Phys. **C59** (1993) 525;
P.A. Raczka and A. Szymacha, Preprint IFT/13/94 [hep–ph/9412236].

[16] A.C. Mattingly, Rice University Preprint DE-FG05-92ER40717-8.
[17] P. M. Stevenson, Phys. Rev. **D23** (1981) 2916; Nucl. Phys. **B231** (1984) 65.
[18] M.R. Pennington and G.G. Ross, Phys. Lett. **102B** (1981) 167.
[19] N.V. Krasnikov and A.A. Pivovarov, Phys. Lett. **B116** (1982) 168.
[20] S.G. Gorishny, A.L. Kataev, and S.A. Larin, Phys. Lett. **B259** (1991) 144.
[21] J.D. Bjorken, Preprint, SLAC–PUB–5103, 1989.
[22] A.V. Radyushkin, Preprint JINR, E2-82-159, 1982.
[23] H.F. Jones and I.L. Solovtsov, Phys. Lett. **B349** (1995) 519.
[24] H.F. Jones and I.L. Solovtsov, Talk given at the Int. Europhysics Conference on High Energy Physics, Brussels, 27 July - 2 August, 1995.
[25] H.F. Jones, I.L. Solovtsov, and O.P. Solovtsova, Phys. Lett. **B359** (1995) 441.
[26] Review of Particle Properties, Phys. Rev. **D50**, Part 1 (1994).
[27] A.L. Kataev, Talk presented at the Symposium on Radiative Correction: Status and Outlook, Preprint CERN-TH.7465/94 [hep-ph/9410308];
S. Bethke, Talk presented at the QCD'94, Montpellier, France, July 7-13, 1994, Preprint PITHA 94/30.