JOINT INSTITUTE FOR NUCLEAR RESEARCH

VII International Conference

SYMMETRY METHODS IN PHYSICS

Dubna, Russia July 10 - 16, 1995

Edited by
A.N.Sissakian
G.S.Pogosyan

Volume 1

Dubna 1996

INTERBASIS EXPANSION FOR THE KALUZA-KLEIN MONOPOLE SYSTEM

C. Grosche,[1,2] G. S. Pogosyan,[3] A. N. Sissakian[3]

[1]*Institut für Theoretische Physik, Technische Universität Clausthal, Sommerfeldstraße 6, 38678 Clausthal-Zellerfeld, Germany*
[2]*II. Institut für Theoretische Physik Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany*
[3]*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia*

Abstract

We study the interbasis expansion of the wave-functions of the Kaluza-Klein monopole system in the parabolic coordinate system with respect to the spherical coordinate system, and vice versa. We show that the coefficients of the expansion are proportional to Clebsch-Gordan coefficients.

1. Introduction

In the framework of quantum mechanics magnetic monopoles have been first discussed by Dirac in his classical paper [1]. He described them as quantized singularities in the electromagnetic field, the quantization condition being

$$2ge = nc\hbar, \qquad (n \in \mathbb{N}) \tag{1.1}$$

(e - electric charge, g - magnetic charge, c - velocity of light), arising from the singlevaluedness requirement of the wave-function. The corresponding Schrödinger equation can be straightforwardly evaluated, and leads to a pure continuous spectrum of an electron moving in the field of a magnetic monopole. More general is the Dyon problem, where a Coulomb-interaction term $\propto eg/r$ is included, and bound states can appear. This problem has been discussed by several authors, see e.g. Barut et al. [2], Jackiw [3], or Zwanziger [4].

More elaborated monopole models have been developed since and monopole solutions seem to be inevitable in grand unified theories [5]. Important examples are the (Bogomolnyi-Prasad-Sommerfield) BPS monopoles, e.g. [6], which move along geodesics in a curved space, and Kaluza-Klein monopoles, e.g. [7]-[10], the latter emerging from the former by means of a static solution, i.e., large spatial separation, of the classical field equations of five-dimensional gravity (Taub-NUT limit, "Euclidean limit"). Then, the relevant metric for the (full) Kaluza-Klein monopole system is given by ($\mathbf{x} = (x, y, z) \in \mathbb{R}^3$) [7]

$$ds^2 = \frac{1}{\Lambda(r)}d\mathbf{x}^2 + \Lambda(r)(dx_5^2 + 4m\,\mathbf{A}\cdot d\mathbf{x})^2 \ , \qquad \mathbf{A} = \frac{\pm 1 - z/r}{r^2 - z^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \ , \tag{1.2}$$

with $x_5 = 4m\psi$; the metric term Λ and the φ-component of the magnetic interaction in spherical coordinates (r, ϑ, φ) are given by

$$\Lambda(r) = \frac{1}{1 + \frac{4m}{r}} \ , \quad A_\varphi = 4m(\pm 1 - \cos\vartheta) \ , \quad A_r = A_\vartheta = 0 \ . \tag{1.3}$$

The additional angular variable ψ describes the relative phase. This form of the Kaluza-Klein monopole system is the simplest solution of the classical field equations. The singularity at the origin vanishes if the coordinate ψ is cyclic with period 4π [8, 9]. The quantity Λ in the metric (g_{ab}) represents the effects of gravity, and $\mathbf{A}$ is identified with the electromagnetic field interaction. $4m$ is the only parameter that characterizes the Kaluza-Klein monopole system, seen as a test particle in the Taub-NUT space, and the coupling $g = 4m < 0$ generates a discrete energy spectrum. Similarly as in the classical example of the o(4), respectively o(3,1) dynamical symmetry algebra in the Kepler problem, the total angular momentum operator $\mathbf{J}$ and a suitable rescaled Pauli-Runge-Lenz vector $\mathbf{K}$ close into an o(4) or o(3,1) algebra, depending on the sign of the energy, which can be extended to an o(4,2) symmetry [2, 8, 10]. On the corresponding homogeneous spaces of the groups O(4) and O(3,1), i.e., the three-dimensional sphere and the three-dimensional hyperboloid, a convenient coordinate space representation may be chosen for perturbation investigations for the discrete spectrum, respectively scattering phenomena in particular channels, e.g. [11] for a review concerning coordinate systems in homogeneous spaces, and their corresponding path integral representations and solutions.

The dynamical symmetry allows for a complete algebraic description of the classical as well as the quantum motions for the Kaluza-Klein monopole system. It was shown by [7] that these specific monopole problems admit due to their symmetry properties a solution in spherical and parabolic coordinates. However, the third separating system, prolate spheroidal coordinates have not been taken into account until now.

By Barut et al. [2] it was found that quantum systems with an O(4,2) symmetry can be related by the Kustaanheimo-Stiefel transformation to a four-dimensional oscillator, a fact which has been extensively exploited in path integral evaluations concerning Kepler-Coulomb, e.g. [12, 13], and Dyon problems [14]–[20], and references therein.

In this contribution we investigate the interbasis expansion of the (discrete and continuous) wave-functions in parabolic coordinates with respect to the spherical basis, and vice versa. We show that the the coefficients in the interbasis expansion are proportional to Clebsch-Gordan coefficients.

2. Quantum Mechanical Solution

If the cyclic variable ψ is separated off, the observables in the (reduced) Kaluza-Klein monopole system are given by a suitably chosen angular momentum operator $\mathbf{J}$ and a Pauli-Runge-Lenz operator $\mathbf{K}$ which can be cast into the form, e.g. [8, 10, 21]–[23]

$$\left.\begin{aligned} \mathbf{J} &= \mathbf{x}\times\boldsymbol{\pi} - 4m\hbar q\frac{\mathbf{x}}{|\mathbf{x}|} , \\ \mathbf{K} &= \frac{1}{2M}(\boldsymbol{\pi}\times\mathbf{J} - \mathbf{J}\times\boldsymbol{\pi}) - 4m\frac{\mathbf{x}}{|\mathbf{x}|}\left(H - \frac{q^2\hbar^2}{M}\right) , \end{aligned}\right\} \tag{2.1}$$

where $\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$, $\mathbf{p} = -i\hbar\nabla$. They satisfy the commutation relations

$$[J_i, J_k] = i\hbar\epsilon_{ikl}J_l, \quad [J_i, K_k] = i\hbar\epsilon_{ikl}K_l, \quad [K_i, K_k] = i\hbar\left(\frac{q^2\hbar^2}{M} - 2H\right)\epsilon_{ikl}J_l. \tag{2.2}$$

Here, summation over repeated indices is understood, and ϵ_{ikl} is totally antisymmetric in the indices $i, k, l = 1, 2, 3$. On the fixed energy eigenspace $H\Psi = E\Psi$ one defines the rescaled Pauli-Lenz-Runge operator $\mathbf{M}$ by means of

$$\left.\begin{array}{ll} \mathbf{M}=\left(\frac{\hbar^2q^2}{M}-2E\right)^{-1/2}\mathbf{K} & \text{for } E<\hbar^2q^2/2M\ , \\ \mathbf{M}=\mathbf{K} & \text{for } E=\hbar^2q^2/2M\ , \\ \mathbf{M}=\left(2E-\frac{\hbar^2q^2}{M}\right)^{-1/2}\mathbf{K} & \text{for } E>\hbar^2q^2/2\mathrm{M}\ . \end{array}\right\} \tag{2.3}$$

The operators $\mathbf{M}$ and $\mathbf{J}$ close to an o(4) algebra for $E<\hbar^2q^2/2M$, to an o(3,1) algebra for $E>\hbar^2q^2/2M$, and to an $o(3)\otimes\mathbb{R}^3$ algebra in the case $E=\hbar^2q^2/2M$, e.g. [8]. The property of five functionally independent observables characterizes the (reduced) Kaluza-Klein monopole system as a super-integrable system [25, 26] in three-dimensional Euclidean space. A complete set of observables of the full system is given by, e.g. [10],

$$\{q,\underline{\mathrm{H}},\mathbf{A}^2,A_3,\mathbf{B}^2,B_3\}\ , \tag{2.4}$$

where $\mathbf{A}=\frac{1}{2}(\mathbf{J}+\mathbf{M})$ and $\mathbf{B}=\frac{1}{2}(\mathbf{J}-\mathbf{M})$.

We consider the spherical and the parabolic coordinate system which separate the Kaluza-Klein monopole system. The *spherical coordinate system* is given by $(r>0, 0<\vartheta<\pi, 0\le\varphi<2\pi)$

$$x=r\sin\vartheta\cos\varphi,\quad y=r\sin\vartheta\sin\varphi,\quad z=r\cos\vartheta. \tag{2.5}$$

The *parabolic coordinate system* has the form $(\xi,\eta>0, 0\le\varphi<2\pi)$

$$x=\xi\eta\cos\varphi,\quad y=\xi\eta\sin\varphi,\quad z=\frac{1}{2}(\xi^2-\eta^2). \tag{2.6}$$

In terms of these coordinates the line element ds^2 takes on the form $(dx_5=4md\psi, \psi\in[0,4\pi))$

$$ds^2=\frac{1}{\Lambda(r)}\Big(dr^2+r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2)\Big)+(4m)^2\Lambda(r)\Big(d\psi+(\pm1-\cos\vartheta)d\varphi\Big)^2 \tag{2.7}$$

$$=\frac{1}{\Lambda(r)}\Big((\xi^2+\eta^2)(d\xi^2+d\eta^2)+\xi^2\eta^2d\varphi^2\Big)+(4m)^2\Lambda(r)\left[d\psi+\left(\pm1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)d\varphi\right]^2 \tag{2.8}$$

In the following we only take into account the "+"-sign which is sufficient for our purposes. Therefore we obtain for the Hamiltonian operator H in the two coordinate systems

Polar Coordinates:

$$H=-\frac{\hbar^2}{2M}\Lambda(r)\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\vartheta^2}+\cot\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right)\right.$$
$$\left.+\left(\frac{1}{(4m\Lambda)^2}+\frac{(1-\cos\vartheta)^2}{r^2\sin^2\vartheta}\right)\frac{\partial^2}{\partial\psi^2}-\frac{2}{r^2}\frac{1-\cos\vartheta}{\sin^2\vartheta}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\psi}\right] \tag{2.9}$$

Parabolic Coordinates:

$$=-\frac{\hbar^2}{2M}\Lambda(r)\left\{\frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)+\frac{1}{\xi^2\eta^2}\frac{\partial^2}{\partial\varphi^2}\right.$$
$$\left.+\left[\frac{1}{(4m\Lambda)^2}+\frac{1}{\xi^2\eta^2}\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)^2\right]\frac{\partial^2}{\partial\psi^2}-\frac{2}{\xi^2\eta^2}\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\psi}\right\}. \tag{2.10}$$

In the spherical coordinate system the time-independent Schrödinger equation

$$H\Psi(r,\vartheta,\varphi,\psi)=E\Psi(r,\vartheta,\varphi,\psi) \tag{2.11}$$

is solved by making the Ansatz for the wave-functions according to

$$\Psi(r,\vartheta,\varphi,\psi)=R(r)Z(\vartheta)\frac{e^{i(\nu\varphi+k\psi)}}{4\pi\sqrt{2|m|}} \ . \tag{2.12}$$

Here are $\nu\in\mathbb{Z}$, $q=k/4m\equiv s/4m$, $2s=0,\pm1,\pm2,\ldots$, and the operator $\hat{q}$, conjugate to $\hat{\psi}$, corresponding to the quantum number q is conserved and identified with the relative electric charge [7, 10, 23].

The functions $R(r)$ and $Z(\vartheta)$ are normalized, and the particular form of Ψ guarantees that the wave-functions are normalized to unity with respect to the scalar product

$$(f,g)=\int_0^\infty\frac{r^2dr}{\Lambda(r)}\int_0^\pi\sin\vartheta d\vartheta\int_0^{2\pi}d\varphi\int_0^{4\pi}4|m|\,d\psi f^*g \ . \tag{2.13}$$

Explicitly, this gives for the angular and radial variables the ordinary differential equations

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\sin\vartheta\frac{dZ}{d\vartheta}+\left[J(J+1)-\frac{\nu^2}{4\sin^2\frac{\vartheta}{2}}-\frac{(\nu-2k)^2}{4\cos^2\frac{\vartheta}{2}}\right]Z=0 \ , \tag{2.14}$$

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr}+\left[\left(\frac{2ME}{\hbar^2}-\frac{k^2}{16m^2}\right)+\frac{1}{r}\left(\frac{8mME}{\hbar^2}-\frac{k^2}{2m}\right)-\frac{J(J+1)}{r^2}\right]R=0 \ , \tag{2.15}$$

where J is the spherical quantum number. We see that a discrete spectrum can only occur for $g=4m<0$, $E<\hbar k^2/32m^2M$. The solution of these differential equations, respectively the wave-functions of the spectral expansion of the corresponding path integral solution, are in terms of Jacobi polynomials in $\cos\vartheta$, and Laguerre polynomials for the discrete and Whittaker functions for the continuous spectrum in the radial variable. Thus we get for $m<0$ the bound state wave-functions $\Psi_{NJ\nu k}$ in spherical coordinates [7, 17, 18, 24]

$$\Psi_{NJ\nu k}(r,\vartheta,\varphi,\psi)=\frac{e^{i\nu\varphi+ik\psi}}{4\pi\sqrt{2|m|}}R_{NJ}(r)\,Z_{J\nu k}(\vartheta) \ , \tag{2.16}$$

where the radial $R_{NJ}(r)$ wave-functions have the form

$$R_{NJ}(r)=\sqrt{\frac{(N-J-1)!}{(N+J)!}}\frac{2\,e^{-r/aN}}{a^{3/2}N^{3/2}(N^2-s^2)^{1/4}}\left(\frac{2r}{aN}\right)^J L_{N-J-1}^{(2J+1)}\left(\frac{2r}{aN}\right), \tag{2.17}$$

and the functions $Z_{J\nu k}(\vartheta)$ are given by

$$Z_{J\nu k}(\vartheta)=\left[\frac{(2J+1)[(J-\frac{1}{2}(|\nu|+|\nu-2k|)]![(J+\frac{1}{2}(|\nu|+|\nu-2k|)]!}{2[J+\frac{1}{2}(|\nu|-|\nu-2k|)]![J-\frac{1}{2}(|\nu|-|\nu-2k|)]!}\right]^{1/2}$$
$$\times\left(\sin\frac{\vartheta}{2}\right)^{|\nu|}\left(\cos\frac{\vartheta}{2}\right)^{|\nu-2k|}P_{J-\frac{1}{2}(|\nu|+|\nu-2k|)}^{(|\nu|,|\nu-2k|)}(\cos\vartheta) \ . \tag{2.18}$$

The bound state energy levels are

$$E_N = \frac{\hbar^2}{(4m)^2M}\sqrt{N^2-s^2}\Big(N-\sqrt{N^2-s^2}\Big) \ , \qquad N=|s|+1,|s|+2,\cdots \ . \tag{2.19}$$

Here $N = n_r + l + \frac{|\nu|+|\nu-2k|}{2} + 1$ is the principal quantum number, l the orbital, and n_r the radial quantum numbers, which have for fixed N and s, the values $0, 1, \cdots N-|s|-1$, respectively. The quantum numbers J and l are related through $J = l + \frac{|\nu|+|\nu-2k|}{2}$. Note, that $a = 1\Big/\Big(N\sqrt{q^2-2E_NM/\hbar^2}\Big) = 4|m|\Big/\Big[N(N-\sqrt{N^2-s^2})\Big]$ is a "Bohr-radius".

The continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2M}(p^2+q^2) \ , \tag{2.20}$$

with largest lower bound $E_0 = \hbar^2q^2/2M = \hbar^2k^2/32m^2M$, and the continuous wave-functions $\Psi_{pJ\nu k}$ are

$$\Psi_{pJ\nu k}(r,\vartheta,\varphi,\psi) = \frac{e^{i\nu\varphi+ik\psi}}{4\pi\sqrt{2|m|}}R_{pJ}(r)\,Z_{J\nu k}(\vartheta) \ , \tag{2.21}$$

with the radial wave-functions $R_{pJ}(r)$ given by

$$R_{pJ}(r) = \frac{|\Gamma[J+1-2i|m|(p^2-q^2)/p]|}{(2J+1)!\sqrt{2\pi r}}e^{\pi|m|(p^2-q^2)/p}M_{\frac{2i|m|(p^2-q^2)}{p},J+\frac{1}{2}}(-2ipr) \ . \tag{2.22}$$

The $M_{\kappa,\lambda}(z) = e^{-x/2}x^{\lambda+1/2}{}_1F_1(\frac{1}{2}-\kappa+\lambda;2\lambda+1;x)$ are Whittaker functions [27, p.1059]. The functions (2.17,2.22) are orthogonal and form a complete set.

In parabolic coordinates one makes the Ansatz

$$\Psi(\xi,\eta,\varphi,\psi) = f_1(\xi)f_2(\eta)\frac{e^{i(\nu\varphi+k\psi)}}{4\pi\sqrt{2|m|}} \ . \tag{2.23}$$

The wave-functions Ψ are normalized according to the scalar product

$$(f,g) = \int_0^\infty d\xi\int_0^\infty d\eta(\xi^2+\eta^2)\frac{\xi\eta}{\Lambda(r)}\int_0^{2\pi}d\varphi\int_0^{4\pi}4|m|d\psi f^*g \ . \tag{2.24}$$

This gives the ordinary differential equations

$$\frac{d^2f_1}{d\xi^2}+\frac{1}{\xi}\frac{df_1}{d\xi}+\left[\left(\frac{2ME}{\hbar}-\frac{k^2}{16m^2}\right)\xi^2-\frac{(\nu-2k)^2}{\xi^2}+\beta_1\right]f_1 = 0 \ , \tag{2.25}$$

$$\frac{d^2f_2}{d\eta^2}+\frac{1}{\eta}\frac{df_2}{d\eta}+\left[\left(\frac{2ME}{\hbar}-\frac{k^2}{16m^2}\right)\eta^2-\frac{\nu^2}{\eta^2}+\beta_2\right]f_2 = 0 \ , \tag{2.26}$$

where $\beta_{1,2}$ are related through

$$\beta_1+\beta_2 = \frac{16mME}{\hbar^2}-\frac{k^2}{m} \ . \tag{2.27}$$

The solutions of these ordinary differential equations, respectively the solution for the corresponding path integral formulation, are for the bound state wave-functions Laguerre polynomials in ξ and η, respectively. We obtain [7, 19] for the bound state wave-functions $(n_1, n_2 \in \mathbb{N}; N = n_1 + n_2 + \frac{1}{2}(|\nu| + |\nu - 2k|) + 1)$

$$\Psi_{n_1 n_2 \nu k}(\xi, \eta, \varphi, \psi) = \frac{e^{i\nu\varphi + ik\psi}}{4\pi\sqrt{2|m|}} \left[\frac{2}{a^3 N^3 \sqrt{N^2 - s^2}} \frac{n_1! n_2!}{(n_1 + |\nu - 2k|)!(n_2 + |\nu|)!}\right]^{1/2}$$
$$\times \left(\frac{\xi^2}{aN}\right)^{\frac{|\nu-2k|}{2}} \left(\frac{\eta^2}{aN}\right)^{\frac{|\nu|}{2}} \exp\left(-\frac{\xi^2 + \eta^2}{2aN}\right) L_{n_1}^{(|\nu-2k|)}\left(\frac{\xi^2}{aN}\right) L_{n_2}^{(|\nu|)}\left(\frac{\eta^2}{aN}\right) , \quad (2.28)$$

with the energy spectrum (2.19). The continuous states $\Psi_{p\beta\nu k}$ are given by [19]

$$\Psi_{p\beta\nu k}(\xi, \eta, \varphi, \psi) = \frac{e^{i\nu\varphi + ik\psi}}{4\pi\sqrt{2|m|}} \exp\left[\frac{\pi|m|}{p}(p^2 - q^2)\right]$$
$$\times \frac{|\Gamma(\frac{1}{2} + |\frac{\nu}{2} - k| - i\beta_1)\Gamma(\frac{1+|\nu|}{2} - i\beta_2)|}{\sqrt{2\pi^2 p}\,\xi\eta |\nu - 2k|!\,|\nu|!} M_{i\beta_1, \frac{|\nu-2k|}{2}}(-ip\xi^2) M_{i\beta_2, \frac{|\nu|}{2}}(-ip\eta^2) , \quad (2.29)$$

where $\beta_{1,2} = \frac{1}{4}\left[4|m|(p - q^2/p) \pm 2\beta/p\right]$, β is the parabolic separation parameter, and E_p as in (2.20). The wave-functions (2.28,2.29) are orthogonal and form a complete set.

3. Interbasis Expansion for the Discrete Basis

Let us consider the interbasis expansion of the parabolic bound state wave-functions (2.28) with respect to the spherical wave-functions (2.16), i.e.,

$$\Psi_{n_1 n_2 \nu k}(\xi, \eta, \varphi, \psi) = \sum_{l=0}^{n_1+n_2} W_{n_1 n_2}^l \Psi_{n_r l \nu k}(r, \vartheta, \varphi, \psi) , \quad (3.1)$$

where $n_1 + n_2 = n_r + l$; we have included the dependence of ψ, and have re-inserted the angular quantum number l and the radial quantum number n_r in the principal quantum number $N = n_r + J + 1 \equiv n_r + l + \frac{1}{2}(|\nu| + |\nu - 2k|) + 1$. The parabolic variables can be expressed in terms of the spherical variables by means of

$$\xi^2 = r + z = r(1 + \cos\vartheta) , \qquad \eta^2 = r - z = r(1 - \cos\vartheta) . \quad (3.2)$$

We consider (3.1) in the limit $r \to \infty$. From the property of the Laguerre polynomials $L_n^{(\alpha)}(x) \to (-1)^n x^2/n!$, as $x \to \infty$, we see that the dependence on r cancels on both sides of (3.1).

Using the orthogonality condition of the angular wave-functions (2.18) we find the following expression for the interbasis coefficients $W_{n_1 n_2}^l$

$$W_{n_1 n_2}^l = \frac{(-1)^l}{2^{n_1+n_2+|\nu|+|\nu-2k|}} \sqrt{\frac{l!(l + |\nu| + |\nu - 2k|)!}{(l + |\nu|)!(l + |\nu - 2k|)!}}$$
$$\times \sqrt{\frac{[l + \frac{1}{2}(|\nu| + |\nu - 2k| + 1)] n_r! (n_r + 2l + |\nu| + |\nu - 2k| + 1)!}{n_1! n_2! (n_1 + |\nu - 2k|)!(n_2 + |\nu|)!}} I_{n_1 n_2}^l , \quad (3.3)$$

with the quantity $I^l_{n_1 n_2}$ given by

$$I^l_{n_1 n_2} = \int_{-1}^{1} dx\,(1+x)^{n_1+|\nu-2k|}(1-x)^{n_2+|\nu|} P_l^{(|\nu|,|\nu-2k|)}(x) \ . \tag{3.4}$$

Using the Rodriguez formula for the Jacobi polynomials [27, p.1035]

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \tag{3.5}$$

and comparing with the integral representation of the Clebsch-Gordan coefficients [28, p.243]

$$\begin{aligned} C^{jm}_{j_1 m_1, j_2 m_2} &= \delta_{m_1+m_2,m}\frac{(-1)^{j_1+m_2-j}}{2^{j_1+j_2+j+1}} \\ &\times\sqrt{\frac{(2j+1)(j+m)!(j_1+j_2+j+1)!(j_1+j_2-j)!}{(j_2-j_1+j)!(j_1-j_2+j)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!(j-m)!}} \\ &\times\int_{-1}^{1} dx\,(1-x)^{j_1-m_1}(1+x)^{j_2-m_2}\frac{\mathrm{d}^{j-m}}{\mathrm{d}x^{j-m}}\left[(1-x)^{j_2-j_1+j}(1+x)^{j_1-j_2+j}\right] \ , \end{aligned} \tag{3.6}$$

yields that the coefficients $W^l_{n_1 n_2}$ of the interbasis expansion (3.1) are given in terms of the Clebsch-Gordan coefficients $C^{jm}_{j_1 m_1, j_2 m_2}$, i.e.,

$$W^l_{n_1 n_2} = (-1)^{n_2} C^{jm}_{j_1 m_1, j_2 m_2} \ , \tag{3.7}$$

where

$$\left.\begin{aligned} j_1 &= \frac{n_1+n_2+|\nu-2k|}{2} \ , \quad j_2 = \frac{n_1+n_2+|\nu|}{2} \ , \quad j = l + \frac{|\nu|+|\nu-2k|}{2} \ , \\ m_1 &= \frac{n_1-n_2+|\nu-2k|}{2} \ , \quad m_2 = \frac{n_2-n_1+|\nu|}{2} \ , \quad m = \frac{|\nu|+|\nu-2k|}{2} \ . \end{aligned}\right\} \tag{3.8}$$

Using the orthogonality condition of the Clebsch-Gordan coefficients, we can invert (3.1) yielding

$$\Psi_{n_r l\nu k}(r,\vartheta,\varphi,\psi) = \sum_{n_1=0}^{n_r+l} W^l_{n_1 n_2}\Psi_{n_1 n_2 \nu k}(\xi,\eta,\varphi,\psi) \ , \tag{3.9}$$

which represents the expansion of the spherical basis with respect to the parabolic basis.

4. Interbasis Expansion for the Continuous Basis

Let us consider the interbasis expansion of the parabolic basis with respect to the spherical basis, i.e.,

$$\Psi_{p\beta\nu k}(\xi,\eta,\varphi,\psi) = \sum_{l=0}^{\infty} W^{p\beta}_{l\nu k}\Psi_{pJ\nu k}(r,\vartheta,\varphi,\psi) \ , \tag{4.1}$$

with the spherical wave-functions $\Psi_{pl\nu k}$ (2.21) and the parabolic wave-functions $\Psi_{p\beta\nu k}$ (2.29). We now consider the following expression

$$W^{p\beta}_{l\nu k} \cdot {}_1F_1(a+b+l; 2l+|\nu|+|\nu-2k|+2; -2ipr) = \frac{(-i)^l}{2^{l+|\nu|+|\nu-2k|}}$$
$$\times\sqrt{\frac{(2l+|\nu|+|\nu-2k|+1)l!(l+|\nu|-|\nu-2k|)!}{16\pi p(l+|\nu|)!(l+|\nu-2k|)!}}\frac{|\Gamma(a)\Gamma(b)|}{|\Gamma(l+a+b)|}$$
$$\times\frac{(2l+(|\nu|+|\nu-2k|)+1)!}{(|\nu|)!(|\nu-2k|)!}\sum_{s,t=0}^{\infty}\frac{(a)_s(b)_t}{(1+|\nu-2k|)_s(1+|\nu|)_t}\frac{(-ipr)^{s+t-l}}{s!t!}\Theta_{st}\ , \qquad (4.2)$$

where $a=\frac{1}{2}(1+|\nu-2k|)-i|m|(p-q^2/p)-i\beta/p$, $b=\frac{1}{2}(1+|\nu|)-i|m|(p-q^2/p)+i\beta/p$ and $(z)_s$ denotes Pochhammer's symbol. The quantity Θ_{st} is given by

$$\Theta_{st} = \int_{-1}^{1}(1+x)^{|\nu-2k|+s}(1-x)^{|\nu|+t}P_l^{(|\nu|,|\nu-2k|)}(x)dx\ . \qquad (4.3)$$

Using the Rodriguez formula (3.5) one shows that Θ_{st} is equal to zero for $s+t\geq l$. Therefore, we can consider the limit $r\to 0$ on both sides of (4.2) and after evaluating $\Theta_{s,l-s}$ obtain

$$W^{p\beta}_{l\nu k} = \frac{(-i)^l}{|\nu-2k|!}\sqrt{\frac{(2l+|\nu|+|\nu-2k|+1)(l+|\nu-2k|)!(l+|\nu|+|\nu-2k|)!}{4\pi p\, l!(l+|\nu|)!}}$$
$$\times\frac{|\Gamma(a)\Gamma(b)|\Gamma(l+b)}{|\Gamma(l+a+b)|\Gamma(b)}\,{}_3F_2\left(\begin{matrix} a & l+|\nu|+|\nu-2k| & -l \\ 1+|\nu-2k| & 1-l-b & \end{matrix}\middle|\,1\right)\ . \qquad (4.4)$$

Using a symmetry transformation for ${}_3F_2(1)$ according to [29]

$${}_3F_2\left(\begin{matrix} -n & b & c \\ d & e & \end{matrix}\middle|\,1\right) = \frac{(d-b)_n}{(a)_n}\,{}_3F_2\left(\begin{matrix} -n & b & e-c \\ e & b-d-n+1 & \end{matrix}\middle|\,1\right) \qquad (4.5)$$

yields

$$W^{p\beta}_{l\nu k} = \frac{(-i)^l}{|\nu-2k|!}\sqrt{\frac{(2l+|\nu|+|\nu-2k|+1)(l+|\nu-2k|)!(l+|\nu|+|\nu-2k|)!}{4\pi p\, l!(l+|\nu|)!}}$$
$$\times\frac{\Gamma(l+a+b)|\Gamma(a)\Gamma(b)|}{\Gamma(a+b)|\Gamma(l+a+b)|}\,{}_3F_2\left(\begin{matrix} a & -l & l+|\nu|+|\nu-2k| \\ 1+|\nu-2k| & a+b & \end{matrix}\middle|\,1\right)\ . \qquad (4.6)$$

Comparing (4.6) with the corresponding formula for the SU(2) Clebsch-Gordan coefficients [28, 30]

$$C^{jm}_{j_1m_1,j_2m_2} = (-1)^{j_1-m_1}\delta_{m,m_1+m_2}\frac{(j_1+j_2-m)!}{(j_2-j_1+m)!}$$
$$\times\sqrt{\frac{(2j+1)(j_2-j_1+j)!(j_1+m_1)!(j_2+m_2)!(j+m)!}{(j_1-m_1)!(j_2-m_2)!(j-m)!(j_1+j_2-j)!(j_1-j_2+j)!(j_1+j_2+j+1)!}}$$
$$\times{}_3F_2\left(\begin{matrix} -j_1+m_1 & -j+m & j+m+1 \\ -j_1-j_2+m & j_2-j_1+m & \end{matrix}\middle|\,1\right)\ , \qquad (4.7)$$

we finally obtain for the coefficients $W_{l\nu k}^{p\beta}$ of the interbasis expansion

$$W_{l\nu k}^{p\beta} = (-1)^{l+a}\sqrt{\frac{\Gamma(a)\Gamma(b)\Gamma(1-a)\Gamma(1-b)}{4\pi p\,\Gamma(a+b)\Gamma(1-a-b)}}\, C_{j_1 m_1, j_2 m_2}^{jm} \ , \tag{4.8}$$

where

$$\left.\begin{aligned} j_1 &= \frac{|\nu| - a - b}{2} , & j_2 &= \frac{|\nu - 2k| - a - b}{2} , & j &= l + \frac{|\nu| + |\nu - 2k|}{2} , \\ m_1 &= \frac{|\nu - 2k| - a + b}{2} , & m_2 &= \frac{|\nu| + a - b}{2} , & m &= \frac{|\nu| + |\nu - 2k|}{2} . \end{aligned}\right\} \tag{4.9}$$

Thus we have established the interbasis expansion of the continuous basis in parabolic coordinates in terms of the spherical basis.

In order to derive the interbasis expansion of the continuous basis in spherical coordinates in terms of the parabolic basis one considers the expression

$$Q_{ll'} = \int W_{l'\nu k}^{p\beta *} W_{l\nu k}^{p\beta} d\beta \ . \tag{4.10}$$

Because the parabolic parameter β may have in general a complex value, one must clarify the range of integration for β in $\mathbb{C}$ for the inverse interbasis expansion. By means of a straightforward but tedious calculation one shows that

$$Q_{ll'} = \int_{\mathbf{R}} W_{l'\nu k}^{p\beta *} W_{l\nu k}^{p\beta} d\beta = \delta_{ll'} \ , \tag{4.11}$$

and thus we have for the inverse interbasis expansion of the spherical basis with respect to the parabolic basis, i.e., the inverse of (4.1) has the form

$$\Psi_{pl\nu k}(r, \vartheta, \varphi, \psi) = \int_{\mathbf{R}} d\beta\, W_{l\nu k}^{p\beta *} \Psi_{p\beta\nu k}(\xi, \eta, \varphi, \psi) \ , \tag{4.12}$$

and the β-integration is taken along the real axis.

5. Summary

In this contribution we have derived the interbasis coefficients of the wave-functions of the Kaluza-Klein monopole system which relate the spherical and parabolic bases with each other. We have found that the coefficients for the discrete as well as for the continuous bases are proportional to Clebsch-Gordan coefficients; in the case of the discrete basis we have found the difference is but a phase factor. An extension of our results to the Dyon problem with an additional scalar potential $\propto \hbar^2\mu^2/2Mr^2$ according to [4, 16] is straightforward and omitted. The interbasis expansion corresponding to the spheroidal basis will be discussed elsewhere.

References

[1] P.A.M.Dirac: *Proc.Roy.Soc.(London)* **A 133** (1932) 60.
[2] A.O.Barut, C.K.E.Schneider and R.Wilson: *J.Math.Phys.* **20** (1979) 2244.

[3] R.Jackiw: *Ann.Phys.(N.Y.)* **129** (1980) 183.
[4] D.Zwanziger: *Phys.Rev.* **176** (1968) 1480.
[5] N.S.Craigie, P.Goddard and W.Nahm (eds.): Monopoles in Quantum Field Theory. (*World Scientific*, Singapore, 1982).
[6] E.B.Bogomol'nyi: *Sov.J.Nucl.Phys.* **24** (1976) 449; E.N.Bogomolnyi and M.S.Marinov: *Sov.J.Nucl.Phys.* **24** (1976) 355; A.M.Polyakov: *JETP Lett.* **20** (1974) 194; M.K.Prasad and C.M.Sommerfield: *Phys.Rev.Lett.* **35** (1975) 760; G.t'Hooft: *Nucl.Phys.* **B 79** (1974) 276.
[7] G.W.Gibbons and N.S.Manton: *Nucl.Phys.* **B 274** (1986) 183.
[8] B.Cordani, G.Fehér and P.A.Horvàthy: *Phys.Lett.* **B 201** (1988) 481.
[9] D.J.Gross and M.J.Perry: *Nucl.Phys.* **B 226** (1983) 29; A.Salam and J.Strathdee: *Ann.Phys.(N.Y.)* **141** (1982) 316; R.D.Sorkin: *Phys.Rev.Lett.* **51** (1983) 87.
[10] L.Gy.Fehér and P.A.Horváthy: *Phys.Lett.* **B 183** (1987) 182, 512 (E).
[11] C.Grosche: Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulæ. (*World Scientific*, Singapore, 1996).
[12] I.H.Duru and H.Kleinert: *Phys.Lett.* **B 84** (1979) 185. *Fortschr.Phys.* **30** (1982) 401.
[13] C.Grosche: *Fortschr.Phys.* **40** (1992) 695.
[14] H.Dürr and A.Inomata: *J.Math.Phys.* **26** (1985) 2231.
[15] L.Chetouani, L.Guechi, M.Letlout and T.F.Hammann: *Nuovo Cimento* **B 105** (1990) 387.
[16] H.Kleinert: *Phys.Lett.* **A 116** (1986) 201.
[17] C.C.Bernido: *Nucl.Phys.* **B 321** (1979) 108.
[18] A.Inomata and G.Junker: *Phys.Lett.* **B 234** (1990) 41; *Phys.Rev.* **D 43** (1991) 1235.
[19] *C.Grosche: J.Phys.A: Math.Gen.* **24** *(1991) 1771.*
[20] C.Grosche and F.Steiner: Table of Feynman Path Integrals. To appear in: *Springer Tracts in Modern Physics* (1996).
[21] N.Katayama: *Nuovo Cimento* **B 108** (1993) 657.
[22] M.Visinescu: *Z.Phys.* **C 60** (1993) 337.
[23] G.W.Gibbons and P.J.Ruback: *Phys.Lett.* **B 188** (1987) 226.
[24] L.Chetouani, L.Guechi, T.F.Hammann and A.Lecheheb: *J.Math.Phys.* **36** (1995) 1016.
[25] N.W.Evans: *Phys.Rev.* **A 41** (1990) 5666.
[26] C.Grosche, G.S.Pogosyan and A.N.Sissakian: *Fortschr.Phys.* **43** (1995) 453.
[27] I.S.Gradshteyn and I.M.Ryzhik: Table of Integrals, Series, and Products. (*Academic Press*, New York, 1980).
[28] D.A.Varshalovich, A.N.Moskalev and V.K.Khersonskii: Quantum Theory of Angular Momentum. (*World Scientific*, Singapore, 1988).
[29] W.N.Bailey: Generalized Hypergeometric Series, Cambridge Tracts No.32 (*Cambridge University Press*, Cambridge, 1935).
[30] M.Kibler, L.G.Mardoyan and G.S.Pogosyan: *Int.J.Quantum Chem.* **52** (1994) 1301.