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Variational Perturbation Theory in φ^4-Model

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Abstract

A nonperturbative method - Variational Perturbation Theory is discussed. A quantity we are interested in is represented by a series, a finite number of terms of which describes not only the region of small coupling constant but well reproduces the strong coupling limit. The method is formulated only in terms of the Gaussian quadratures and diagrams are used of the conventional perturbation theory. Its efficiency is demonstrated for the quantum-mechanical anharmonical oscillator. We investigate the renormalization for φ^4-model in the variational perturbation theory. The nonperturbative β-function is derived in the framework of the proposed approach. The obtained result is in agreement with four-loop approximation and have the asymptotic behavior as $g^{3/2}$ for a large coupling constant.

A central problem of quantum field theory is going beyond the scope of standard perturbation theory. A great amount of study is devoted to the development of nonperturbative methods. Many nonperturbative approaches make use of a variational procedure for finding the leading contribution. The idea of applying the variational methods in quantum field theory began long ago [1,2] (see also [3,4]). In the last few years approaches with different modifications have produced many applications. In this paper, we consider a method of the variational perturbation theory (VPT) [5–7]. In the framework of VPT it is possible to represent the investigated quantity in the form of a series and it is possible to influence the properties of convergence of this series through certain parameters of variational type. Thus it will become possible to make the optimization of VPT series from the viewpoint of better approximation of a value. Our method is formulated in terms of the Gaussian functional quadratures (like in perturbation theory). Also, we shall construct the VPT so that for its N-th order only those diagrams will be required that compose the N-th order of standard perturbation theory.

Here we will apply the VPT method to Green functions of the φ^4-model in the Euclidean d-dimensional space. To this end we write the 2ν-point function

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in the form

$$G_{2\nu} = \int D\varphi \{\varphi^{2\nu}\} \exp(-S[\varphi]), \tag{1}$$

where

$$\{\varphi^n\} = \varphi(x_1)...\varphi(x_n)$$

and the functional of action looks as follows:

$$\begin{aligned} S[\varphi] &= S_0[\varphi] + \frac{m^2}{2} S_2[\varphi] + \lambda S_4[\varphi], \\ S_0[\varphi] &= \frac{1}{2}\int dx\,(\partial\varphi)^2, \qquad S_p[\varphi] = \int dx\,\varphi^p. \end{aligned} \tag{2}$$

We shall construct a VPT series by using the following Gaussian functional quadratures

$$\begin{aligned} &\int D\varphi \exp\{-[\frac{1}{2} < \varphi\hat{K}\varphi > + < \varphi J >]\} = \\ &= \left(\det \frac{\hat{K}}{-\partial^2 + m^2}\right)^{-1/2} \exp[\frac{1}{2} < J\hat{K}^{-1}J >]. \end{aligned} \tag{3}$$

The VPT series for the Green functions (1) is constructed in the following way:

$$G_{2\nu} = \sum_{n=0}^{\infty} G_{2\nu,n}, \tag{4}$$

$$G_{2\nu,n} = \frac{(-1)^n}{n!} \int D\varphi \{\varphi^{2\nu}\} (\lambda S_4[\varphi] - \tilde{S}[\varphi])^n \exp(-S_0[\varphi] - \frac{m^2}{2} S_2[\varphi] - \tilde{S}[\varphi]). \tag{5}$$

The variational functional $\tilde{S}[\varphi]$ will be taken to be dependent on certain parameters, but the total sum (4) surely will not depend on these parameters. Their choice can be such as to provide the expansion (4) being optimal (see refs. [5–7]).

The functional $\tilde{S}[\varphi]$ should be defined so that the terms of the VPT series (4) be calculable, i.e. the form of $\tilde{S}[\varphi]$ should be such that the functional integral in (5) can be reduced to the Gaussian qnadratures (3). This requirement does not mean that the functional $\tilde{S}[\varphi]$ must be quadratic in fields. We can pass to the Gaussian functional integral by using the Fourier transformation.

We choose here, for example, the sum of harmonic and anharmonic functionals being $\tilde{S}[\varphi]$,i.e.:

$$\tilde{S}[\varphi] = \frac{M^2}{2} S_2[\varphi] + \theta^2 S_2^2[\varphi], \tag{6}$$

where M and θ are the certain parameters through which the VPT series is optimized. We obtain

$$G_{2\nu,n} = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{1}{l!(n-k-l)!}\int_{-\infty}^{\infty}\frac{du}{2\sqrt{\pi}}\exp(-\frac{u^2}{4}) \times \theta^{2l}(M^2-m^2)^{n-k-l}(-\frac{\partial}{\partial M^2})^{n+l-k}\tilde{g}^{(k)}_{2\nu}(\chi^2), \quad (7)$$

where

$$\tilde{g}^{(k)}_{2\nu,n}(\chi^2) = \frac{1}{k!}\int D\varphi\{\varphi^{2\nu}\}(-\lambda S_4[\varphi])^k\exp\{-[S_0[\varphi]+\frac{\chi^2}{2}S_2[\varphi]]\}. \quad (8)$$

The latter expression can be written as follows

$$\tilde{g}^{(k)}_{2\nu,n}(\chi^2) = \det\Big[\frac{-\partial^2+\chi^2}{-\partial^2+m^2}\Big]^{-1/2} g^{(k)}_{2\nu,n}(\chi^2), \quad (9)$$

where $g^{(k)}_{2\nu,n}(\chi^2)$ are calculated on the basis of diagrams of the $k-th$ order of conventional perturbation theory with the propagator $\Delta(p,\chi^2)=(p^2+\chi^2)^{-1}$. A new mass parameter χ^2 is depended on u and variational parameters M^2 and θ. Thus, the $N-th$ order of the VPT expansion (4) can be constructed with the same diagrams as the conventional perturbation $N-th$ order is made up.

Let us consider a case of the quantum-mechanical anharmonic oscillator (AO) as an example of exploiting the VPT method. The AO from a point of view of the path integral formalism is a one-dimensional φ^4-model. The connection between the ground state energy E_0 and the dimensionless four-point Green function $G_4(0,0,0,0)$ takes the form

$$\frac{\partial E_0}{\partial\lambda} = \lambda^{-2/3}G_4\,. \quad (10)$$

For calculating Green function G_4 we will use the two-parameters anharmonical VPT functional

$$\tilde{S}[\varphi] = [\theta S_0[\varphi]+\chi S_2[\varphi]]^2\,. \quad (11)$$

The application of the asymptotic optimization that requires the contribution of the remote terms in the VPT series to be minimal allows one to find the relation between the parameter θ and χ : $16\theta\chi^3=9$. The remaining variational parameter is fixed on the basis of a finite number of VPT expansion terms. For the ground state energy in the first order of VPT we get the strong coupling expansion

$$E_0^{(1)} = \lambda^{1/3}[0.663+0.1407\omega^2-0.0085\omega^4+\cdots]\,, \quad (12)$$

where the dimensionless parameter $\omega^2 = m^2\lambda^{-2/3}$. We have to compare the obtained result with the exact value [8]

$$E_0^{exact} = \lambda^{1/3}[0.668 + 0.1437\omega^2 - 0.0088\omega^4 + \cdots] . \tag{13}$$

We can also calculate the mass parameter μ^2 connected with the two-point Green function: $\mu^{-2} = G_2(p = 0)$. In the strong coupling limit we obtain $\mu^2 = 3.078\lambda^{2/3}$, whereas the exact value is $\mu^2_{exact} = 3.009\lambda^{2/3}$. We can estimate the energy of the first excited level E_1. Defining the energy shift $\mu_1 = E_1 - E_0$ and using the spectral representation for the propagator we arrive at the following estimate for μ_1 :$\mu_1 \leq \mu_1^{(+)}$, where

$$\mu_1^{(+)} = 2G_2(x = 0)/G_2(p = 0) . \tag{14}$$

By analogy with the sum rules, we may expect a sufficiently rapid saturation of the spectral representation, which brings μ_1 and $\mu_1^{(+)}$ closer to each other. In the first order of the one-parameter VPT in the strong coupling limit we get $\mu_1^{(+)} = 1.763\lambda^{1/3}$, whereas exact value is $\mu_1^{exact} = 1.726\lambda^{1/3}$ [8]. The effective potential and corresponded numerical characteristics for AO was computed in [6].

Let us consider the renomalization procedure. The mass less φ^4-model in four dimensions has the Euclidean action

$$S[\varphi] = S_0[\varphi] + S_I[\varphi] \quad , \tag{15}$$

where

$$S_0[\varphi] = \frac{1}{2}\int dx\ \varphi(-\partial^2)\ \varphi \quad , \tag{16}$$

$$S_I[\varphi] = \frac{(4\pi)^2}{4!}\ g \int dx\ \varphi^4 \quad . \tag{17}$$

As is well known, the series of perturbation theory for generating functional of the Green's functions

$$W[J] = \int D\varphi\ \exp\{-S[\varphi] + \int dx\ J \cdot \varphi\} \quad . \tag{18}$$

diverges. A formal argument consists in a meaningless functional integral for negative coupling constant. The function $W[J]$ as function of g is not the analytic function at $g = 0$. The concrete asymptotic behavior of higher-order terms of the perturbation theory can be determined by the functional saddle-point method (the large parameter is the number of the order term) [9–12]. The main contribution to the functional integral (18) comes from the configurations

of fields φ which correspond to the positive power of the large saddle-point parameter. However, in this case, the functional (17) can not be considered as the perturbative term in the comparison with expression (16). This displays in divergence of the perturbation series.

The idea of the VPT method consists in the organization of a new effective functional interaction S_I'. We expect that this functional can be considered as a small value when compared with a new functional S_0'. For the realization of this idea we must be careful about the possibility to make the calculation. Practically, we must use only the Gaussian functional integrals, i.e. the form of $\tilde{S}[\varphi]$ should be such that the functional integral in (18) can be reduced to Gaussian quadratures.

Let us consider the VPT–functional

$$\tilde{S}[\varphi] = \theta^2 S_0^2[\varphi] \tag{19}$$

and rewrite the total action (15) as

$$S[\varphi] = S_0'[\varphi] + \eta\, S_I'[\varphi] \quad , \tag{20}$$

where

$$S_0'[\varphi] = S_0[\varphi] + \tilde{S}[\varphi] \quad , \tag{21}$$

and

$$S_I'[\varphi] = S_I[\varphi] - \tilde{S}[\varphi] \quad . \tag{22}$$

In this case, the expansion of expression (18) is carried out in powers of η. After all calculations we should put $\eta = 1$. The parameter θ^2 in eq. (19) is a parameter of variational type. The initial functional (18) certainly does not depend on this parameter. We may take the θ^2 so as to provide the best approximation with a finite number of VPT series terms. The different methods of the optimization were considered in refs. [5-7].

It is convenient to define the new parameter t by the relation

$$\theta^2 = 4\, C_s\, \frac{(4\pi)^2}{4!}\, g \cdot t \quad . \tag{23}$$

Here $C_s = 4!/(16\pi)^2$ is a constant entering into Sobolev inequality (see, for example, refs. [13,14] and also ref. [15])

$$\int dx\ \varphi^4 \le C_s \left[\int dx\ \varphi\, (-\partial^2)\, \varphi \right]^2 \quad . \tag{24}$$

The parameter t is fixed if we require the contribution of higher order terms of the VPT series to be minimal. This way of determining a variational parameter, is called the asymptotic optimization of VPT series, gives the value $t = 1$ [7].

After expansion in the powers of η we obtain that the remainder contains the $\tilde{S}[\varphi]$ in the exponent and consequently, we have a nongaussian form of the functional integral. However, the problem is easily solved by implementing the Fourier transformation. As a result, Green's function $G_{2\nu}$ in N-th order of VPT take the following form

$$G_{2\nu}^{(N)} = \int_0^\infty d\alpha\ \alpha^{\nu-1} \exp(-\alpha - \theta^2\alpha^2) \sum_{n=0}^{N} \eta^n \alpha^{2n} \sum_{k=0}^{n} \frac{(\theta^2)^{n-k}}{(n-k)!} \frac{g_{2\nu}^k}{\Gamma(2k+\nu)} \quad . \quad (25)$$

Here functions $g_{2\nu}^k$ are ordinary perturbative coefficients for Green's function $G_{2\nu}$. To calculate them, the standard Feynman diagrams can be used.

It should be stressed that the expansion of expression (25) in powers of coupling constant g contains all powers of g. The first N terms of this expansion coincide with N terms of perturbative series.

Let us consider the procedure of renormalization. Instead of field φ and coupling constant g we introduce the bare field φ_0 and bare coupling constant g_0. The field φ_0 is connected with the renormalized field by relation: $\varphi_0 = Z^{1/2}\varphi$. The divergence constants Z and g_0 are obtained from VPT expansion. The constant Z can be calculated by the propagator G_2. We will be employing the constant Z in the first order of VPT series. From eq. (25) we find

$$Z^{(1)} = \Gamma(1)\, J_1(\theta_0^2) + \eta\, \theta_0^2\, \Gamma(3)\, J_3(\theta_0^2) \quad , \quad (26)$$

where we define

$$J_\nu(\theta^2) = \frac{1}{\Gamma(\nu)} \int_0^\infty d\alpha\ \alpha^{\nu-1} \exp(-\alpha - \alpha^2\theta^2) \quad . \quad (27)$$

The function $J_\nu(\theta^2)$ is normalized by the condition $J_\nu(0) = 1$. The connected part of four-point Green's function in the second order of VPT has the form

$$-G_4^{(2)}(\mu^2) = \eta\, g_0\, J_4(\theta_0^2) + \eta^2 \Big[g_0\, \frac{\theta_0^2}{1!} \frac{\Gamma(6)}{\Gamma(4)} J_6(\theta_0^2) - \frac{3}{2}\, g_0^2\, J_6(\theta_0^2) \ln \frac{\Lambda^2}{\mu^2} \Big] \quad . \quad (28)$$

In this expression we wrote out only the divergence part which we need in the following. We use the renormalization scheme with symmetric normalization point μ^2 . For the bare coupling constant g_0 we write down the VPT expansion $g_0 = g(1 + \eta\,\alpha + \ldots)$. An analogy to this one introduces also the VPT expansions for θ_0^2 and $J_\nu(\theta_0^2)$. The divergence coefficient α is defined by expressions (26), (28) and the demand of the function finite $-Z^2 G_4(\mu^2)$. If we change the normalization point $\mu \rightarrow \mu'$ and use that the bare coupling constant

independent on μ we find the connection between g and g'

$$g' = g + \eta\,\beta(g)\,\ln\frac{\mu'^2}{\mu^2} \quad , \tag{29}$$

where Gell-Mann–Low function is expressed as

$$\beta(g) = \frac{(\frac{3}{2}\,g^2)\,J_6(\theta^2)/J_4(\theta^2)}{1 - \theta^2\{[\Gamma(6)\,J_6(\theta^2)/\Gamma(4)\,J_4(\theta^2)] - 2\,[\Gamma(3)\,J_3(\theta^2)/\Gamma(1)\,J_1(\theta^2)]\}}. \tag{30}$$

Here the parameter θ^2 is connected with the renormalized coupling constant g by eq.(23) with the optimal value $t = 1$.

The expression of β-function (30) in the perturbation series contains all powers of the coupling constant g. It is interesting to compare the first coefficients of the VPT β-function (30) with the well-known values of perturbation theory. From (30) we get

$$\beta(g) = 1.5\,g^2 - 2.25\,g^3 + 14.59\,g^4 - 132.95\,g^5 + \ldots \quad . \tag{31}$$

In the considering mass less case, we use conterterms which contain only divergent parts. In the framework of the dimensional regularization this conforms only the pole part for conterterms [16]. Corresponding β-function in four-loop approximation looks as follows [17]

$$\beta_{perturb.}(g) = 1.5\,g^2 - 2.83\,g^3 + 16.27\,g^4 - 135.80\,g^5 + \ldots \quad . \tag{32}$$

Note that for the construction of β-function (30) we used only the first order of VPT. For this approximation the expressions (31) and (32) have sufficient agreement.

As follows from expression (30), the β-function has the monotonous increase and does not have the ultraviolet stable point. For a large coupling constant, β- function has the asymptotic behavior

$$\beta(g) \simeq 2.99\,g^{3/2} \quad . \tag{33}$$

The degree of g in eq. (33) is more than the linear increase of β-function that was obtained in ref. [18], and is less than the square increase that was found in ref. [19].

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