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THE STRUCTURE OF THE HURWITZ TRANSFORMATION

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Abstract

It is shown that the Hurwitz transformation consists of two structural elements: the Levi–Civita transformation and the $SU(2)$ transformation acting in the space of Cayley-Klein parameters.

1 Introduction

The Levi-Civita [1], Kustaanheimo-Stiefel [2] and Hurwitz [3, 4] transformations are elegant and useful mathematical constructions. These transformations made it possible to solve a lot of problems: spinor regularization of the equations of celestial mechanics [5], the problem of Coulomb-oscillator correspondence in quantum mechanics [6], some problems of quantum chemistry [7], functional integration [8], quantum field theory [9], boson calculus [10] and geometrical quantization [11].

Hurwitz transformation (H) maps the 8-dimensional u-space onto the 5-dimensional x-space:

$$\begin{aligned}
x_0 &= u_0^2+u_1^2+u_2^2+u_3^2-u_4^2-u_5^2-u_6^2-u_7^2\\
x_1 &= 2(u_0u_4-u_1u_5-u_2u_6-u_3u_7)\\
x_2 &= 2(u_0u_5+u_1u_4-u_2u_7+u_3u_6)\\
x_3 &= 2(u_0u_6+u_1u_7+u_2u_4-u_3u_5)\\
x_4 &= 2(u_0u_7-u_1u_6+u_2u_5+u_3u_4)
\end{aligned} \tag{1}$$

Here x_j and u_j denote Cartesian coordinates in x- and in u-spaces respectively.

The algebraic structure of the H is such that the Euler's identity holds valid:

$$r^2 \equiv x_0^2+x_1^2+\cdots+x_4^2=(u_0^2+u_1^2+\cdots+u_7^2)^2 \equiv u^4 \tag{2}$$

In the particular case $u_1 = u_2 = u_3 = u_5 = u_6 = u_7 = 0$, the H turns into the Levi-Civita transformation ($x_2 = x_3 = x_4 = 0$)

$$\begin{aligned}
x_0 &= u_0^2-u_4^2\\
x_1 &= 2u_0u_4
\end{aligned} \tag{3}$$

Recently, the Hurwitz transformation has been connected with non-associative algebras [12] and with the Fock-Bargmann-Schwinger representation [13]. Further, the Cayley-Klein [14] and the Eulerian [15] parametrizations have been carried out and nonbilinear version of the Hurwitz transformation has been developed [16].

In spite of these achievements up to now there is no clearness in understanding of the H's structure, except the connection with Clifford algebra [12]. In this note, we discuss the structure of the H-transformation. Below, we will obtain a representation in which the H is determined by two structural elements: the Levi-Civita and the $SU(2)$ transformation. Spaces where these transformations operate will be specified somewhat later.

2 The Conformal Structure of the H

The coordinate x_0 in (1) is distinguished in structure from other coordinates of the x-space. The coordinates u can be classified into the groups (u_0, u_1, u_2, u_3) and (u_4, u_5, u_6, u_7) according to the role they play in forming the coordinate x_0. Therefore, we will separate the coordinate x_0 from the coordinates x_1, x_2, x_3, x_4 and introduce the complex-valued coordinates

$$\begin{aligned} z_1 &= x_1 + ix_2 \quad , \quad z_3 = x_3 + ix_4 \\ v_0 &= u_0 + iu_1 \quad , \quad v_2 = u_2 + iu_3 \\ v_4 &= u_4 + iu_5 \quad , \quad v_6 = u_6 + iu_7 \end{aligned} \tag{4}$$

Let us introduce the complex vectors $\mu = (z_1, z_3)$, $f = (v_0, v_2)$ and $g = (v_4, v_6)$ and denote by $|\mu|$, $|f|$ and $|g|$ there modules:

$$\begin{aligned} |\mu| &= (z_1^* z_1 + z_3^* z_3)^{1/2} \\ |f| &= (v_0^* v_0 + v_2^* v_2)^{1/2} \\ |g| &= (v_4^* v_4 + v_6^* v_6)^{1/2} \end{aligned} \tag{5}$$

With the exception of x_0, transformation (1) can be represented in more compact form

$$\begin{pmatrix} z_1 \\ z_3 \end{pmatrix} = 2 \begin{pmatrix} v_0 & -v_2^* \\ v_2 & v_0^* \end{pmatrix} \begin{pmatrix} v_4 \\ v_6 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} z_1 \\ z_3^* \end{pmatrix} = 2 \begin{pmatrix} v_4 & -v_6 \\ v_6^* & v_4^* \end{pmatrix} \begin{pmatrix} v_0 \\ v_2^* \end{pmatrix} \tag{7}$$

Relations (6) and (7) are dual to each other in the sense that the transformation matrix in (6) and the column which it acts on depend on the coordinates (u_0, u_1, u_2, u_3) and (u_4, u_5, u_6, u_7) respectively; whereas the transformation matrix in (7) is determined by the coordinates (u_4, u_5, u_6, u_7) and the corresponding column,- by the coordinates (u_0, u_1, u_2, u_3).

The matrix of this transformation is unitary with weight, i.e.

$$\begin{pmatrix} v_0^* & v_2^* \\ -v_2 & v_0 \end{pmatrix} \begin{pmatrix} v_0 & -v_2^* \\ v_2 & v_0^* \end{pmatrix} = |f|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} v_4^* & v_6 \\ -v_6^* & v_4 \end{pmatrix} \begin{pmatrix} v_4 & -v_6 \\ v_6^* & v_4^* \end{pmatrix} = |g|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

Now, it is followed from (6)-(9), that

$$(z_1^*, z_3^*)\begin{pmatrix} z_1 \\ z_3 \end{pmatrix} = 4|f|^2(v_4^*, v_6^*)\begin{pmatrix} v_4 \\ v_6 \end{pmatrix}$$

$$(z_1^*, z_3)\begin{pmatrix} z_1 \\ z_3^* \end{pmatrix} = 4|g|^2(v_0^*, v_2)\begin{pmatrix} v_0 \\ v_2^* \end{pmatrix}$$

Combining these expressions with the formula for x_0, we have

$$\begin{aligned} x_0 &= |f|^2 - |g|^2 \\ |\mu| &= 2|f||g| \end{aligned} \tag{10}$$

Comparing (10) with (9) we conclude that pairs $(x_0, |\mu|)$ and $(|f|, |g|)$ are connected with each other by means of Levi-Civita transformation which realize the conformal mapping of a complex quarter plane $|f| + i|g|$ on the complex half plane $x_0 + i|\mu|$.

$$x_0 + i|\mu| = (|f| + i|g|)^2$$

3 The SU(2) Structure of the H

To separate the second structural element of the H it is necessary to turn to the new coordinates. Let us introduce the coordinates $(x_0, |\mu|, a_1, a_3)$, instead of (x_0, z_1, z_3), by the following way:

$$\begin{pmatrix} z_1 \\ z_3 \end{pmatrix} = |\mu|\begin{pmatrix} a_1 \\ a_3 \end{pmatrix}, \tag{11}$$

As it is followed from (5) and (11), the complex numbers a_1 and a_3 obey to the condition

$$|a_1|^2 + |a_3|^2 = 1$$

Thus, we have only three independent real coordinates. In the same way (v_0, v_2) and (v_4, v_6) can be changed by $(|f|, a_0, a_2)$ and $(|g|, a_4, a_6)$:

$$\begin{pmatrix} v_0 \\ v_2 \end{pmatrix} = |f|\begin{pmatrix} a_0 \\ a_2 \end{pmatrix}, |a_0|^2 + |a_2|^2 = 1 \tag{12}$$

$$\begin{pmatrix} v_4 \\ v_6 \end{pmatrix} = |g|\begin{pmatrix} a_4 \\ a_6 \end{pmatrix}, |a_4|^2 + |a_6|^2 = 1 \tag{13}$$

The complex numbers a_j have the meaning of the Ceyley-Klein parameters in the 4-dimensional spaces (z_1, z_2), (v_0, v_2) and (v_4, v_6). Now, let us substitute (10)-(13) to transformations (6) and (7) and use the second relation (10). We arrive to the following transformations:

$$\begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 & -a_2^* \\ a_2 & a_0^* \end{pmatrix}\begin{pmatrix} a_4 \\ a_6 \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} a_1 \\ a_3^* \end{pmatrix} = \begin{pmatrix} a_4 & -a_6 \\ a_6^* & a_4^* \end{pmatrix} \begin{pmatrix} a_0 \\ a_2^* \end{pmatrix} \tag{15}$$

As we can see from (11) – (13), the matrices in (14) and (15) are unitary and unimodular, so that the second structural element of the H is the group of $SU(2)$ transformation.

4 The Euler's Coordinates

The change of the Cartesian coordinates by the Ceyley-Klein parameters allows us to find the $SU(2)$ structure of the H-transformation. However, it is often convenient to deal with independent variables rather than with parameters a_j.

The wide arbitrariness are in the choice of these variables. Below we use the representation:

$$a_1 = \cos\frac{\beta}{2} e^{-i\frac{\alpha+\gamma}{2}}\ , \quad a_3 = \sin\frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}}\ ,$$

$$a_0 = \cos\frac{\beta_1}{2} e^{-i\frac{\alpha_1+\gamma_1}{2}}\ , \quad a_2 = \sin\frac{\beta_1}{2} e^{i\frac{\alpha_1-\gamma_1}{2}}\ ,$$

$$a_4 = \cos\frac{\beta_2}{2} e^{-i\frac{\alpha_2+\gamma_2}{2}}\ , \quad a_6 = \sin\frac{\beta_2}{2} e^{i\frac{\alpha_2-\gamma_2}{2}}\ ,$$

which demonstrated the spinor character of the H-transformation. Here α, β, γ and $\alpha_j, \beta_j \gamma_j$ denote the Euler's angles in x- and u-spaces. These angles vary in the following ranges:

$$0 \le \alpha < 2\pi\ , 0 \le \beta \le \pi\ , -2\pi \le \gamma < 2\pi$$

$$0 \le \alpha_j < 2\pi\ , 0 \le \beta_j \le \pi\ , -2\pi \le \gamma_j < 2\pi$$

Substituting these formula into transformations (14) and (15) we arrive at the spinor representation

$$\begin{pmatrix} \cos\frac{\beta}{2} e^{-i\frac{\alpha+\gamma}{2}} \\ \sin\frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}} \end{pmatrix} = \mathcal{R}(\alpha_1, \beta_1, \gamma_1) \begin{pmatrix} \cos\frac{\beta_2}{2} e^{-i\frac{\alpha_2+\gamma_2}{2}} \\ \sin\frac{\beta_2}{2} e^{i\frac{\alpha_2-\gamma_2}{2}} \end{pmatrix} \tag{16}$$

$$\begin{pmatrix} \cos\frac{\beta}{2} e^{-i\frac{\alpha+\gamma}{2}} \\ \sin\frac{\beta}{2} e^{i\frac{\gamma-\alpha}{2}} \end{pmatrix} = \mathcal{R}(\gamma_2, \beta_2, \alpha_2) \begin{pmatrix} \cos\frac{\beta_1}{2} e^{-i\frac{\alpha_1+\gamma_1}{2}} \\ \sin\frac{\beta_1}{2} e^{i\frac{\gamma_1-\alpha_1}{2}} \end{pmatrix} \tag{17}$$

Here $\mathcal{R}$ is the matrix of finite rotations given by

$$\mathcal{R}(a, b, c) = \begin{pmatrix} \cos\frac{b}{2} e^{-i\frac{a+c}{2}} & -\sin\frac{b}{2} e^{-i\frac{a-c}{2}} \\ \sin\frac{b}{2} e^{i\frac{a-c}{2}} & \cos\frac{b}{2} e^{i\frac{a+c}{2}} \end{pmatrix}$$

We see that, in the space of angles $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ the H transformation acts according to the scheme [15]:

$$S^3(\alpha,\beta,\gamma) = \mathcal{R}(\alpha_1,\beta_1,\gamma_1)\mathcal{R}(\alpha_2,\beta_2,\gamma_2)S^3(0,0,0)\,,$$

$$S^3(\gamma,\beta,\alpha) = \mathcal{R}(\gamma_2,\beta_2,\alpha_2)\mathcal{R}(\gamma_1,\beta_1,\alpha_1)S^3(0,0,0)\,,$$

Thus, from the given triplets $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$, the H constructs the matrices of the finite rotations R_1 and R_2 which transform the northern pole of the three-dimensional unit sphere into a point of the same sphere belonging to the x-space.

5 The General Structure of the H

Let us turn to the most general formulation of the H-transformation [3,4]:

$$\begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} u_0 & u_1 & u_2 & u_3 & -u_4 & -u_5 & -u_6 & -u_7 \\ u_1 & -u_0 & u_3 & -u_2 & -u_5 & u_4 & u_7 & -u_6 \\ u_2 & -u_3 & -u_0 & u_1 & -u_6 & -u_7 & u_4 & u_5 \\ u_3 & u_2 & -u_1 & -u_0 & -u_7 & u_6 & -u_5 & u_4 \\ u_4 & -u_5 & -u_6 & -u_7 & u_0 & -u_1 & -u_2 & -u_3 \\ u_5 & u_4 & -u_7 & u_6 & u_1 & u_0 & u_3 & -u_2 \\ u_6 & u_7 & u_4 & -u_5 & u_2 & -u_3 & u_0 & u_1 \\ u_7 & -u_6 & u_5 & u_4 & u_3 & u_2 & -u_1 & u_0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} \qquad (18)$$

The discovered above structural elements are hided in formula (18). However, it is possible pass to the representation where (10), (14) and (15) act as the j o i n t transformation, mapping u-space on x-space.

For this purpose it is necessary to introduce instead of Cartesian coordinates the complex coordinates $(x_0 + i|\mu|, a_1, , a_3)$ and $(|f| + i|g|, a_0, a_2, a_4, a_6)$.

In result the H-transformation takes the form:

$$\begin{pmatrix} x_0 + i|\mu| \\ a_1 \\ a_3 \end{pmatrix} = \begin{pmatrix} |f| + i|g| & 0 & 0 \\ 0 & a_0 & -a_2^* \\ 0 & a_2 & -a_0^* \end{pmatrix} \begin{pmatrix} |f| + i|g| \\ a_4 \\ a_6 \end{pmatrix}, \qquad (19)$$

$$\begin{pmatrix} x_0 + i|\mu| \\ a_1 \\ a_3^* \end{pmatrix} = \begin{pmatrix} |f| + i|g| & 0 & 0 \\ 0 & a_4 & -a_6 \\ 0 & a_6^* & a_4^* \end{pmatrix} \begin{pmatrix} |f| + i|g| \\ a_0 \\ a_2^* \end{pmatrix}, \qquad (20)$$

Starting from (19) and (20) we can, without any difficulty, return to the formula (1). In this sense, our result is equivalent to (18), i.e. to the general form of the H-transformation. Moreover, the expressions (19) and (20) include, in contrast to (18), the information about conformal and $SU(2)$ structure of the H explicitly.

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