JOINT INSTITUTE FOR NUCLEAR RESEARCH

International Workshop

FINITE DIMENSIONAL INTEGRABLE SYSTEMS

Dubna, Russia July 18-21 1994

Edited by

A.N.Sissakian

G.S.Pogosyan

Dubna 1995

THE EULERIAN PARAMETERIZATION OF THE HURWITZ TRANSFORMATION

L.G. Mardoyan,[1] A.N. Sissakian,[2] V.M. Ter-Antonyan[2]

[1] *Yerevan State University, 375049 Yerevan, Armenia.*
[2] *JINR Dubna, 141980 Russia.*

Abstract

A parameterization in which the Hurwitz transformation is equivalent to the Eulerian addition of the angles is suggested. It is shown, that in this parameterization, the reduction of the state vectors is given by the addition theorem for the Wigner's D-functions.

1 Introduction

The wonderful theorem in algebra is by Hurwitz [1]: the equation

$$x_0^2 + x_1^2 + \cdots + x_{n-1}^2 = (u_0^2 + u_1^2 + \cdots + u_{N-1}^2)^2$$

has a solution bilinear in u_j only for the following pairs of numbers $(N,n) = (2,2), (4,3), (8,5)$. We write this solution as

$$x_j = a_{jlm} u_l u_m \tag{1}$$

The expression (1) may be interpreted as a bilinear transformation that maps one Euclidean space into another. Three remarkble circumstances are connected with the transformation (1):

a. For $(N,n) = (2,2)$, $(N,n) = (4,3)$, $(N,n) = (8,5)$ the transformation (1) coincides with the Levi-Civita [2], Kustaanheimo-Stiefel [3] and Hurwitz transformations [4], respectively.

b. For $N = 2, 4, 8$ this transformation may be associated with the algebras of complex numbers, quaternions and octanions [5].

c. The transformation (1) establish the connection between two fundamental problems of mechanics, the oscillator and Kepler's problems.

The latter circumstance is especially interesting for the applications. It has played a prominent role in celestial mechanics [6], has been used in quantum mechanics [7],quantum field theory [8,9,10] and in quantum chemistry [11].

In this paper, we will consider the case $(N,n) = (8,5)$, i.e. the Hurwitz transformation [12]. The explicit form of (1) in this case is

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2 \\
x_1 &= 2(u_0u_4 - u_1u_5 - u_2u_6 - u_3u_7) \\
x_2 &= 2(u_0u_5 + u_1u_4 - u_2u_7 + u_3u_6) \\
x_3 &= 2(u_0u_6 + u_1u_7 + u_2u_4 - u_3u_5) \\
x_4 &= 2(u_0u_7 - u_1u_6 + u_2u_5 + u_3u_4)
\end{aligned} \tag{2}$$

The transformation (2) expresses the Cartesian coordinates through the Cartesian ones. This representation is convenient for the work with the Laplacian since the latter is of the simplest form in Cartesian coordinates. However, the symmetry of the problems can require other coordinates. Does the parameterization (in u - space) exist different from the Cartesian one which is transformed by (2) into the parameterization convenient for solving the reduced (x - space) problem?

In this paper, the parameterization that bring us out of the Cartesian representation of the Hurwitz transformation and satisfies the above condition is proposed.

2 Parameterization

Let us introduce the coordinates $\omega, \lambda, \alpha_f, \beta_f, \gamma_f, \alpha_g, \beta_g, \gamma_g$ instead of the Cartesian coordinates u_j as

$$\begin{aligned} u_0 + iu_1 &= f(\omega, \lambda) \cos\frac{\beta_f}{2} e^{i\frac{\alpha_f+\gamma_f}{2}} \\ u_2 + iu_3 &= f(\omega, \lambda) \sin\frac{\beta_f}{2} e^{i\frac{\alpha_f-\gamma_f}{2}} \\ u_4 + iu_5 &= g(\omega, \lambda) \cos\frac{\beta_g}{2} e^{i\frac{\alpha_g+\gamma_g}{2}} \\ u_6 + iu_7 &= g(\omega, \lambda) \sin\frac{\beta_g}{2} e^{i\frac{\alpha_g-\gamma_g}{2}} \end{aligned} \tag{3}$$

The new coordinates may vary in the following ranges

$$0 \le \beta_f, \beta_g \le \pi,\ 0 \le \alpha_f, \alpha_g < 4\pi; 0 \le \gamma_f, \gamma_g < 2\pi$$

For a definite type of the coordinates (polar, parabolic, elliptic, etc.), ω and λ can be determined completely. For the consideration below, the functions $f(\omega, \lambda)$ and $g(\omega, \lambda)$ will remain rather arbitrary: only the connection with the coordinates u_j is fixed

$$\begin{aligned} f(\omega, \lambda) &= (u_0^2 + u_1^2 + u_2^2 + u_3^2)^{1/2} \\ g(\omega, \lambda) &= (u_4^2 + u_5^2 + u_6^2 + u_7^2)^{1/2} \end{aligned} \tag{4}$$

In the coordinates (3) differential elements of the length, volume and the Laplace operator have the following forms

$$\begin{aligned} dl_8^2 &= df^2 + dg^2 + \frac{f^2}{4} dl_f^2 + \frac{g^2}{4} dl_g^2 \\ dV_8 &= f^3 g^3 df\, dg\, d\Omega_f\, d\Omega_g \\ \Delta_8 &= \frac{1}{f^3}\frac{\partial}{\partial f}\left(f^3 \frac{\partial}{\partial f}\right) + \frac{1}{g^3}\frac{\partial}{\partial g}\left(g^3 \frac{\partial}{\partial g}\right) - \frac{4}{f^2}\hat{J}_f^2 - \frac{4}{g^2}\hat{J}_g^2 \end{aligned} \tag{5}$$

where

$$dl_a^2 = d\alpha_a{}^2 + d\beta_a{}^2 + d\gamma_a{}^2 + 2\cos\beta_a d\alpha_a d\gamma_a$$

$$d\Omega_a = \frac{1}{8}\sin\beta_a d\beta_a d\alpha_a d\gamma_a$$

$$\hat{j}_a^2 = -\left[\frac{\partial^2}{\partial\beta_a^2} + \cot\beta_a\frac{\partial}{\partial\beta_a} + \frac{1}{\sin^2\beta_a}\left(\frac{\partial^2}{\partial{\alpha_a}^2} - 2\cos\beta_a\frac{\partial^2}{\partial\alpha_a\partial\gamma_a} + \frac{\partial^2}{\partial{\gamma_a}^2}\right)\right]$$

and $a = f, g$.

3 The eulerian rotations

Let us examine to which coordinates the Hurwitz transformation transform the coordinates (3).

After substituting (3) into (2), we have

$$\begin{aligned} x_0 &= f^2(\omega,\lambda) - g^2(\omega,\lambda) \\ x_j &= 2f(\omega,\lambda)g(\omega,\lambda)\hat{x}_j \end{aligned} \tag{6}$$

Here $j = 1,2,3,4$, and $\hat{x}_j$ are given by the expressions

$$\hat{x}_1 = \cos\frac{\beta_f}{2}\cos\frac{\beta_g}{2}\cos\frac{\alpha_f+\alpha_g+\gamma_f+\gamma_g}{2} - \sin\frac{\beta_f}{2}\sin\frac{\beta_g}{2}\cos\frac{\alpha_f-\alpha_g-\gamma_f+\gamma_g}{2}$$

$$\hat{x}_2 = \cos\frac{\beta_f}{2}\cos\frac{\beta_g}{2}\sin\frac{\alpha_f+\alpha_g+\gamma_f+\gamma_g}{2} + \sin\frac{\beta_f}{2}\sin\frac{\beta_g}{2}\sin\frac{\alpha_f-\alpha_g-\gamma_f+\gamma_g}{2}$$

$$\hat{x}_3 = \cos\frac{\beta_f}{2}\sin\frac{\beta_g}{2}\cos\frac{\alpha_f-\alpha_g+\gamma_f+\gamma_g}{2} + \sin\frac{\beta_f}{2}\cos\frac{\beta_g}{2}\cos\frac{\alpha_f+\alpha_g-\gamma_f+\gamma_g}{2}$$

$$\hat{x}_4 = -\cos\frac{\beta_f}{2}\sin\frac{\beta_g}{2}\sin\frac{\alpha_f-\alpha_g+\gamma_f+\gamma_g}{2} + \sin\frac{\beta_f}{2}\cos\frac{\beta_g}{2}\sin\frac{\alpha_f+\alpha_g-\gamma_f+\gamma_g}{2}$$

Let us introduce the following coordinates

$$\begin{aligned} x_0 &= f^2(\omega,\lambda) - g^2(\omega,\lambda) \\ x_1 + ix_2 &= 2f(\omega,\lambda)g(\omega,\lambda)\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}} \\ x_3 + ix_4 &= 2f(\omega,\lambda)g(\omega,\lambda)\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}} \end{aligned} \tag{7}$$

In these coordinates we have

$$\begin{aligned} dl_5^2 &= \frac{\xi+\eta}{4\xi}d\xi^2 + \frac{\xi+\eta}{4\eta}d\eta^2 + \frac{\xi\eta}{4}dl^2 \\ dV_5 &= \frac{\xi\eta}{4}(\xi+\eta)d\xi\eta d\Omega \\ \Delta_5 &= \frac{4}{\xi+\eta}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial}{\partial\xi}\right) + \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial}{\partial\eta}\right)\right] - \frac{4}{\xi\eta}\hat{j}^2 \end{aligned} \tag{8}$$

where $\xi = 2f^2(\omega,\lambda), \eta = 2g^2(\omega,\lambda)$ and $dl^2, d\Omega$ and J^2 may be obtained from the expressions (5) through the substitution $(\alpha_a, \beta_a, \gamma_a) \to (\alpha, \beta, \gamma)$.
We require the identity of the coordinates (7) with (6). As a result, we arrive at the system of trigonometric equations:

$$\begin{aligned}
\cos\frac{\beta}{2}e^{\frac{i}{2}(\alpha+\gamma)} &= \cos\frac{\beta_f}{2}\cos\frac{\beta_g}{2}e^{\frac{i}{2}(\alpha_f+\alpha_g+\gamma_f+\gamma_g)} - \sin\frac{\beta_f}{2}\sin\frac{\beta_g}{2}e^{\frac{i}{2}(\alpha_f-\alpha_g-\gamma_f+\gamma_g)}\\
\sin\frac{\beta}{2}e^{\frac{i}{2}(\alpha-\gamma)} &= \cos\frac{\beta_f}{2}\sin\frac{\beta_g}{2}e^{-\frac{i}{2}(\alpha_f-\alpha_g+\gamma_f+\gamma_g)} + \sin\frac{\beta_f}{2}\cos\frac{\beta_g}{2}e^{\frac{i}{2}(\alpha_f+\alpha_g-\gamma_f+\gamma_g)}
\end{aligned}$$

After solution we obtain

$$\begin{aligned}
\cot(\alpha-\alpha_g) &= \cos\beta_g\cot(\alpha_f+\gamma_g) + \cot\beta_f\frac{\sin\beta_g}{\sin(\alpha_f+\gamma_g)}\\
\cos\beta &= \cos\beta_f\cos\beta_g - \sin\beta_f\sin\beta_g\cos(\alpha_f+\gamma_g) \qquad (9)\\
\cot(\gamma-\gamma_f) &= \cos\beta_f\cot(\alpha_f+\gamma_g) + \cot\beta_g\frac{\sin\beta_f}{\sin(\alpha_f+\gamma_g)}
\end{aligned}$$

The relations (9) represent the basic result of this paper: the Hurwitz transformation includes the transformation $(f+ig) \to (f+ig)^2$ and the Euler's addition of the triplets of angles $(\alpha_f, \beta_f, \gamma_f)$ and $(\alpha_g, \beta_g, \gamma_g)$ into the triplet of angles (α, β, γ). In this sense, we call the parameterization (3) E u l e r i a n.

4 Correspondence of the solutions

We see from (5) and (8) that the Δ_8 and Δ_5 operators contain the squares of the momentum operators. If in the Schrödinger equation the form of the potential allows us to separate the variables connected with the operators J_f^2, J_g^2 and J^2, then the solutions will possess the universal dependence on the angles entering into these operators.

The coordinates ω and λ may be fixed, i.e. we can consider hyperspheres S^6 and S^3 in u- and x- spaces respectively. As a result, the solutions

$$\begin{aligned}
\psi_6 &= \left(\frac{2j_f+1}{2\pi^2}\right)^{1/2}\left(\frac{2j_g+1}{2\pi^2}\right)^{1/2} D^{j_f}_{m_f,m'_f}(\alpha_f,\beta_f,\gamma_f) D^{j_g}_{m_g,m'_g}(\alpha_g,\beta_g,\gamma_g)\\
\psi_3 &= \left(\frac{2j+1}{2\pi^2}\right)^{1/2} D^j_{m,m'}(\alpha,\beta,\gamma) \qquad (10)
\end{aligned}$$

are obtained. The symbol D in right sides means the Wigner's D-function [11].

For the angles entering into (9) the addition theorem [13]

$$D^j_{m,m'}(\alpha,\beta,\gamma) = \sum_{k=-j}^{j} D^j_{m,k}(\alpha_f,\beta_f,\gamma_f) D^j_{k,m'}(\alpha_g,\beta_g,\gamma_g) \qquad (11)$$

is valid.

As a consequence, o n l y a subset ($j_f = j_g = j$) of ψ_6 functions participates in the "construction" of the ψ_3 function. We see that for the spheres $S^6 = S^3_f \otimes S^3_g$ and S^3 which are connected through the Hurwitz transformation, the correspondence between the solutions of the Schrödinger equation on these spheres is given by the addition theorem (11).

5 Conclusion

In this paper we introduce the Eulerian parameterization for the Hurwitz transformation and derive the following results:

a. The Hurwitz transformation in the Eulerian parameterization contains, as a "key element" the Eulerian addition of the angles $(\alpha_f, \beta_f, \gamma_f) \oplus (\alpha_g, \beta_g, \gamma) = (\alpha, \beta, \gamma)$.

b. The Eulerian parameterization generates the universal dependence of the solutions of the Schrödinger equation on the angles $(\alpha_f, \beta_f, \gamma_f)$, $(\alpha_g, \beta_g, \gamma_g)$ and (α, β, γ).

c. In the Eulerian parameterization, correspondence between the solutions in x- and u - spaces is achieved due to the addition theorem for the Wigner's D-functions.

An extension of the Hurwitz transformation beyound the scope of the Cartesian coordinates was considered in [14] from entirely different positions. The correspondence between the hyperspherical coordinates resulted from the refusal from the bilinearity of the transformation.

One of us (L.G.Mardoyan) expresses his gratitude to the Organizing Committee and Dr. G.S.Pogosyan for the support which made possible his participation in this workshop.

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