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QUANTUM MOTION ON THE THREE DIMENSIONAL SPHERE. ELLIPSOIDAL BASES.

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1 Introduction

As is known [1, 2], in the three-dimensional space of constant positive curvature there exist 6 orthogonal systems of coordinates admitting a complete separation of variables in the Hamilton–Jacobi equation or in the Helmholtz equation. These are hyperspherical, cylindrical, sphero-conical, two elliptic cylindrical and the ellipsoidal systems of coordinates. The most complex of these systems of coordinates is the ellipsoidal one, which contains all the rest five in the limiting case [3]. The present paper is devoted to the solution of the Helmholtz equation on the three-dimensional sphere in the ellipsoidal system of coordinates.

2 The ellipsoidal coordinates

The algebraic form of the ellipsoidal system of coordinates is [1]

$$\begin{aligned} x_1^2 &= \frac{(\rho_1-a_1)(\rho_2-a_1)(\rho_3-a_1)}{(a_4-a_1)(a_3-a_1)(a_2-a_1)} \\ x_2^2 &= \frac{(\rho_1-a_2)(\rho_2-a_2)(\rho_3-a_2)}{(a_4-a_2)(a_3-a_2)(a_1-a_2)} \\ x_3^2 &= \frac{(\rho_1-a_3)(\rho_2-a_3)(\rho_3-a_3)}{(a_4-a_3)(a_2-a_3)(a_1-a_3)} \\ x_4^2 &= \frac{(\rho_1-a_4)(\rho_2-a_4)(\rho_3-a_4)}{(a_1-a_4)(a_2-a_4)(a_3-a_4)} \end{aligned} \qquad (1)$$

where $0 \le a_1 \le \rho_1 \le a_2 \le \rho_2 \le a_3 \le \rho_3 \le a_4$. The coordinate surfaces on which $\rho_i = const.$ are obtained as a result of intersection of the three-dimensional unit sphere $x_1^2+x_2^2+x_3^2+x_4^2=1$ with three families of conic surfaces

$$\frac{x_1^2}{\rho_i-a_1}+\frac{x_2^2}{\rho_i-a_2}+\frac{x_3^2}{\rho_i-a_3}+\frac{x_4^2}{\rho_i-a_4}=0 \quad (i=1,2,3) \qquad (2)$$

and represent complete families of confocal nonruled, ruled and nonruled ellipsoids [1].

Relation (1) connecting the Cartesian and ellipsoidal coordinates are not in the one-to-one correspondence as ρ_i depend only on $(x_1^2, x_2^2, x_3^2, x_4^2)$ and, consequently, take the same values at 16 points $(\pm x_1, \pm x_2, \pm x_3, \pm x_4)$. To obtain a one-to-one correspondence between the Cartesian and ellipsoidal coordinates, as in the case of elliptic system of coordinates on the two-dimensional sphere [4], one can introduce uniformised variables γ, μ, ν determin the position of the point on the three-dimensional sphere by the following relations:

$$\rho_1 = a_1 + (a_2 - a_1)\cos^2\mu, \quad \rho_2 = a_2 + (a_3 - a_2)\sin^2\nu, \quad \rho_3 = a_3 + (a_4 - a_3)\sin^2\gamma,$$

As a result, the ellipsoidal system of coordinates can be written down in the trigonome form as

$$\begin{aligned}
x_1 &= \frac{1}{(k_1^2+k_2^2)^{1/2}}\sqrt{(k_1^2 + k_2^2\sin^2\nu)(1 - k_3^2\cos^2\gamma)}\cos\mu \\
x_2 &= \frac{1}{(k_2^2+k_3^2)^{1/2}}\sqrt{k_2^2 + k_3^2\sin^2\gamma)}\sin\mu\sin\nu \\
x_3 &= \frac{1}{(k_1^2+k_2^2)^{1/2}}\sqrt{(k_2^2 + k_1^2\sin\mu^2)}\cos\nu\sin\gamma \\
x_4 &= \frac{1}{(k_2^2+k_3^2)^{1/2}}\sqrt{(k_3^2 + k_2^2\cos\nu^2)(1 - k_1^2\cos\mu^2)}\cos\gamma
\end{aligned}$$

where $0 \le \nu < 2\pi$, $0 \le \mu \le \pi$, $0 \le \gamma < \pi$ and

$$k_1^2 = \frac{a_2 - a_1}{a_4 - a_1}, \quad k_2^2 = \frac{a_3 - a_2}{a_4 - a_1}, \quad k_3^2 = \frac{a_4 - a_3}{a_4 - a_1}, \quad k_1^2 + k_2^2 + k_3^2 = 1.$$

As is seen from the definition (4), the ellipsoidal system of coordinates is determi by three parameters and the binding condition (5). It is the most general system coordinates which turns into simpler coordinates at particular values of the parameter [3]. In particular cases $k_1^2 = 0$ and $k_2^2 = 0$ the ellipsoidal system of coordinates turns i ellipso-cylindrical systems of coordinates of type I and II, respectively. Further vanish of the parameter k_2^2 or k_3^2 may result, respectively, in the spherical or cylindrical syst of coordinates. Then, if we let k_1^2 and k_2^2 tend to zero simultaneously and the ra $k_1^2/(k_1^2 + k_2^2)$ is put finite equal to k^2, one can easily see that the ellipsoidal system coordinates degenerates into the sphero-conic one and upon substitution $k^2 = 0$ or k^2 = turns into the spherical system of coordinates. In more detail these transitions are gi in the table.

3 Separation of variables and integrals of motion

The Helmholtz or Schrödinger equation for a particle motion on the three-dimensio sphere of the unit radius can be written down as

$$\Delta\Psi + J(J+2)\Psi = 0, \quad J = 0, 1, 2, \cdots$$

where Δ is the Laplace operator determined as follows:

$$\Delta = -(L^2 + N^2)$$

and L_i and N_i are six generators of the group $O(4)$

$$L_i = -i\varepsilon_{ikl}x_l\frac{\partial}{\partial x_k}, \quad N_i = -i(x_i\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_i}), \quad i = 1, 2, 3,$$

Table: The degenerations of the ellipsoidal coordinate system

ellipso-cylindric system type I	$x_1 = \sqrt{1-k_3^2\cos^2\xi}\sin\theta\cos\varphi$ $x_2 = \sqrt{1-k_3^2\cos^2\xi}\sin\theta\sin\varphi$ $x_3 = \cos\theta\sin\xi$ $x_4 = \sqrt{1-k_2^2\sin\theta}\cos\xi$	$k_1^2 \to 0$ $k_2^2 + k_3^2 = 1$	$\mu \to \varphi$ $\nu \to \theta$ $\gamma \to \xi$
ellipso-cylindric system type II	$x_1 = \sqrt{1-k_3^2\cos^2\xi}\cos\varphi$ $x_2 = \sin\xi\sin\varphi\sin\theta$ $x_3 = \sin\xi\sin\varphi\cos\theta$ $x_4 = \cos\xi\sqrt{1-k_1^2\cos^2\varphi}$	$k_2^2 \to 0$ $k_1^2 + k_3^2 = 1$	$\mu \to \varphi$ $\nu \to \theta$ $\gamma \to \xi$
Spherical system	$x_1 = \sin\chi\sin\theta\cos\varphi$ $x_2 = \sin\chi\sin\theta\sin\varphi$ $x_3 = \sin\chi\cos\theta$ $x_4 = \cos\chi$	$k_1^2 \to 0$ $k_2^2 \to 0$ $k_1^2/(k_1^2+k_2^2) = 0$	$\mu \to \varphi$ $\nu \to \theta$ $\gamma \to \chi$
sphero - conical system	$x_1 = \sin\chi\sqrt{1-k'^2\cos^2\theta}\cos\varphi$ $x_2 = \sin\chi\sin\theta\sin\varphi$ $x_3 = \sin\chi\cos\theta\sqrt{1-k^2\cos\varphi}$ $x_4 = \cos\chi$	$k_1^2 \to 0$ $k_2^2 \to 0$ $k_1^2/(k_1^2+k_2^2) = k^2$ $k_2 + k'^2 = 1$	$\mu \to \varphi$ $\nu \to \theta$ $\gamma \to \chi$
Cylindric system	$x_1 = \sin\alpha\cos\varphi_1$ $x_2 = \sin\alpha\sin\varphi_1$ $x_3 = \cos\alpha\sin\varphi_2$ $x_4 = \cos\alpha\cos\varphi_2$	$k_1^2 \to 0$ $k_4^2 \to 0$	$\mu \to \varphi_1$ $\nu \to \alpha$ $\gamma \to \varphi_2$

which obey the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_j, \quad [N_i, N_j] = i\epsilon_{ijk}L_j, \quad [L_i, N_j] = i\epsilon_{ijk}N_j, \tag{9}$$

If in the Helmholtz equation (7) one passes to the ellipsoidal system of coordinates, after the substitution $\psi(\rho_1, \rho_2, \rho_3) = \psi_1(\rho_1)\psi_2(\rho_2)\psi_3(\rho_3)$ and introduction of ellipsoidal separation constants λ_1, λ_2 one arrives at three identical differential equations

$$4\sqrt{P(\rho_i)}\frac{d}{d\rho_i}\sqrt{P(\rho_i)}\frac{d\psi_i}{d\rho_i} + \left\{J(J+2)\rho_i^2 - \lambda_1\rho_i - \lambda_2\right\}\psi_i = 0, \quad i = 1,2,3 \tag{10}$$

where $P(\rho) = (\rho-a_1)(\rho-a_2)(\rho-a_3)(\rho-a_4)$. Equation (10), derived by separating variables in the ellipsoidal system of coordinates, is the generalized Lame' equation and falls into a class of equations of the Fuchsian type with five singularities [5] $\{a_1, a_2, a_3, a_4, \infty\}$; moreover, (a_1, a_2, a_3, a_4) are elementary singularities with indices (0,1/2) and a point at infinity is regular.

Each of the separated equations (10) contains besides hypermoment J also two constants λ_1 and λ_2 depending in the general case on four dimensional parameters a_1, a_2, a_3, a_4 or k_1, k_2, k_3 determining singularities of the given equation. Therefore, unlike the standard one-dimensional spectral problem, the main difficulty consists in calculating simultaneously (or quantizing) the energy spectrum of both the ellipsoidal separation constants.

Let us explicitly write down the operators (ellipsoidal integrals of motion) Λ_1 and Λ_2 whose eigenvalues are the ellipsoidal separation constants λ_1 and λ_2. Eliminating the hypermomemt J from the system of equations (10), we derive for Λ_1 and Λ_2 as functions of the parameters $a = (a_1, a_2, a_3, a_4)$, the following expressions in the ellipsoidal variables ρ_i:

$$\begin{aligned}\Lambda_1(a) = & - \frac{4(\rho_3+\rho_2)\sqrt{P(\rho_1)}}{(\rho_3-\rho_1)(\rho_2-\rho_1)}\frac{\partial}{\partial\rho_1}\sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1} - \frac{4(\rho_3+\rho_1)\sqrt{P(\rho_2)}}{(\rho_3-\rho_2)(\rho_1-\rho_2)}\frac{\partial}{\partial\rho_2}\sqrt{P(\rho_2)}\frac{\partial}{\partial\rho_2} \\ & - \frac{4(\rho_2+\rho_1)\sqrt{P(\rho_3)}}{(\rho_2-\rho_3)(\rho_1-\rho_3)}\frac{\partial}{\partial\rho_3}\sqrt{P(\rho_3)}\frac{\partial}{\partial\rho_3} \qquad (11)\end{aligned}$$

$$\begin{aligned}\Lambda_2(a) = & \frac{4\rho_3\rho_2\sqrt{P(\rho_1)}}{(\rho_3-\rho_1)(\rho_2-\rho_1)}\frac{\partial}{\partial\rho_1}\sqrt{P(\rho_1)}\frac{\partial}{\partial\rho_1} + \frac{4\rho_3\rho_1\sqrt{P(\rho_2)}}{(\rho_3-\rho_2)(\rho_1-\rho_2)}\frac{\partial}{\partial\rho_2}\sqrt{P(\rho_2)}\frac{\partial}{\partial\rho_2} \\ & + \frac{4\rho_2\rho_1\sqrt{P(\rho_3)}}{(\rho_2-\rho_3)(\rho_1-\rho_3)}\frac{\partial}{\partial\rho_3}\sqrt{P(\rho_3)}\frac{\partial}{\partial\rho_3} \qquad (12)\end{aligned}$$

Passing from the variables ρ_i to the Cartesian ones, we arrive at the following expression for the ellipsoidal integrals of motion

$$\begin{aligned}\Lambda_1(a) &= (a_1+a_4)L_1^2 + (a_2+a_4)L_2^2 + (a_3+a_4)L_3^2 \\ &+ (a_2+a_3)N_1^2 + (a_1+a_3)N_2^2 + (a_1+a_2)N_3^2 \qquad (13)\end{aligned}$$

$$\Lambda_2(a) = -a_1a_4L_1^2 - a_2a_4L_2^2 - a_3a_4L_3^2 - a_2a_3N_1^2 - a_1a_3N_2^2 - a_1a_2N_3^2 \qquad (14)$$

Instead of the system of operators (10) and (14) it is more convenient to use new operators $\hat{\lambda}$ and $\hat{\mu}$, that depend on three parameters k_1^2, k_2^2, k_3^2, (only two of them being independent, according to (5)) and are connected with the old Λ_1 and Λ_2 according to

$$\hat{\lambda} = (a_4-a_1)^{-1}\{\Lambda_1(a) - 2a_2\Delta\}, \quad \hat{\mu} = (a_4-a_1)^{-2}\{a_2\Lambda_1(a) + \Lambda_2(a) - a_2^2\Delta\}. \qquad (15)$$

Thus, the ellipsoidal basis is the system of three operators $\{\mathcal{L} = -\Delta, \hat{\lambda}, \hat{\mu}\}$ where

$$\begin{aligned}\hat{\lambda}(k_1^2,k_2^2,k_3^2) &= k_3^2L_1^2 + (k_3^2+k_1^2)L_2^2 + L_3^2 + k_1^2N_1^2 - k_2^2N_3^2 + (k_2^2-k_1^2)\mathcal{L} \qquad (16) \\ \hat{\mu}(k_1^2,k_2^2,k_3^2) &= k_1^2(k_2^2+k_3^2)L_1^2 - k_2^2(k_2^2+k_3^2)L_3^2 + k_1^2k_2^2N_2^2\end{aligned}$$

From the system of operators (16) one can easily derive for particular values of the parameters k_1^2, k_2^2 and k_3^2 all possible, or equivalent to them, sets of diagonal operators $\{\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2\}$, corresponding to different bases for free motion on the three-dimensional sphere.

I. The case $k_1^2 \to 0,\ k_2^2 + k_3^2 = 1$. **Ellipso-cylindrical basis I.**

$$\begin{aligned}\mathcal{L}_1 &= \hat{\lambda}(0,k_2^2,k_3^2) = k_3^2L^2 + k_2^2(L_3^2 - N_3^2) + k_2^2\mathcal{L}, \\ \mathcal{L}_2 &= \hat{\mu}(0,k_2^2,k_3^2) = -k_2^2L_3^2 \qquad (17)\end{aligned}$$

II. The case $k_2^2 \to 0,\ k_1^2 + k_3^2 = 1$. **Ellipso-cylindrical basis II.**

$$\begin{aligned}\mathcal{L}_1 &= \hat{\lambda}(k_1^2, 0, k_3^2) = L^2 + k_1^2(N_1^2 - L_1^2) - k_1^2\mathcal{L},\\ \mathcal{L}_2 &= \hat{\mu}(k_1^2, 0, k_3^2) = k_1^2 k_3^2 L_1^2\end{aligned} \tag{18}$$

III. The case $k_1^2 = k_3^2 \to 0,\ k_2^2 = 1$ **Cylindrical basis.**

$$\begin{aligned}\mathcal{L}_1 &= \hat{\lambda}(0, k_2^2, 0) = \mathcal{L} + L_3^2 - N_3^2,\\ \mathcal{L}_2 &= \hat{\mu}(0, k_2^2, 0) = -L_3^2\end{aligned} \tag{19}$$

IY. The case $k_1^2 = k_2^2 \to 0\ ,\ k_3^2 = 1\ \ k_1^2/(k_1^2 + k_2^2) = k^2$. **Sphero-conical basis.**

$$\begin{aligned}\mathcal{L}_1 &= \hat{\lambda}(0, 0, k_3^2) = L^2,\\ \mathcal{L}_2 &= \lim_{\substack{k_1^2 \to 0\\ k_2^2 \to 0}} \frac{\hat{\mu}(k_1^2, k_2^2, k_3^2)}{k_1^2 + k_2^2} = k^2 L_1^2 - k'^2 L_3^2\end{aligned} \tag{20}$$

Y. The case $k_1^2 = k_2^2 \to 0\ ,\ k_3^2 = 1\ \ k_1^2/(k_1^2 + k_2^2) = 0$. **Spherical basis.**

$$\begin{aligned}\mathcal{L}_1 &= \hat{\lambda}(0, 0, k_3^2) = L^2,\\ \mathcal{L}_2 &= \lim_{\substack{k_1^2 \to 0\\ k_2^2 \to 0}} \frac{\hat{\mu}(k_1^2, k_2^2, k_3^2)}{k_1^2 + k_2^2} = -L_3^2\end{aligned} \tag{21}$$

Thus, by means of different limiting conditions of the parameters (k_1^2, k_2^2, k_3^2) we have obtained all five nonequivalent sets of operators corresponding to separation of variables in the Helmholtz equation on the three-dimensional sphere in simpler systems of coordinates.

4 Solution of the ellipsoidal equation

Let us construct solutions of the generalized Lame' equation. Search for the ellipsoidal wave function $\psi(\rho)$ as an expansion in series round one of the singularities a_2

$$\psi(\rho) = (\rho - a_1)^{\frac{\alpha_1}{2}}(\rho - a_2)^{\frac{\alpha_2}{2}}(\rho - a_3)^{\frac{\alpha_3}{2}}(\rho - a_4)^{\frac{\alpha_4}{2}} \sum_{t=0}^{\infty} b_t \left(\frac{\rho - a_2}{a_4 - a_1}\right)^t \tag{22}$$

where ${\alpha_i}^2 = \alpha_i$. Substituting (22) into the generalized Lame' equation (22) we derive three-term recurrence relations for the expansion coefficients b_t

$$\beta_t b_{t+1} + \{\mu - \gamma_t\} b_t + \{\lambda - \delta_t\} b_{t-1} + \omega_t b_{t-2} = 0 \tag{23}$$

where

$$\lambda = (a_4 - a_1)^{-1}[2a_2 J(J+2) - \lambda_1], \quad \mu = (a_4 - a_1)^{-2}[{a_2}^2 J(J+2) - a_2\lambda_1 - \lambda_2]$$

$$\begin{aligned}\gamma_t &= [4t^2 + (4t+1)(\alpha_2 + \alpha_4) + 2\alpha_2\alpha_4]k_1^2 k_2^2 + [4t^2 + (4t+1)(\alpha_2 + \alpha_3)\\ &+ 2\alpha_2\alpha_3]k_1^2(k_2^2 + k_3^2) - [4t^2 + (4t+1)(\alpha_2 + \alpha_1) + 2\alpha_2\alpha_1]k_2^2(k_2^2 + k_3^2)\end{aligned}$$

$$\begin{aligned}\delta_t = & - 2[\alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 + (t+1)(\alpha_2+\alpha_3+\alpha_4) + (t+1)(t-1)]k_1^2 \\ & + 2[\alpha_2\alpha_1 + \alpha_2\alpha_4 + \alpha_1\alpha_4 + (t+1)(\alpha_2+\alpha_1+\alpha_4) + (t+1)(t-1)]k_2^2 \\ & + 2[\alpha_2\alpha_1 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + (t+1)(\alpha_2+\alpha_1+\alpha_3) + (t+1)(t-1)](k_2^2+k_3^2) \\ \omega_t = & J(J+2) - [4(t-1)(t-2) + \{4(t-2)+3]\sum_i \alpha_i + \sum_{i\neq j}\alpha_i\alpha_j\} \\ \beta_t = & [4(t+1)(t+\alpha_2+1/2)]k_1^2k_2^2(k_2^2+k_3^2)\end{aligned}$$

Now we consider polynomial solutions of the ellipsoidal equation (10). Let all the coefficients of the four-term recurrence relation (23) starting from b_{N+1} be zero at any integer N, i.e.

$$b_{N+1} = b_{N+2} = b_{N+3} = \dots\dots\dots = 0 \tag{24}$$

Then, from the recurrence equation (23) under substitution $t = N+2$ and from the condition $b_N \neq 0$, we have

$$J = 2N + \sum_i \alpha_i \tag{25}$$

As a result, a polynomial solution of the ellipsoidal equation (10) can be written down as

$$\psi(\rho) = (\rho - a_1)^{\frac{\alpha_1}{2}}(\rho - a_2)^{\frac{\alpha_2}{2}}(\rho - a_3)^{\frac{\alpha_3}{2}}(\rho - a_4)^{\frac{\alpha_4}{2}}\sum_{t=0}^{N} b_t\left(\frac{\rho - a_2}{a_4 - a_1}\right)^t \tag{26}$$

where b_t obeys the following four-term recurrence relation:

$$\beta_t b_{t+1} + \{\mu - \gamma_t\}b_t + \{\lambda - \delta_t\}b_{t-1} + 4[N - t + 2][N + t - 1 + \sum_i \alpha_i]b_{t-2} = 0 \tag{27}$$

Now we have to solve the problem of eigenvalues of the constants λ, μ. Let us write down the four-term recurrence relation (27) as a system of homogeneous equations

$$\begin{array}{ccccccccc} (\gamma_0-\mu)b_0 & + & \beta_0 b_1 & & & & & = 0 \\ (\delta_1-\lambda)b_0 & + & (\gamma_1-\mu)b_1 & + & \beta_1 b_2 & & & = 0 \\ \omega_2 b_0 & + & (\delta_2-\lambda)b_1 & + & (\gamma_2-\mu)b_2 & + & \beta_2 b_3 & = 0 \\ \dots & & \dots & & \dots & & \dots & \dots \\ \dots & & \dots & & \dots & & \dots & \dots \\ \omega_{N-1}b_{N-3} & + & (\delta_{N-1}-\lambda)b_{N-2} & + & (\gamma_{N-1}-\mu)b_{N-1} & + & \beta_{N-1}b_N & = 0 \\ & & \omega_N b_{N-2} & + & (\delta_N-\lambda)b_{N-1} & + & (\gamma_N-\mu)b_N & = 0 \\ & & & & \omega_{N+1}b_{N-1} & + & (\delta_{N+1}-\lambda)b_N & = 0 \end{array} \tag{28}$$

As is seen from (28), the homogeneous system obtained is a redetermined one since the number of equations $N+2$ is larger than the number of unknowns, and the corresponding matrix is rectangular. As concerns a homogeneous system of equations of this type, it is known that a necessary and sufficient condition for the existence of a nontrivial solution is equality to zero of all determinants of order $(N+1)$ [6]. However, as it is proved in the appendix, for a system of equations of the type (28) it is sufficient that two determinants, resulting from the system (28) by eliminating the last and the next to last rows, be equal to zero.

Now let q_1, q_2, q_3 be integers equal to the number of zeroes of the ellipsoidal wave function (28) in the intervals (a_1, a_2), (a_2, a_3) and (a_3, a_4). As the general number of zeroes of the polynomial (23) in the interval (a_1, a_4) equals N, the ellipsoidal quantum numbers q_1, q_2, q_3 are connected with each other by a simple relation

$$q_1 + q_2 + q_3 = N, \quad q_i = 0, 1, ...N, \quad (i = 1, 2, 3) \tag{29}$$

and can be chosen to enumerate ellipsoidal wave functions and ellipsoidal separation constants $\{\lambda, \mu\}$. As a result, we get that at a fixed N there exist $(N+1)(N+2)/2$ pairs of different values of $\{\lambda, \mu\}$, and depending on parity of the hypermoment J, the following sixteen polynomials are given as an ellipsoidal wave function:

$$\begin{aligned}
uE^{2N}_{q_1q_2q_3}(\rho; a_i) &= \sum_{t=0}^{N} b_t^{(0,0,0,0)}(\rho - a_2)^t, \quad J = 2N \\
cE^{2N+1}_{q_1q_2q_3}(\rho; a_i) &= \sum_{t=0}^{N} b_t^{(0,1,0,0)}(\rho - a_2)^{t+1/2}, \quad J = 2N+1 \\
sE^{2N+1}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{\rho - a_1}\sum_{t=0}^{N} b_t^{(1,0,0,0)}(\rho - a_2)^t, \quad J = 2N+1 \\
dE^{2N+1}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{\rho - a_3}\sum_{t=0}^{N} b_t^{(0,0,1,0)}(\rho - a_2)^t, \quad J = 2N+1 \\
pE^{2N+1}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{\rho - a_4}\sum_{t=0}^{N} b_t^{(0,0,0,1)}(\rho - a_2)^t, \quad J = 2N+1 \\
csE^{2N+2}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{\rho - a_1}\sum_{t=0}^{N} b_t^{(1,1,0,0)}(\rho - a_2)^{t+1/2}, \quad J = 2N+2 \\
cdE^{2N+2}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{\rho - a_3}\sum_{t=0}^{N} b_t^{(0,1,1,0)}(\rho - a_2)^{t+1/2}, \quad J = 2N+2 \\
cpE^{2N+2}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{\rho - a_4}\sum_{t=0}^{N} b_t^{(0,1,0,1)}(\rho - a_2)^{t+1/2}, \quad J = 2N+2 \\
sdE^{2N+2}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{(\rho - a_1)(\rho - a_3)}\sum_{t=0}^{N} b_t^{(1,0,1,0)}(\rho - a_2)^t, \quad J = 2N+2 \\
spE^{2N+2}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{(\rho - a_1)(\rho - a_2)}\sum_{t=0}^{N} b_t^{(1,0,0,1)}(\rho - a_2)^t, \quad J = 2N+2 \\
dpE^{2N+2}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{(\rho - a_3)(\rho - a_4)}\sum_{t=0}^{N} b_t^{(0,0,1,1)}(\rho - a_2)^t, \quad J = 2N+2 \\
csdE^{2N+3}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{(\rho - a_1)(\rho - a_3)}\sum_{t=0}^{N} b_t^{(1,1,1,0)}(\rho - a_2)^{t+1/2}, \quad J = 2N+3 \\
cspE^{2N+3}_{q_1q_2q_3}(\rho; a_i) &= \sqrt{(\rho - a_1)(\rho - a_4)}\sum_{t=0}^{N} b_t^{(1,1,0,1)}(\rho - a_2)^{t+1/2}, \quad J = 2N+3
\end{aligned}$$

$$cdpE^{2N+3}_{q_1q_2q_3}(\rho;a_i) = \sqrt{(\rho-a_3)(\rho-a_4)}\sum_{t=0}^{N} b_t^{(0,1,1,1)}(\rho-a_2)^{t+1/2}, \qquad J=2N+3$$

$$sdpE^{2N+3}_{q_1q_2q_3}(\rho;a_i) = \sqrt{(\rho-a_1)(\rho-a_3)(\rho-a_4)}\sum_{t=0}^{N} b_t^{(1,0,1,1)}(\rho-a_2)^{t}, \qquad J=2N+3$$

$$csdpE^{2N+4}_{q_1q_2q_3}(\rho;a_i) = \sqrt{(\rho-a_1)(\rho-a_3)(\rho-a_4)}\sum_{t=0}^{N} b_t^{(1,1,1,1)}(\rho-a_2)^{t+1/2}, \qquad J=2N+4$$

5 Ellipsoidal basis

According to the afore-said in sect.4, the ellipsoidal basis is divided into sixteen classes

$$\Psi^{(0,0,0,0)}_{N,q_1,q_2,q_3} = C^{(0,0,0,0)} uE^{2N}_{q_1q_2q_3}(\rho_1;a_i) uE^{2N}_{q_1q_2q_3}(\rho_2;a_i) uE^{2N}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(J+2)(J+4)}{8}$$

$$\Psi^{(1,0,0,0)}_{N,q_1,q_2,q_3} = C^{(1,0,0,0)} sE^{2N+1}_{q_1q_2q_3}(\rho_1;a_i) sE^{2N+1}_{q_1q_2q_3}(\rho_2;a_i) sE^{2N+1}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N+1, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(J+1)(J+3)}{8}$$

$$\Psi^{(0,1,0,0)}_{N,q_1,q_2,q_3} = C^{(0,1,0,0)} cE^{2N+1}_{q_1q_2q_3}(\rho_1;a_i) cE^{2N+1}_{q_1q_2q_3}(\rho_2;a_i) cE^{2N+1}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N+1, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(J+1)(J+3)}{8}$$

$$\Psi^{(0,0,1,0)}_{N,q_1,q_2,q_3} = C^{(0,0,1,0)} dE^{2N+1}_{q_1q_2q_3}(\rho_1;a_i) dE^{2N+1}_{q_1q_2q_3}(\rho_2;a_i) dE^{2N+1}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N+1, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(J+1)(J+3)}{8}$$

$$\Psi^{(0,0,0,1)}_{N,q_1,q_2,q_3} = C^{(0,0,0,1)} pE^{2N+1}_{q_1q_2q_3}(\rho_1;a_i) pE^{2N+1}_{q_1q_2q_3}(\rho_2;a_i) pE^{2N+1}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N+1, \quad D=\frac{(N+1)(N+2)}{2}=\frac{(J+1)(J+3)}{8}$$

$$\Psi^{(1,1,0,0)}_{N,q_1,q_2,q_3} = C^{(1,1,0,0)} csE^{2N+2}_{q_1q_2q_3}(\rho_1;a_i) csE^{2N+2}_{q_1q_2q_3}(\rho_2;a_i) csE^{2N+2}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N+2, \quad D=\frac{(N+1)(N+2)}{2}=\frac{J(J+2)}{8}$$

$$\Psi^{(1,0,1,0)}_{N,q_1,q_2,q_3} = C^{(1,0,1,0)}\, sdE^{2N+2}_{q_1q_2q_3}(\rho_1;a_i)\, sdE^{2N+2}_{q_1q_2q_3}(\rho_2;a_i)\, sdE^{2N+2}_{q_1q_2q_3}(\rho_3;a_i),$$
$$n=2N+2, \quad D=\frac{(N+1)(N+2)}{2}=\frac{J(J+2)}{8}$$

$$\Psi^{(0,1,1,0)}_{N,q_1,q_2,q_3} = C^{(0,1,1,0)}\, cdE^{2N+2}_{q_1q_2q_3}(\rho_1;a_i)\, cdE^{2N+2}_{q_1q_2q_3}(\rho_2;a_i)\, cdE^{2N+2}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N+2, \quad D=\frac{(N+1)(N+2)}{2}=\frac{J(J+2)}{8}$$

$$\Psi^{(0,1,0,1)}_{N,q_1,q_2,q_3} = C^{(0,1,0,1)}\, cpE^{2N+2}_{q_1q_2q_3}(\rho_1;a_i)\, cpE^{2N+2}_{q_1q_2q_3}(\rho_2;a_i)\, cpE^{2N+2}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N+2, \quad D=\frac{(N+1)(N+2)}{2}=\frac{J(J+2)}{8}$$

$$\Psi^{(1,0,0,1)}_{N,q_1,q_2,q_3} = C^{(1,0,0,1)}\, spE^{2N+2}_{q_1q_2q_3}(\rho_1;a_i)\, spE^{2N+2}_{q_1q_2q_3}(\rho_2;a_i)\, spE^{2N+2}_{q_1q_2q_3}(\rho_3;a_i),$$
$$J=2N+2, \quad D=\frac{(N+1)(N+2)}{2}=\frac{J(J+2)}{8}$$

$$
\begin{aligned}
\Psi^{(0,0,1,1)}_{N,q_1,q_2,q_3} &= C^{(0,0,1,1)}\, dpE^{2N+2}_{q_1q_2q_3}(\rho_1;a_i)\, dpE^{2N+2}_{q_1q_2q_3}(\rho_2;a_i)\, dpE^{2N+2}_{q_1q_2q_3}(\rho_3;a_i),\\
& \quad J = 2N+2, \quad D = \frac{(N+1)(N+2)}{2} = \frac{J(J+2)}{8}\\
\Psi^{(1,1,1,0)}_{N,q_1,q_2,q_3} &= C^{(1,1,1,0)}\, csdE^{2N+3}_{q_1q_2q_3}(\rho_1;a_i)\, csdE^{2N+3}_{q_1q_2q_3}(\rho_2;a_i)\, csdE^{2N+3}_{q_1q_2q_3}(\rho_3;a_i),\\
& \quad J = 2N+3, \quad D = \frac{(N+1)(N+2)}{2} = \frac{(J-1)(J+1)}{8}\\
\Psi^{(1,1,0,1)}_{N,q_1,q_2,q_3} &= C^{(1,1,0,1)}\, cspE^{2N+3}_{q_1q_2q_3}(\rho_1;a_i)\, cspE^{2N+3}_{q_1q_2q_3}(\rho_2;a_i)\, cspE^{2N+3}_{q_1q_2q_3}(\rho_3;a_i),\\
& \quad J = 2N+3, \quad D = \frac{(N+1)(N+2)}{2} = \frac{(J-1)(J+1)}{8}\\
\Psi^{(0,1,1,1)}_{N,q_1,q_2,q_3} &= C^{(0,1,1,1)}\, cdpE^{2N+3}_{q_1q_2q_3}(\rho_1;a_i)\, cdpE^{2N+3}_{q_1q_2q_3}(\rho_2;a_i)\, cdpE^{2N+3}_{q_1q_2q_3}(\rho_3;a_i),\\
& \quad J = 2N+3, \quad D = \frac{(N+1)(N+2)}{2} = \frac{(J-1)(J+1)}{8}\\
\Psi^{(1,0,1,1)}_{N,q_1,q_2,q_3} &= C^{(1,0,1,1)}\, sdpE^{2N+3}_{q_1q_2q_3}(\rho_1;a_i)\, sdpE^{2N+3}_{q_1q_2q_3}(\rho_2;a_i)\, sdpE^{2N+3}_{q_1q_2q_3}(\rho_3;a_i),\\
& \quad J = 2N+3, \quad D = \frac{(N+1)(N+2)}{2} = \frac{(J-1)(J+1)}{8}\\
\Psi^{(1,1,1,1)}_{N,q_1,q_2,q_3} &= C^{(1,1,1,1)}\, csdpE^{2N+3}_{q_1q_2q_3}(\rho_1;a_i)\, csdpE^{2N+3}_{q_1q_2q_3}(\rho_2;a_i)\, csdpE^{2N+3}_{q_1q_2q_3}(\rho_3;a_i),\\
& \quad J = 2N+4, \quad D = \frac{(N+1)(N+2)}{2} = \frac{J(J-2)}{8}
\end{aligned}
$$

Here D is the number of states at a given value of the hypermoment J. The multiplicity of degeneracy of energy levels is determined by a sum of all states of even or odd fixed J and is correspondingly equal to $(J+1)$.

The coefficients $C^{(i,j,k,l)}$ where $i,j,k,l = 0,1$ are determined from the normalization condition of the ellipsoidal basis

$$
\frac{1}{8}\int_{a_1}^{a_2}\int_{a_2}^{a_3}\int_{a_3}^{a_4}\left[\Psi^{(i,j,k,l)}_{N,q_1,q_2,q_3}(\rho_1,\rho_2,\rho_3)\right]^2 \frac{(\rho_2-\rho_1)(\rho_3-\rho_2)(\rho_3-\rho_1)}{\sqrt{-P(\rho_1)P(\rho_2)P(\rho_3)}}\, d\rho_1\, d\rho_2\, d\rho_3 = 1 \quad (30)
$$

6 Mathematical supplement

Let us find out conditions for the existence of solutions of the homogeneous system (22) which satisfy the requirement $b_N \neq 0$. Rewrite the system (22) in the matrix form having in advance divided the jth equation into $\omega_j \neq 0$; $j = 0,1,\cdots,N+1$. For this purpose we introduce a rectangular matrix $P = ||p_{ij}||$, where $i = 0,\cdots,N+1$; $j = 0,\cdots,N$ and

$$
\begin{aligned}
p_{ii} &= (\gamma_i - \mu)/\omega_i, \quad i = 0,1,\cdots,N;\\
p_{i,i+1} &= \beta_i/\omega_i, \quad i = 0,1,\cdots,N-1;\\
p_{i,i-1} &= (\delta_i - \lambda)/\omega_i, \quad i = 1,\cdots,N+1;\\
p_{i,i-2} &= 1, \quad i = 2,\cdots,N+1;
\end{aligned} \qquad (31)
$$

$p_{ij} = 0$, at $i \leq j-2$ and $i \geq j+3$. We derive the equation

$$
Pb = 0 \qquad (32)
$$

where $b = (b_0, b_1, \cdots, b_N)^T$.

Denoting by P_1 the matrix obtained from the matrix P by eliminating the first row, we get the first condition for the existence of nontrivial solutions of the system (32)

$$det P_1 = 0 \tag{33}$$

To get the second condition and to solve the system (31), let us consider the latter without the first two equations and transfer the elements of the last column to the right-hand side. We derive the following inhomogeneous system of equations

$$P_2 \tilde{b} = f \tag{34}$$

where P_2 - is the matrix obtained from P by eliminating the first two rows and the last column, $\tilde{b} = (b_0, b_1, \cdots, b_{N-1})^T$, $f = (f_1, f_2, \cdots, f_N)^T$, $f_i = 0$, $i = 1, \cdots, N-3$;. $f_i = p_{i+1,N} b_N$; $i = N-2, N-1, N$. Note that P_2 - is the upper triangle matrix, which has units on its principal diagonal. Let us consider minors of the matrix P with the corresponding signs

$$s_{ij} = (-1)^{i+j} \begin{vmatrix} p_{i+1,i} & p_{i+1,i+1} & \cdots & p_{i+1,j-1} \\ p_{i+2,i} & p_{i+2,i+1} & \cdots & p_{i+2,j-1} \\ \cdots & \cdots & \cdots & \cdots \\ p_{j,i} & p_{j,i+1} & \cdots & p_{j,j-1} \end{vmatrix} \tag{35}$$

at $0 \le i < j \le N+1$. It is to be mentioned that since $p_{ij} = 0$ at $i \le j-2$ and $i \ge j+3$, and $p_{i,i-2} = 1$, the following relations hold:

$$s_{ij} = -p_{j,j-1} s_{i,j-1} - p_{j-1,j-1} s_{i,j-2} - p_{j-2,j-1} s_{i,j-3} \tag{36}$$

$$s_{ij} = -p_{i+1,i} s_{i+1,j} - p_{i+1,i+1} s_{i+2,j} - p_{i+1,i+2} s_{i+3,j} \tag{37}$$

Let us treat the upper triangle matrix $S_2 = P_2^{-1}$. It follows from the lemma below that the elements of this matrix, which are above the principal diagonal, satisfy relations (36) and the principal diagonal of the matrix has units.

Lemma. Let $A = ||a_{ij}||_{i,j=1}^n$ be an upper triangle matrix with units on the principal. Then $B = A^{-1} = ||b_{ij}||_{i,j=1}^n$ is also an upper triangle matrix with units on the principal diagonal, and at $i < j$

$$b_{ij} = (-1)^{i+j} \begin{vmatrix} a_{i,i+1} & a_{i,i+2} & \cdots & a_{i,j-1} & a_{ij} \\ 1 & a_{i+1,i+2} & \cdots & p_{i+1,j-1} & a_{i+1,j} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & a_{j-1,j} \end{vmatrix} \tag{38}$$

Proof of the lemma. For $i \le j$ we have

$$\sum_{k=i}^{j} a_{ik} b_{kj} = \delta_{ij} \tag{39}$$

Hence for $i < j$

$$b_{ij} = -\sum_{k=i+1}^{j} a_{ik} b_{kj} \tag{40}$$

In the same manner the recurrence formula is derived from the expansion of the determinant (38) over the last column. Since formulae (38) and (40) coincide at $J+1$, the lemma is proved.

Thus,

$$\tilde{b} = S_2 f \tag{41}$$

Choosing as b_N an arbitrary nonzero number, we obtain from (41) a vector b that satisfies all the equations of the system (32) starting from the third one. By virtue of (41) the vector b thus chosen also satisfies the second equation of the system (32). We get

$$p_{00} \sum_{j=1}^{N} s_{1j} f_j + p_{01} \sum_{j=1}^{N} s_{2j} f_j = 0, \tag{42}$$

which is equivalent to the equality

$$\begin{gathered} p_{00}(s_{1,N-2}p_{N-1,N} + s_{1,N-1}p_{N,N} + s_{1,N}p_{N+1,N}) \\ +p_{01}(s_{2,N-2}p_{N-1,N} + s_{2,N-1}p_{N,N} + s_{2,N}p_{N+1,N}) \end{gathered}$$

Using the recurrence relation (36) we get

$$p_{00}s_{1,N+1} + p_{01}s_{2,N+1} = 0 \tag{43}$$

Theorem. For the system (28) to have solutions, for which $b_N \neq 0$, it is necessary and sufficient to satisfy the conditions (33) and (43). If these conditions are fulfilled, the system (28) for any $b_N \neq 0$ has a solution determined by formula (41).

Thus, to determine eigenvalues of the ellipsoidal separation constant $\{\lambda, \mu\}$ one has to solve the system of two algebraic equations

$$\begin{cases} det P_1 = 0, \\ p_{00}s_{1,N+1} + p_{01}s_{2,N+1} = 0. \end{cases}$$

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