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Path Integral Discussion for Smorodinsky-Winternitz Potentials: II. The Two- and Three-Dimensional Sphere

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Abstract

Steps towards path integral formulations for Smorodinsky-Winternitz potentials, respectively systems with accidental degeneracies, on the two- and three-dimensional sphere, and a complete classification of super-integrable systems on spaces of constant curvature are presented. We mention all coordinate systems which separate the Smorodinsky-Winternitz potentials on a sphere, and state the corresponding path integral formulations. Whereas in many coordinate systems explicit path integral solutions are not possible, we list in all soluble cases the path integral in terms of the propagator, respectively the spectral expansion into the wave functions and the energy spectrum.

1. Introduction

In a previous publication we have extensively discussed Smorodinsky-Winternitz potentials in two- and three-dimensional Euclidean space. The present paper is devoted to the discussion of the corresponding case of the two- and three-dimensional sphere. It is a continuation and generalization of the flat space cases in order to find super-integrable potentials in spaces of constant curvature. Smorodinsky, Winternitz et al. started a systematic search to find and classify potential problems in two and three dimensional Euclidean space which can be seen as non-central generalizations of the Coulomb-, the harmonic oscillator and radial barrier potentials. The classification scheme starts from the consideration in which way integrals of motion (in classical mechanics), respectively additional operators corresponding to these integrals of motion, i.e. the observables which commute with the Hamiltonian, are related to the separability of the potential problem in more than one coordinate systems. In two dimensional Euclidean space R^2 [23] there are four potentials which satisfy the requirements. In three dimensional Euclidean space R^3 Smorodinsky, Winternitz et al. [70], and recently Evans [19] found all potentials which separate in more than one coordinate system.

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The generalization of the oscillator and the Coulomb potential in a curved space have been discussed by several authors. The harmonic oscillator must be constructed in such a way that the flat space limit yields the well-known results of the wave functions and the energy-spectrum $E_n = \hbar\omega(n + d/2)$ (d dimension). The proper generalization is the so-called HIGGS-oscillator [44]:

$$V_{\text{Higgs}}(\vec{s}) = \frac{M}{2}\,\omega^2 R^2\,\frac{s_1^2 + s_2^2 + s_3^2}{s_4^2}\,. \tag{1}$$

This quantum mechanical problem has been discussed by several authors, e.g. BONATOS et al. [4], GRANOVSKY et al. [25], KATAYAMA [55], LEEMON [66], and POGOSYAN et al. [85].

The constant-curvature case for the Coulomb potential has been known for a long time (INFELD [46], INFELD and SCHILD [47], STEVENSON [94] and SCHRÖDINGER [88]) and has the form

$$V_C(\vec{s}) = -\frac{\alpha}{R}\,\frac{s_4}{\sqrt{s_1^2 + s_2^2 + s_3^2}}\,. \tag{2}$$

The limit $R \to \infty$ yields the well-known form of the Coulomb potential. The corresponding quantum mechanical problem has been discussed by GRANOVSKY et al. [26], KIBLER et al. [57], KUROCHKIN and OTCHIK [63], OTCHIK and RED'KOV [80], PERVUSHIN et al. [87], and VINITSKY et al. [97]. Path integral discussions of the Coulomb potential in a curved space are due to BARUT et al. [5, 6] and GROSCHE [29].

The study of these systems on the sphere was the beginning of the generalization of Smorodinsky-Winternitz potentials on spaces of constant curvature. These investigations show that it is possible to construct a harmonic oscillator, a Coulomb problem on a sphere, and a ring-shaped oscillator and a HARTMANN potential [42] as well. As will be shown all these potentials (and some others which do not have a particular name) separate on the two- and three-dimensional sphere, respectively, in more that one coordinate system.

However, there seems at first sight to be less structure as in flat space. In two-dimensional Euclidean space, there are four super-integrable potentials known [23, 38], and in three-dimensional Euclidean space there are five maximally super-integrable potentials and eight minimally super-integrable potentials [19, 38, 70]. In this paper we are dealing with the most obvious generalizations of the flat space cases, and we find for the two-dimensional sphere *two* Smorodinsky-Winternitz potentials, and on the three-dimensional sphere *six* Smorodinsky-Winternitz potentials. Therefore the question arises, where are the remaining constant curvature counterparts of the Euclidean cases? In R^2 there are four, and in R^2 there are eleven orthogonal coordinate systems [76], whereas in $S^{(2)}$ there are two, and on $S^{(3)}$ six coordinate systems [77]. Again, one must ask, where are the remaining constant curvature counterparts of the flat space cases. In other words: What is a coordinate system on a sphere? A solution of this problem would give us the possibility for the explicit construction of *all super-integrable* potentials on spheres. For instance, we do not know yet what is the HOLT potential [43], or the Stark-effect on a sphere.

In the following we do not distinguish between minimally and maximally super-integrable potentials on spheres. This would require a proper definition of super-integrability on spaces of constant (positive) curvature and coordinate systems. This is not to our disposal yet. We denote any potential on the sphere which is separable in *more than one* coordinate system as a Smorodinsky-Winternitz potential.

Our paper is therefore designed as a starting point for a systematic investigation of the number of Smorodinsky-Winternitz potentials (and the number of coordinate systems) on spaces of constant curvature. Such a study seems not to exist until now. Potentials like these appear in many applications in physics, and are important in the study of the physical relevance of particular space-time geometries. For instance, the sphere, the pseudosphere (double-sheeted hyperboloid) and a Anti-DeSitter geometry [61] allow the incorporation of a singular potential like the Coulomb potential. The single-sheeted hyperboloid does not admit this [35]. Our line of reasoning to solve the corresponding quantum mechanical problems will be the path integral technique of FEYNMAN [20, 21]. As we see, in comparison to the flat space, we will be mostly concerned with the path integral solution and identities arising from the (modified) Pöschl-Teller path integral, i.e., the solutions are of the hypergeometric type with hypergeometric functions and Jacobi polynomials. In flat space, Pöschl-Teller systems were only appearing on a sub-level, the radial wave functions were almost all of the confluent hypergeometric type, i.e., we had exponentials, Bessel functions, parabolic cylinder functions, Whittaker functions, Hermite- and Laguerre-polynomials as wave functions. For instance, in the case of the Coulomb problem in flat space extensive studies by means of path integrals, where all these features appear, have been done by many authors, among them e.g., CARPIO-BERNIDO et al. [7]–[10], CASTRIGIANO and STÄRK [11], CHETOUANI et al. [12, 13, 14], DURU and KLEINERT [17, 18, 58], GROSCHE et al. [31, 33, 34, 38, 41], INOMATA [49], PAK and SÖKMEN [82], STEINER [92], and STORCHAK [96].

A closer look on the corresponding wave functions on spheres reveals that the hypergeometric wave function is not the only kind of solution. In most cases it is even not the most general one, because the most general solution is based on the Lamé wave equation [76, 83] which has a very complicated structure (however it contains all other cases as limiting, respectively particular cases). This fact that one can only deal with solutions in terms of Pöschl-Teller wave functions has the consequence that the path integration (if possible) is much simpler as in the flat space cases. For instance, space-time transformations are required (cf. DURU and KLEINERT [17, 18], FISCHER et al. [22], GROSCHE and STEINER [34, 40, 41, 91, 93], JUNKER [52], KLEINERT [58, 59], PAK and SÖKMEN [81], PELSTER and WUNDERLIN [84], STORCHAK [95], YOUNG and DEWITT-MORETTE [90]) only in the cases of the Coulomb- and Hartmann potentials, and to show the separability of a problem in elliptical (and related) coordinates. This is due to the symmetry properties of the quantum motion on spheres. The oscillator on the two-dimensional sphere has a group structure which can be put into connection with the six-dimensional sphere. This property is in particular useful in the case of elliptical coordinates, because the corresponding wave functions also appear as matrix elements on $S^{(6)}$, however, with analytically continued quantum numbers. This will be shortly sketched in appendix B.

Therefore the path integration in the soluble cases is not very difficult and we can rely on results already present in the literature. The path integral is used as an analytical tool for obtaining in the soluble cases the solutions in an easy way, and our emphasis lies on the problem of the separability in coordinate systems and super-integrability on spheres.

The case of Smorodinsky-Winternitz potentials on spaces of constant negative curvature, i.e., on the two- and three-dimensional pseudosphere, will be treated in a forthcoming publication [39].

The remaining part of our paper is organized as follows: In the next Section we shortly introduce the coordinate systems on the two- and three-dimensional sphere. In comparison to Ref. [36] the notation and notion of the coordinate systems will be improved. In the third and in the fourth Section we discuss the Smorodinsky-Winternitz potentials on the two- and three-dimensional sphere, respectively. The fifth section is devoted to a summary and discussion of our findings. In appendix A we state how we define the lattice of our path integral formulations. We also mention the two path integral solutions we need in our discussion, the path integral solution of the Pöschl-Teller, and of the modified

Pöschl-Teller potential. In appendix B we discuss a particular path integral formulation of the free motion on the six-dimensional sphere which is useful in the discussion of the Higgs oscillator on the sphere in elliptic coordinates.

2. Coordinate Systems on Spaces with Constant Positive Curvature

The coordinates on the D-dimensional sphere are denoted by the vector $\vec{s}=(s_1,\ldots,s_D)$. The basic building blocks of separable coordinate systems are the D-systems *elliptic* coordinates [53]

$$s_j^2=R^2\frac{\prod_{i=1}^D(\varrho_i-e_j)}{\prod_{j\neq i}(e_i-e_j)},\qquad (j=1,\ldots,D+1),\quad \sum_{j=1}^{D+1}s_j^2=R^2, \tag{3}$$

corresponding to a metric

$$ds^2=-\frac{1}{4k}\sum_{i=1}^D\frac{1}{P_D(\varrho_i)}\left[\prod_{j\neq i}(\varrho_i-\varrho_j)\right](d\varrho_i)^2,\quad P_D(\varrho)=\prod_{i=1}^{D+1}(\varrho-e_i) \tag{4}$$

($k>0$ curvature). In order to find the possible explicit coordinate systems one must pay attention to the requirement that, (i) the metric must be positive definite, (ii) the variables $\{\varrho\}_{i=1}^D$ should vary in such a way that they correspond to a coordinate patch which is compact. There is a unique solution to these requirements given by

$$e_1<\varrho_1<\ldots<e_D<\varrho_D<e_{D+1}. \tag{5}$$

2.1. The Two-Dimensional Sphere

Let us first consider the coordinate systems on the two-dimensional sphere [62, 67, 68, 90, 98]. First, we have the

2.1.1. Polar Coordinates:

$$\begin{aligned} s_1&=R\sin\theta\cos\phi, && 0\leqq\theta\leqq\pi,\\ s_2&=R\sin\theta\sin\phi, && 0\leqq\phi<2\pi,\\ s_3&=R\cos\theta. \end{aligned} \tag{6}$$

These are the usual two-dimensional polar coordinates on the sphere. The momentum operators have the form

$$p_\theta=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta\right),\quad p_\phi=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\phi}. \tag{7}$$

For the Hamiltonian we get

$$\begin{aligned} -\frac{\hbar^2}{2MR^2}\Delta_{S^{(2)}}&=-\frac{\hbar^2}{2MR^2}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\\ &=\frac{1}{2MR^2}\left(p_\theta^2+\frac{1}{\sin^2\theta}p_\phi^2\right)-\frac{\hbar^2}{8MR^2}\left(1+\frac{1}{\sin^2\theta}\right). \end{aligned} \tag{8}$$

A separable potential must have the form

$$V(\theta, \phi) = V_1(\theta) + \frac{V_2(\phi)}{\sin^2\theta}. \tag{9}$$

2.1.2. Elliptic Coordinates: Second, we consider the elliptic coordinate system which reads in algebraic form as follows ($a_1 \leqq \varrho_1 \leqq a_2 \leqq \varrho_2 \leqq a_3$)

$$s_1^2 = R^2 \frac{(\varrho_1 - a_1)(\varrho_2 - a_1)}{(a_2 - a_1)(a_3 - a_1)}, \quad s_2^2 = R^2 \frac{(\varrho_1 - a_2)(\varrho_2 - a_2)}{(a_3 - a_2)(a_1 - a_2)}, \quad s_3^2 = R^2 \frac{(\varrho_1 - a_3)(\varrho_2 - a_3)}{(a_1 - a_3)(a_2 - a_3)}. \tag{10}$$

If we put $\varrho_1 = a_1 + (a_2 - a_1)\,\mathrm{sn}^2(\mu, k)$ and $\varrho_2 = a_2 + (a_3 - a_2)\,\mathrm{cn}^2(\nu, k')$, where $\mathrm{sn}(\mu, k)$, $\mathrm{cn}(\mu, k)$ and $\mathrm{dn}(\mu, k)$ are the Jacobi elliptic functions with modulus k, we obtain for the coordinates $\vec{s}$ on the sphere

$$\begin{aligned} s_1 &= R\,\mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k'), \quad -K \leqq \mu \leqq K, \\ s_2 &= R\,\mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k'), \quad -2K' \leqq \nu \leqq 2K', \\ s_3 &= R\,\mathrm{dn}(\mu, k)\,\mathrm{sn}(\nu, k'), \end{aligned} \tag{11}$$

where

$$k^2 = \frac{a_2 - a_1}{a_3 - a_1} = \sin^2 f, \quad k'^2 = \frac{a_3 - a_2}{a_3 - a_1} = \cos^2 f, \quad k^2 + k'^2 = 1. \tag{12}$$

K and K' are the complete elliptic integrals, and $2fR$ is the interfocus distance on the upper semisphere of the ellipses on the sphere. Note the relations $\mathrm{cn}^2\alpha + \mathrm{sn}^2\alpha = 1$ and $\mathrm{dn}^2\alpha = 1 - k^2\,\mathrm{sn}^2\alpha$. In the following we omit the moduli k and k' of the Jacobi elliptic functions if it is obvious that the variable μ goes with k and ν goes with k'. The momentum operators are

$$p_\mu = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\mu} - \frac{k^2\,\mathrm{sn}\,\mu\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}\right), \quad p_\nu = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\nu} - \frac{k'^2\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu\,\mathrm{dn}\,\nu}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}\right), \tag{13}$$

and the Hamiltonian has the form (here and in the following we use an ordering prescription which we have called product ordering, cf. [27] and appendix A)

$$\begin{aligned} -\frac{\hbar^2}{2MR^2}\Delta_{S^{(2)}} &= -\frac{\hbar^2}{2MR^2}\frac{1}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right) \\ &= \frac{1}{2MR^2}\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}}\,(p_\mu^2 + p_\nu^2)\,\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}}. \end{aligned} \tag{14}$$

Note that $\Delta V \equiv 0$. A potential which is separable in these coordinates must have the form

$$V(\vec{s}) = \frac{V_1(\varrho_1) + V_2(\varrho_2)}{\varrho_2 - \varrho_1} = \frac{V_1(\mu) + V_2(\nu)}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}. \tag{15}$$

As in flat space we can get another elliptic coordinate system, where one of the foci is on the s_3-axis which is obtained after a rotation about the s_2-axis by means of

$$\vec{s} \mapsto R(f)\vec{s} \tag{16}$$

with the matrix $R(f)$ given by

$$R(f)=\begin{pmatrix} \cos f & 0 & \sin f \\ 0 & 1 & 0 \\ -\sin f & 0 & \cos f \end{pmatrix}. \tag{17}$$

This gives the elliptic II system which has the form [51]

$$\begin{aligned} s_1 &= R[k' \operatorname{sn}(\mu, k) \operatorname{dn}(\nu, k') + k \operatorname{dn}(\mu, k) \operatorname{sn}(\nu, k')], \\ s_2 &= R \operatorname{cn}(\mu, k) \operatorname{cn}(\nu, k'), \\ s_3 &= R[k' \operatorname{dn}(\mu, k) \operatorname{sn}(\nu, k') - k \operatorname{sn}(\mu, k) \operatorname{dn}(\nu, k')]. \end{aligned} \tag{18}$$

2.2. The Three-Dimensional Sphere

On the sphere $S^{(3)}$ we have six coordinate systems which are given by [53, 54, 77]

2.2.1. Spherical Coordinates:

$$\begin{aligned} s_1 &= R \sin\chi \sin\theta \cos\phi, \qquad 0<\theta, \chi<\pi, \\ s_2 &= R \sin\chi \sin\theta \sin\phi, \qquad 0 \leqq \phi < 2\pi, \\ s_3 &= R \sin\chi \cos\theta. \\ s_4 &= R \cos\chi. \end{aligned} \tag{19}$$

The momentum operators are

$$p_\chi = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\chi} + \cot\chi\right), \quad p_\theta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right), \tag{20}$$

and $p_\phi = -\mathrm{i}\hbar\partial_\phi$. For the Hamiltonian we obtain

$$\begin{aligned} &-\frac{\hbar^2}{2MR^2}\Delta_{S^{(3)}} \\ &= \frac{\hbar^2}{2MR^2}\left[\frac{\partial^2}{\partial\chi} + 2\cot\chi\frac{\partial}{\partial\chi} + \frac{1}{\sin^2\chi}\left(\frac{\partial^2}{\partial\theta} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi}\right)\right] \\ &= \frac{1}{2MR^2}\left(p_\chi^2 + \frac{p_\theta^2}{\sin^2\chi} + \frac{p_\phi^2}{\sin^2\chi\sin^2\theta}\right) - \frac{\hbar^2}{8MR^2}\left(4 + \frac{1}{\sin^2\chi} + \frac{1}{\sin^2\chi\sin^2\theta}\right). \end{aligned} \tag{21}$$

A separable potential must have the form

$$V(\vec{s}) = V_1(\chi) + \frac{V_2(\theta)}{\sin^2\chi} + \frac{V_3(\phi)}{\sin^2\chi\sin^2\theta}. \tag{22}$$

2.2.2. Cylindrical Coordinates:

$$\begin{aligned} s_1 &= \sin\theta\cos\phi_1, & 0<\theta<\pi/2, \\ s_2 &= \sin\theta\sin\phi_1, & 0\leqq\phi_{1,2}<2\pi, \\ s_3 &= \cos\theta\cos\phi_2, & \\ s_4 &= \cos\theta\sin\phi_2. & \end{aligned} \tag{23}$$

The momentum operators are

$$p_\theta = \frac{\hbar}{i}\left(\frac{\partial}{\partial\theta} + \cot\theta - \tan\theta\right), \tag{24}$$

and $p_{\phi_{1,2}} = -i\hbar\partial_{\phi_{1,2}}$. Therefore we have for the Hamiltonian

$$\begin{aligned} -\frac{\hbar^2}{2MR^2}\Delta_{S^{(3)}} &= -\frac{\hbar^2}{2MR^2}\left[\frac{\partial^2}{\partial\theta^2} + 2(\cot\theta - \tan\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi_1^2} + \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\phi_2^2}\right] \\ &= \frac{1}{2MR^2}\left(p_\theta^2 + \frac{p_{\phi_1}^2}{\sin^2\theta} + \frac{p_{\phi_2}^2}{\cos^2\theta}\right) - \frac{\hbar^2}{8MR^2}\left(4 + \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right). \end{aligned} \tag{25}$$

A separable potential must have the form

$$V(\vec{s}) = V_1(\theta) + \frac{V_2(\phi_1)}{\sin^2\theta} + \frac{V_3(\phi_2)}{\cos^2\theta}. \tag{26}$$

2.2.3. Sphero-Elliptic Coordinates:

$$\begin{aligned} s_1 &= R\sin\chi\,\mathrm{sn}(\mu,k)\,\mathrm{dn}(\nu,k'), & -K\leqq\mu\leqq K, \\ s_2 &= R\sin\chi\,\mathrm{cn}(\mu,k)\,\mathrm{cn}(\nu,k'), & -2K'\leqq\nu\leqq 2K', \\ s_3 &= R\sin\chi\,\mathrm{dn}(\mu,k)\,\mathrm{sn}(\nu,k'), & 0\leqq\chi\leqq\pi, \\ s_4 &= R\cos\chi, & \end{aligned} \tag{27}$$

in the notation of 2.1.2. For the momentum operators we have

$$p_\chi = \frac{\hbar}{i}\left(\frac{\partial}{\partial\chi} + \cot\chi\right), \tag{28}$$

together with p_μ and p_ν as in (13). We have for the Hamiltonian

$$\begin{aligned} &-\frac{\hbar^2}{2MR^2}\Delta_{S^{(3)}} \\ &= -\frac{\hbar^2}{2MR^2}\left[\frac{\partial^2}{\partial\chi^2} - 2\cot\chi\frac{\partial}{\partial\chi} + \frac{1}{\sin^2\chi(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right)\right] \\ &= \frac{1}{2MR^2}\left(p_\chi^2 + \frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}}\,\frac{p_\mu^2 + p_\nu^2}{\sin^2\chi}\,\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}}\right) - \frac{\hbar^2}{2MR^2}. \end{aligned} \tag{29}$$

A separable potential must have the form

$$V(\vec{s})=V_1(\chi)+\frac{V_2(\mu)+V_3(v)}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2 v}. \tag{30}$$

2.2.4. Elliptic Cylindrical Coordinates 1: The first elliptical cylindrical coordinate system in algebraic form is given by ($a_1\leqq\varrho_1\leqq a_2\leqq\varrho_2\leqq a_3$)

$$\left.\begin{aligned}
s_1^2&=R^2\frac{(\varrho_1-a_1)(\varrho_2-a_1)}{(a_2-a_1)(a_3-a_1)}\cos^2\phi,\\
s_2^2&=R^2\frac{(\varrho_1-a_1)(\varrho_2-a_1)}{(a_2-a_1)(a_3-a_1)}\sin^2\phi,\\
s_3^2&=R^2\frac{(\varrho_1-a_2)(\varrho_2-a_2)}{(a_3-a_2)(a_1-a_2)},\\
s_4^2&=R^2\frac{(\varrho_1-a_3)(\varrho_2-a_3)}{(a_1-a_3)(a_2-a_3)}.
\end{aligned}\right\} \tag{31}$$

In terms of the Jacobi elliptic functions we have ($-K\leqq\mu\leqq K,\ -2K'\leqq v\leqq 2K',\ 0\leqq\phi<2\pi$)

$$\left.\begin{aligned}
s_1&=R\operatorname{sn}(\mu,k)\operatorname{dn}(v,k')\cos\phi,\\
s_2&=R\operatorname{sn}(\mu,k)\operatorname{dn}(v,k')\sin\phi,\\
s_3&=R\operatorname{cn}(\mu,k)\operatorname{cn}(v,k'),\\
s_4&=R\operatorname{dn}(\mu,k)\operatorname{sn}(v,k').
\end{aligned}\right\} \tag{32}$$

For $R\to\infty$ the oblate spheroidal coordinate system in $\mathbb{R}^3$ is recovered. The momentum operators are given by

$$p_\mu=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\mu}-\frac{k^2\operatorname{sn}\mu\operatorname{cn}\mu\operatorname{dn}\mu}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2 v}+\frac{\operatorname{cn}\mu\operatorname{dn}\mu}{2\operatorname{sn}\mu}\right), \tag{33}$$

$$p_v=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial v}-\frac{k'^2\operatorname{sn}v\operatorname{cn}v\operatorname{dn}v}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2 v}-k'^2\frac{\operatorname{sn}v\operatorname{cn}v}{2\operatorname{dn}v}\right), \tag{34}$$

and $p_\phi=-\mathrm{i}\hbar\partial_\phi$. Therefore we have for the Hamiltonian

$$\begin{aligned}
-\frac{\hbar^2}{2MR^2}\Delta_{S^{(3)}}=-\frac{\hbar^2}{2MR^2}\Bigg[&\frac{1}{(k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2 v)}\left(\frac{\partial^2}{\partial\mu^2}+\frac{\operatorname{cn}\mu\operatorname{dn}\mu}{\operatorname{sn}\mu}\frac{\partial}{\partial\mu}\right.\\
&\left.+\frac{\partial^2}{\partial v^2}-k'^2\frac{\operatorname{sn}v\operatorname{cn}v}{\operatorname{dn}v}\frac{\partial}{\partial v}\right)+\frac{1}{\operatorname{sn}^2\mu\operatorname{dn}^2 v}\frac{\partial^2}{\partial\phi^2}\Bigg]
\end{aligned} \tag{35}$$

$$\begin{aligned}
&=\frac{\hbar^2}{2MR^2}\frac{1}{\sqrt{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2 v}}(p_\mu^2+p_v^2)\frac{1}{\sqrt{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2 v}}\\
&\quad-\frac{\hbar^2}{8MR^2}\left[4+\frac{1}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2 v}\left(\frac{\operatorname{cn}^2\mu\operatorname{dn}^2\mu}{\operatorname{sn}^2\mu}+k'^4\frac{\operatorname{sn}^2 v\operatorname{cn}^2 v}{\operatorname{dn}^2 v}\right)\right].
\end{aligned} \tag{36}$$

A potential separable in these coordinates must have the form

$$V(\vec{s})=\frac{V_1(\mu)+V_2(\nu)}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}+\frac{V_3(\phi)}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}. \tag{37}$$

2.2.5. Elliptic Cylindrical Coordinates 2: The second elliptical cylindrical coordinate system in algebraic form is given by ($a_1\leqq\varrho_1\leqq a_2\leqq\varrho_2\leqq a_3$)

$$\left.\begin{aligned}
s_1^2&=R^2\frac{(\varrho_1-a_2)(\varrho_2-a_2)}{(a_3-a_2)(a_1-a_2)}\cos^2\phi,\\
s_2^2&=R^2\frac{(\varrho_1-a_2)(\varrho_2-a_2)}{(a_3-a_2)(a_1-a_2)}\sin^2\phi,\\
s_3^2&=R^2\frac{(\varrho_1-a_1)(\varrho_2-a_1)}{(a_2-a_1)(a_3-a_1)},\\
s_4^2&=R^2\frac{(\varrho_1-a_3)(\varrho_2-a_3)}{(a_2-a_3)(a_1-a_3)}.
\end{aligned}\right\} \tag{38}$$

In terms of the Jacobi elliptic functions we have ($-K\leqq\mu\leqq K$, $-2K'\leqq\nu\leqq 2K'$, $0\leqq\phi<2\pi$)

$$\left.\begin{aligned}
s_1&=R\,\mathrm{cn}(\mu,k)\,\mathrm{cn}(\nu,k')\cos\phi,\\
s_2&=R\,\mathrm{cn}(\mu,k)\,\mathrm{cn}(\nu,k')\sin\phi,\\
s_3&=R\,\mathrm{sn}(\mu,k)\,\mathrm{dn}(\nu,k'),\\
s_4&=R\,\mathrm{dn}(\mu,k)\,\mathrm{sn}(\nu,k').
\end{aligned}\right\} \tag{39}$$

For $R\to\infty$ the prolate spheroidal coordinate system in R^3 is recovered. The momentum operators are given by

$$p_\mu=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\mu}-\frac{k^2\,\mathrm{sn}\,\mu\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}-\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{2\,\mathrm{cn}\,\mu}\right), \tag{40}$$

$$p_\nu=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\nu}-\frac{k'^2\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu\,\mathrm{dn}\,\nu}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}-\frac{\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu}{2\,\mathrm{cn}\,\nu}\right), \tag{41}$$

and $p_\phi=-\mathrm{i}\hbar\partial_\phi$. Therefore we have for the Hamiltonian

$$\begin{aligned}
-\frac{\hbar^2}{2MR^2}\Delta_{S^{(3)}}=-\frac{\hbar^2}{2MR^2}\Bigg[&\frac{1}{(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)}\left(\frac{\partial^2}{\partial\mu^2}-\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{cn}\,\mu}\frac{\partial}{\partial\mu}\right.\\
&\left.+\frac{\partial^2}{\partial\nu^2}-\frac{\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu}{\mathrm{cn}\,\nu}\frac{\partial}{\partial\nu}\right)+\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\frac{\partial^2}{\partial\phi^2}\Bigg]
\end{aligned} \tag{42}$$

$$\begin{aligned}
&=\frac{\hbar^2}{2MR^2}\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}}(p_\mu^2+p_\nu^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}}\\
&\quad-\frac{\hbar^2}{8MR^2}\left[4+\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}\left(\frac{\mathrm{sn}^2\mu\,\mathrm{dn}^2\mu}{\mathrm{cn}^2\mu}+\frac{\mathrm{sn}^2\nu\,\mathrm{dn}^2\nu}{\mathrm{cn}^2\nu}\right)\right].
\end{aligned} \tag{43}$$

A potential separable in these coordinates must have the form

$$V(\vec{s})=\frac{V_1(\mu)+V_2(\nu)}{k^2\operatorname{cn}^2\mu+k'^2\operatorname{cn}^2\nu}+\frac{V_3(\phi)}{\operatorname{cn}^2\mu\operatorname{cn}^2\nu}. \tag{44}$$

Similarly as in the two-dimensional we can also perform a rotation of the coordinates $\vec{s}$ on $S^{(3)}$ by means of $\vec{s}\mapsto R(f)\vec{s}$ with the matrix $R(f)$ given by

$$R(f)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos f&\sin f\\0&0&-\sin f&\cos f\end{pmatrix}. \tag{45}$$

This gives the rotated elliptic cylindrical II system which has the form

$$\left.\begin{aligned}s_1&=R\operatorname{cn}(\mu,k)\operatorname{cn}(\nu,k')\cos\phi,\\s_2&=R\operatorname{cn}(\mu,k)\operatorname{cn}(\nu,k')\sin\phi,\\s_3&=R\big[k'\operatorname{sn}(\mu,k)\operatorname{dn}(\nu,k')+k\operatorname{dn}(\mu,k)\operatorname{sn}(\nu,k')\big],\\s_4&=R\big[k'\operatorname{dn}(\mu,k)\operatorname{sn}(\nu,k')-k\operatorname{sn}(\mu,k)\operatorname{dn}(\nu,k')\big].\end{aligned}\right\} \tag{46}$$

2.2.6. Ellipsoidal Coordinates: Let us consider the coordinate system defined by

$$\begin{aligned}&\frac{s_1^2}{\varrho_i-a_1}+\frac{s_2^2}{\varrho_i-a_2}+\frac{s_3^2}{\varrho_i-a_3}+\frac{s_4^2}{\varrho_i-a_4}=0,\qquad(i=1,2,3),\\&\text{and}\quad s_1^2+s_2^2+s_3^2+s_4^2=R^2.\end{aligned} \tag{47}$$

Explicitly ($a_1<\varrho_1<a_2<\varrho_2<a_3<\varrho_3<a_4$):

$$\left.\begin{aligned}s_1^2&=R^2\frac{(\varrho_1-a_2)(\varrho_2-a_1)(\varrho_3-a_1)}{(a_2-a_1)(a_3-a_1)(a_4-a_1)},\\s_2^2&=R^2\frac{(\varrho_1-a_2)(\varrho_2-a_2)(\varrho_3-a_2)}{(a_1-a_2)(a_3-a_2)(a_4-a_2)},\\s_3^2&=R^2\frac{(\varrho_1-a_3)(\varrho_2-a_3)(\varrho_3-a_3)}{(a_1-a_3)(a_2-a_3)(a_4-a_3)},\\s_4^2&=R^2\frac{(\varrho_1-a_4)(\varrho_2-a_4)(\varrho_3-a_4)}{(a_1-a_4)(a_2-a_4)(a_3-a_4)}.\end{aligned}\right\} \tag{48}$$

Unfortunately, the ellipsoidal system is a two-parametric system and to write it in terms of elliptic functions is difficult and cumbersome because we need two moduli. The metric tensor is given by

$$(g_{ab})=\operatorname{diag}\left(\frac{(\varrho_1^2-\varrho_2^2)(\varrho_1^2-\varrho_3^2)}{P(\varrho_1^2)},\frac{(\varrho_2^2-\varrho_1^2)(\varrho_2^2-\varrho_3^2)}{P(\varrho_2^2)},\frac{(\varrho_3^2-\varrho_1^2)(\varrho_3^2-\varrho_2^2)}{P(\varrho_3^2)}\right). \tag{49}$$

The momentum operators have the form

$$\left.\begin{aligned}
p_{\varrho_1}&=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\varrho_1}+\frac{1}{2}\,\frac{1}{\varrho_1-\varrho_2}+\frac{1}{2}\,\frac{1}{\varrho_1-\varrho_3}-\frac{1}{4}\,\frac{P'(\varrho_1)}{P(\varrho_1)}\right),\\
p_{\varrho_2}&=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\varrho_2}+\frac{1}{2}\,\frac{1}{\varrho_2-\varrho_1}+\frac{1}{2}\,\frac{1}{\varrho_2-\varrho_3}-\frac{1}{4}\,\frac{P'(\varrho_2)}{P(\varrho_2)}\right),\\
p_{\varrho_3}&=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\varrho_3}+\frac{1}{2}\,\frac{1}{\varrho_3-\varrho_1}+\frac{1}{2}\,\frac{1}{\varrho_3-\varrho_2}-\frac{1}{4}\,\frac{P'(\varrho_3)}{P(\varrho_3)}\right).
\end{aligned}\right\}\tag{50}$$

We obtain for the Hamiltonian

$$\begin{aligned}
&-\frac{\hbar^2}{2MR^2}\Delta_{E(3)}\\
&=-\frac{2\hbar^2}{M}\left[\frac{\sqrt{P(\varrho_1)}}{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}\frac{\partial}{\partial\varrho_1}\sqrt{P(\varrho_1)}\frac{\partial}{\partial\varrho_1}\right.\qquad(51)\\
&\qquad\left.+\frac{\sqrt{P(\varrho_2)}}{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}\frac{\partial}{\partial\varrho_2}\sqrt{P(\varrho_2)}\frac{\partial}{\partial\varrho_2}+\frac{\sqrt{P(\varrho_3)}}{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}\frac{\partial}{\partial\varrho_3}\sqrt{P(\varrho_3)}\frac{\partial}{\partial\varrho_3}\right]\\
&=\frac{1}{2MR^2}\left[\sum_{i=1}^{3}(g^{\varrho_i\varrho_i}p_{\varrho_i}^2)_{\mathrm{Weyl}}+\Delta V_{MP}(\{\varrho\})\right],\qquad(52)
\end{aligned}$$

where we have chosen in this case the Weyl ordering prescription for position and momentum operators [24, 40, 65, 74, 79]. $\Delta V_{MP}(\{\varrho\})$ is determined by the first term in (178) and reads as

$$\Delta V_{\varrho_1}(\{\varrho\})=\frac{\hbar^2}{8MR^2}\,\frac{A_{,\varrho_1\varrho_1}AB(A+B)-A_{,\varrho_1}^2(A^2+B^2)}{\hbar^2A^3B^3},\tag{53}$$

$\hbar^2=(a_1^2-a_2^2)^2/(a_1^2-a_3^2)$. By cyclic permutation we have for $\Delta V_{\varrho_2}=\Delta V_{\varrho_1}(A\to B, B\to C)$ and $\Delta V_{\varrho_3}=\Delta V_{\varrho_2}(B\to C, C\to A)$, together with $A=\varrho_1-\varrho_2$, $B=\varrho_2-\varrho_3$, $C=\varrho_3-\varrho_1$. A potential separable in ellipsoidal coordinates reads as

$$V(\vec{s})=\frac{(\varrho_2-\varrho_3)V_1(\varrho_1)+(\varrho_1-\varrho_3)V_2(\varrho_2)+(\varrho_1-\varrho_2)V_3(\varrho_3)}{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}.\tag{54}$$

3. Path Integral Formulation of the Smorodinsky-Winternitz Potentials on the Two-Dimensional Sphere

In table 1 we list the Smorodinsky-Winternitz potentials on the two-dimensional sphere together with the separating coordinate systems. The cases where an explicit path integration is possible are underlined.

Table 1

Smorodinsky-Winternitz potentials on the two-dimensional sphere

Potential $V(\vec{s})$	Coordinate System
$V_1=\frac{M}{2}\omega^2R^2\frac{s_1^2+s_2^2}{s_3^2}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2}+\frac{k_2^2-\frac{1}{4}}{s_2^2}\right)$	Spherical Elliptic
$V_2=-\frac{\alpha}{R}\frac{s_3}{\sqrt{s_1^2+s_2^2}}+\frac{\hbar^2}{4M\sqrt{s_1^2+s_2^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{k_2^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}-s_1}\right)$	Spherical Elliptic II

3.1. We consider the potential ($k_{1,2}>0$)

$$V_1(\vec{s})=\frac{M}{2}\omega^2R^2\frac{s_1^2+s_2^2}{s_3^2}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2}+\frac{k_2^2-\frac{1}{4}}{s_2^2}\right) \tag{55}$$

which in spherical and elliptic coordinates has the form

Spherical:

$$V_1(\vec{s})=\frac{M}{2}\omega^2R^2\tan^2\theta+\frac{\hbar^2}{2MR^2\sin^2\theta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\phi}\right) \tag{56}$$

Elliptic:

$$=\frac{M}{2}\omega^2R^2\frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}+\frac{\hbar^2}{2MR^2}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}+\frac{k_2^2-\frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right). \tag{57}$$

Both potentials are separable in the two coordinate systems on the sphere $S^{(2)}$. In the case of the Kepler problem the coordinate system must be rotated. We have for V_1 the path integral representations

$K^{(V_1)}(\vec{s}'',\vec{s}';T)$

Spherical:

$$=\int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\mathscr{D}\theta(t)\sin\theta\int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathscr{D}\phi(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}R^2(\dot\theta^2+\sin^2\theta\dot\phi^2-\omega^2\tan^2\theta)\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sin^2\theta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\phi}-\frac{1}{4}\right)-\frac{1}{4}\right)\right]dt\right\} \tag{58}$$

Elliptic, $\lambda_2^2=(1+4M^2\omega^2R^4/\hbar^2)/4$:

$$=\frac{1}{R^2}\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathscr{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{M}{2}R^2\left((k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2)-\omega^2\frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right)\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2MR^2}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right)\right]dt\right\} \tag{59}$$

$$=\frac{\mathrm{e}^{\mathrm{i}M\omega^2R^2T/2\hbar}}{R^2}\int\limits_{\mathbb{R}}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int\limits_0^\infty ds''\int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''}\mathscr{D}\mu(s)\int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''}\mathscr{D}\nu(s)$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{M}{2}R^2(\dot\mu^2+\dot\nu^2)-E(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)\right.\right. \tag{60}$$
$$\left.\left.-\frac{\hbar^2}{2MR^2}\left((\lambda_2^2-\tfrac14)\left(\frac{1}{\mathrm{sn}^2\nu}-\frac{1}{\mathrm{dn}^2\mu}\right)+(k_2^2-\tfrac14)\left(\frac{1}{\mathrm{cn}^2\mu}-\frac{1}{\mathrm{cn}^2\nu}\right)\right)\right]ds\right\}.$$

The path integral in spherical coordinates in easily evaluated by successively using the path integral identity of the Pöschl-Teller potential and applying the technique of separation of variables [27, 28, 38]. Let us denote $\lambda_1=2m\pm k_1\pm k_2+1$ and $\lambda_2^2=(1+4M^2\omega^2R^4/\hbar^2)/4$. We obtain with the principal quantum number $N=l+m$ $((\phi,\theta)\in[0,\frac{\pi}{2}])$

$$K^{(V_1)}(\vec s'',\vec s';T)=\sum_{m=0}^\infty\sum_{N=0}^\infty\Psi_{N,m}(\theta'',\phi'';R)\,\Psi_{N,m}(\theta',\phi';R)\,\mathrm{e}^{-\mathrm{i}E_NT/\hbar}, \tag{61}$$

$$\Psi_{N,m}(\theta,\phi;R)=(\sin\theta)^{-1/2}\,\Phi_m^{(\pm k_2,\pm k_1)}(\phi)\,S_{N,m}(\theta;R) \tag{62}$$

$$\Phi_m^{(\pm k_2,\pm k_1)}(\phi)=\left[2(1+2m\pm k_1\pm k_2+1)\frac{m!\,\Gamma(m\pm k_1\pm k_2+1)}{\Gamma(1+m\pm k_1)\,\Gamma(1+m\pm k_2)}\right]^{1/2}$$
$$\times(\sin\phi)^{1/2\pm k_2}(\cos\phi)^{1/2\pm k_1}\,P_m^{(\pm k_2,\pm k_1)}(\cos2\phi) \tag{63}$$

$$S_{N,m}(\theta;R)=\frac1R\left[2(1+2N+\lambda_1+\lambda_2-2m)\frac{(N-m)!\,\Gamma(1+N-m+\lambda_1+\lambda_2)}{\Gamma(1+N-m+\lambda_1)\,\Gamma(1+N-m+\lambda_2)}\right]^{1/2}$$
$$\times(\sin\theta)^{\lambda_1+1/2}(\cos\theta)^{\lambda_2+1/2}\,P_{N-m}^{(\lambda_2,\lambda_1)}(\cos2\theta), \tag{64}$$

$$E_N=\frac{\hbar^2}{2MR^2}\left[(2N\pm k_1\pm k_2+2)(2N\pm k_1\pm k_2+3)+(2\lambda_2-1)(2N\pm k_1\pm k_2+2)\right]. \tag{65}$$

Let us discuss shortly the cases $0<k_i\leqq\frac12$ $(i=1,2)$, and the degeneracies of the Higgs oscillator on the sphere. If $0<k_{1,2}\leqq\frac12$ we have for each $N=l+m$ four possibilities of parities of the levels, i.e. $(\pm,\pm)$; for the cases $0<k_1\leqq\frac12$ and $k_2>\frac12$ or $0<k_2\leqq\frac12$ and $k_1>\frac12$ we have for each N two possibilities: $(\pm)$; for $k_{1,2}>\frac12$ there is only one possibility: $(+)$. In all cases the degeneracy is $d=N+1=2j+1$ $(j=0,\frac12,1,\ldots)$, coinciding with the dimensions of all irreducible representations of the group $SU(2)$. This is exactly the same behaviour as in the two-dimensional flat-space case [23, 38].

In the limit $R \to \infty$ ($\lambda_2 \to M\omega R^2/\hbar$) we obtain

$$E_N \cong \hbar\omega(2N + 2 \pm k_1 \pm k_2), \tag{66}$$

which is the correct behaviour for the corresponding two-dimensional maximally super-integrable Smorodinsky-Winternitz potential in $\mathbb{R}^2$ [38]. In the limiting case $\omega \to 0$ ($\lambda_2 \to \frac{1}{2}$) we obtain

$$E_N = \frac{\hbar^2}{2MR^2}(2N \pm k_1 \pm k_2 + 2)(2N \pm k_1 \pm k_2 + 3) = \frac{\hbar^2}{2M}\frac{l(l+1)}{R^2}, \tag{67}$$

($l = 2N \pm k_1 \pm k_2 + 2$) which corresponds to the case where just a radial part is present which has the same feature as the spectrum of the free motion on $S^{(2)}$.

In elliptic coordinates we have by means of the path integral identity of appendix B the following formal solution

$$K^{(V_1)}(\vec{s}'', \vec{s}'; T) = [M(\mu', \nu')\, M(\mu'', \nu'')]^{-1/2}$$
$$\mathop{PS}_{m} \sum_{N=0}^{\infty} e^{-iE_N T/\hbar}\, \Psi_{N,m}^{(k_1,k_2,k_3)}(\mu'', \nu'')\, \Psi_{N,m}^{(k_1,k_2,k_3)*}(\mu', \nu'), \tag{68}$$

with the principal quantum number $N \in \mathbb{N}_0$, $M(\mu, \nu)$ as in appendix B, and the same energy spectrum as in (65).

Note that we have performed explicitly a time-transformation in (60) to show the separability in these coordinates. However, only the formal expansion (68) is known.

3.2. We consider the potential ($k_{1,2} > 0$)

$$V_2(\vec{s}) = -\frac{\alpha}{R}\frac{s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\hbar^2}{4M\sqrt{s_1^2+s_2^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}+s_1} + \frac{k_2^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}-s_1}\right), \tag{69}$$

which in spherical and elliptic II coordinates has the form

Spherical:

$$V_2(\vec{s}) = -\frac{\alpha}{R}\cot\theta + \frac{\hbar^2}{8MR^2\sin^2\theta}\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\frac{\phi}{2}} + \frac{k_1^2-\frac{1}{4}}{\sin^2\frac{\phi}{2}}\right) \tag{70}$$

Elliptic II:

$$= -\frac{\alpha}{R}\frac{k'\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu - k\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu} + \frac{\hbar^2}{2MR^2(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)}$$
$$\times\left(\frac{(k_1^2+k_2^2-\frac{1}{2})k'^2 + (k_2^2-k_1^2)k'\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{cn}^2\mu} + \frac{(k_1^2+k_2^2-\frac{1}{2})k^2 + (k_2^2-k_1^2)k\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu}{\mathrm{cn}^2\nu}\right). \tag{71}$$

We state the two path integral representations, where we have used in the elliptic case the rotated version

$$K^{(V_2)}(\vec{s}\,'', \vec{s}\,'; T)$$

Spherical:

$$= \frac{1}{R^2} \int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathcal{D}\theta(t) \sin\theta \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t) \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2)+\frac{\alpha}{R}\cot\theta \right.\right.$$

$$\left.\left. - \frac{\hbar^2}{8MR^2\sin^2\theta}\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\frac{\phi}{2}}+\frac{k_1^2-\frac{1}{4}}{\sin^2\frac{\phi}{2}}-\frac{1}{4}\right)\right]dt+\frac{i\hbar T}{8MR^2}\right\} \tag{72}$$

Elliptic II:

$$= \frac{1}{R^2} \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)\,(\dot{\mu}^2+\dot{\nu}^2)+\frac{\alpha}{R}\,\frac{k'\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu-k\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}\right.\right.$$

$$-\frac{\hbar^2}{2MR^2(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)}\left(\frac{(k_1^2+k_2^2-\frac{1}{2})k'^2+(k_2^2-k_1^2)k'\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{cn}^2\mu}\right.$$

$$\left.\left.\left.+\frac{(k_1^2+k_2^2-\frac{1}{2})k^2+(k_2^2-k_1^2)k\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu}{\mathrm{cn}^2\nu}\right)\right]dt\right\}. \tag{73}$$

Evaluating the path integral in spherical coordinates, the ϕ path integration is of the form of the symmetric Pöschl-Teller potential yielding

$$K^{(V_2)}(\vec{s}\,'', \vec{s}\,'; T)$$

$$= \frac{1}{2R^2}(\sin\theta'\sin\theta'')^{-1/2}\,e^{i\hbar T/8MR^2}\sum_{m=0}^{\infty}\Phi_m^{(\pm k_1,\,\pm k_2)}\left(\frac{\phi''}{2}\right)\Phi_m^{(\pm k_1,\,\pm k_2)}\left(\frac{\phi'}{2}\right)$$

$$\times\int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\mathcal{D}\theta(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{MR^2}{2}\,\dot{\theta}^2+\frac{\alpha}{R}\cot\theta-\frac{\hbar^2}{2MR^2}\,\frac{\lambda^2-\frac{1}{4}}{\sin^2\theta}\right)dt\right\}, \tag{74}$$

where $\lambda = m + (1 \pm k_1 \pm k_2)/2$ and the Pöschl-Teller wave functions $\Phi_m^{(\pm k_1,\,\pm k_2)}$. The remaining θ-path integration, denoted by $K_m^{(\alpha)}(T)$ in the following, is of the form of a trigonometric version of the Manning-Rosen potential [71], which in turn can be transformed into the path integral problem of the modified Pöschl-Teller problem [5, 28]. In order to do this we perform the transformation [5]

$$e^r = \tanh(i\theta/2) \tag{75}$$

together with the time transformation $dt = f(r)\,ds$ and $f(r) = R^2\tanh^2 r$. The transformation (75) can be interpreted as a one-dimensional version of the Kustaanheimo-Stiefel transformation in flat space [17, 18, 64]. This gives the space-time transformed path integral formulæ ($A = -i\alpha/R$)

$$K_m^{(\alpha)}(\theta'', \theta'; T) = \int_{\mathbb{R}} \frac{dE}{2\pi \mathrm{i}} e^{-\mathrm{i}TE/\hbar} G_m^{(\alpha)}(r'', r'; E) \tag{76}$$

$$G_m^{(\alpha)}(r'', r'; E) = \frac{\mathrm{i}}{\hbar} [f(r'') f(r')]^{1/4} \int_0^\infty ds'' K_m^{(\alpha)}(r'', r'; s'') \tag{77}$$

$$\tilde{K}_m^{(\alpha)}(r'', r'; s'') = e^{\mathrm{i}s''(E+A)R^2/\hbar} \int_{r(0)=r'}^{r(s'')=r''} \mathscr{D}r(s)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar} \int_0^{s''} \left(\frac{M}{2} R^2 \dot{r}^2 - \hbar^2 \frac{2MR^2(E-A)/\hbar^2 + \frac{1}{4}}{2MR^2 \cosh^2 r} - \hbar^2 \frac{4\lambda^2 - \frac{1}{4}}{2MR^2 \sinh^2 r}\right) ds\right]. \tag{78}$$

Inserting now the path integral solution of the modified Pöschl-Teller potential, cf. appendix A, we find that the result has the form

$$K(\theta'', \theta', \phi'', \phi'; T) = \sum_{m=0}^{\infty} \sum_{N=0}^{\infty} e^{-\mathrm{i}E_N T/\hbar} \Psi_{N,m}^*(\theta', \phi'; R) \Psi_{N,m}(\theta'', \phi''; R), \tag{79}$$

with the wave functions and the energy-spectrum given by ($\tilde{N} = N + \lambda + \frac{1}{2}$, $a = \hbar^2/M\alpha$, $\sigma_N = R/a\tilde{N}\,(\phi, \theta) \in [0, \pi]$)

$$\Psi_{N,m}(\theta, \phi; R) = (\sin\theta)^{-1/2} \Phi_m^{(\pm k_1, \pm k_2)}(\phi) S_{N,m}(\theta; R) \tag{80}$$

$$\Phi_m^{(\pm k_1, \pm k_2)}(\phi) = \left[(1 + 2m \pm k_1 \pm k_2) \frac{m!\, \Gamma(1 + m \pm k_1 \pm k_2)}{\Gamma(1 + m \pm k_1)\, \Gamma(1 + n \pm k_2)}\right]^{1/2}$$
$$\times \left(\sin\frac{\phi}{2}\right)^{1/2 \pm k_2} \left(\cos\frac{\phi}{2}\right)^{1/2 \pm k_1} P_m^{(\pm k_2, \pm k_1)}(\cos\phi) \tag{81}$$

$$S_{N,m}(\theta; R) = \frac{1}{\Gamma(2\lambda + 1)} \left[\frac{\sigma_N^2 + \tilde{N}^2}{R^2 \tilde{N}^2} \frac{\Gamma(\tilde{N} + \lambda + \frac{1}{2})\, \Gamma(\mathrm{i}\sigma_N + \lambda + \frac{1}{2})}{N!\, \Gamma(\mathrm{i}\sigma_N - \lambda + \frac{1}{2})}\right]^{1/2}$$
$$\times (2\sin\theta)^{\lambda + 1/2} \exp[\mathrm{i}\theta(\mathrm{i}\sigma_N - N)]\, {}_2F_1(-N, \lambda + \tfrac{1}{2} + \mathrm{i}\sigma_N; 2\lambda + 1; 1 - e^{2\mathrm{i}\theta}), \tag{82}$$

$$E_N = \hbar^2 \frac{\tilde{N}^2 - \frac{1}{4}}{2MR^2} - \frac{M\alpha^2}{2\hbar^2 \tilde{N}^2}. \tag{83}$$

Cf. [5] for the flat space limit. The path integral in elliptic II coordinates is intractable.

4. Path Integral Formulation of the Smorodinsky-Winternitz Potentials on the Three-Dimensional Sphere

In table 2 we list the Smorodinsky-Winternitz potentials on the three-dimensional sphere together with the separating coordinate systems. The cases where an explicit path integration is possible are underlined.

Table 2
Smorodinsky-Winternitz potentials on the three-dimensional sphere

Potential $V(\vec{s})$	Coordinate System
$V_1=\frac{M}{2}\omega^2R^2\frac{s_1^2+s_2^2+s_3^2}{s_4^2}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2}+\frac{k_2^2-\frac{1}{4}}{s_2^2}+\frac{k_3^2-\frac{1}{4}}{s_3^2}\right)$	Spherical Cylindrical Sphero-Elliptic Elliptic-Cylindrical I Elliptic-Cylindrical II Ellipsoidal
$V_2=-\frac{\alpha}{R}\frac{s_4}{\sqrt{s_1^2+s_2^2+s_3^2}}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2}+\frac{k_2^2-\frac{1}{4}}{s_2^2}\right)$	Spherical Sphero-Elliptic Elliptic-Cylindrical II*
$V_3=\frac{\hbar^2}{2M}\left[\frac{1}{\sqrt{s_1^2+s_2^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{k_2^2-\frac{1}{4}}{\sqrt{s_1^2+s_2^2}-s_1}\right)+\frac{k_3^2-\frac{1}{4}}{s_3^2}\right]$	Spherical Cylindrical
$V_4=\frac{M}{2}\omega^2R^2\frac{s_1^2+s_2^2+s_3^2}{s_4^2}+\frac{\hbar^2}{2M}\left(\frac{k_3^2-\frac{1}{4}}{s_3^2}+\frac{F(s_2/s_1)}{s_1^2+s_2^2}\right)$	Spherical Cylindrical Elliptic-Cylindrical I Elliptic-Cylindrical II
$V_5=-\frac{\alpha}{R}\frac{s_4}{\sqrt{s_1^2+s_2^2+s_3^2}}+\frac{\hbar^2}{2M(s_1^2+s_2^2)}\left[\frac{\beta s_3}{\sqrt{s_1^2+s_2^2+s_3^2}}+F\left(\frac{s_2}{s_1}\right)\right]$	Spherical Elliptic-Cylindrical II*
$V_6=F(\chi)+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2}+\frac{k_2^2-\frac{1}{4}}{s_2^2}+\frac{k_3^2-\frac{1}{4}}{s_3^2}+\frac{k_4^2-\frac{1}{4}}{s_4^2}\right)$	Spherical Sphero-Elliptic

* after rotation

4.1. We consider the potential ($k_{1,2,3}>0$)

$$V_1(\vec{s})=\frac{M}{2}\omega^2R^2\frac{s_1^2+s_2^2+s_3^2}{s_4^2}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2}+\frac{k_2^2-\frac{1}{4}}{s_2^2}+\frac{k_3^2-\frac{1}{4}}{s_3^2}\right), \tag{84}$$

which reads in the six separating coordinate systems as follows

Spherical:

$$V_1(\vec{s}) = \frac{M}{2}\,\omega^2 R^2 \tan^2\chi + \frac{\hbar^2}{2MR^2\sin^2\chi}\left[\frac{k_3^2-\frac{1}{4}}{\cos^2\theta} + \frac{1}{\sin^2\theta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\phi}\right)\right] \tag{85}$$

Cylindrical:

$$\begin{aligned} &= \frac{M}{2}\,\omega^2 R^2 \frac{1-\cos^2\theta\sin^2\phi_2}{\cos^2\theta\sin^2\phi_2} \\ &\quad + \frac{\hbar^2}{2MR^2}\left[\frac{1}{\sin^2\theta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi_1} + \frac{k_2^2-\frac{1}{4}}{\sin^2\phi_1}\right) + \frac{1}{\cos^2\theta}\,\frac{k_3^2-\frac{1}{4}}{\cos^2\phi_2}\right] \end{aligned} \tag{86}$$

Sphero-Elliptic:

$$= \frac{M}{2}\,\omega^2 R^2 \tan\chi^2 + \frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu} + \frac{k_2^2-\frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu} + \frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\right) \tag{87}$$

Elliptic-Cylindrical I:

$$= \frac{M}{2}\,\omega^2 R^2 \frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu} + \frac{\hbar^2}{2MR^2}\left[\frac{1}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\phi}\right) + \frac{k_3^2-\frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right] \tag{88}$$

Elliptic-Cylindrical II:

$$= \frac{M}{2}\,\omega^2 R^2 \frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu} + \frac{\hbar^2}{2MR^2}\left[\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi} + \frac{k_2^2-\frac{1}{4}}{\sin^2\phi}\right) + \frac{k_3^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\right] \tag{89}$$

Ellipsoidal ($a_{ij} = a_i - a_j$):

$$\begin{aligned} &= \frac{M}{2}\,\omega^2 R^2 \left[a_{14}a_{24}a_{34}\left(\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\,\frac{1}{\varrho_3-a_4}\right.\right. \\ &\quad \left.\left. + \frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\,\frac{1}{\varrho_2-a_4} + \frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\,\frac{1}{\varrho_1-a_4}\right) - 1\right) \\ &\quad + \frac{\hbar^2}{2MR^2}\left\{\frac{1}{(\varrho_1-\varrho_3)(\varrho_2-\varrho_3)}\right. \\ &\quad \times\left[a_{31}a_{21}a_{41}\frac{k_1^2-\frac{1}{4}}{\varrho_3-a_1} + a_{12}a_{32}a_{42}\frac{k_2^2-\frac{1}{4}}{\varrho_3-a_2} + a_{13}a_{23}a_{43}\frac{k_3^2-\frac{1}{4}}{\varrho_3-a_3}\right] \\ &\quad + \frac{1}{(\varrho_1-\varrho_2)(\varrho_3-\varrho_2)}\left[a_{31}a_{21}a_{41}\frac{k_1^2-\frac{1}{4}}{\varrho_2-a_1} + a_{12}a_{32}a_{42}\frac{k_2^2-\frac{1}{4}}{\varrho_2-a_2} + a_{13}a_{23}a_{43}\frac{k_3^2-\frac{1}{4}}{\varrho_2-a_3}\right] \\ &\quad \left. + \frac{1}{(\varrho_2-\varrho_1)(\varrho_3-\varrho_1)}\left[a_{31}a_{21}a_{41}\frac{k_1^2-\frac{1}{4}}{\varrho_1-a_1} + a_{12}a_{32}a_{42}\frac{k_2^2-\frac{1}{4}}{\varrho_1-a_2} + a_{13}a_{23}a_{43}\frac{k_3^2-\frac{1}{4}}{\varrho_1-a_3}\right]\right\}. \end{aligned} \tag{90}$$

We have the path integral representations

$K^{(V_1)}(\vec{s}'', \vec{s}'; T)$

Spherical:

$$= \frac{1}{R^3} \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t) \sin^2\chi \int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathcal{D}\theta(t) \sin\theta \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot{\chi}^2+\sin^2\chi(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)-\omega^2\tan^2\chi)\right.\right.$$

$$\left.\left. -\frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{k_3^2-\frac{1}{4}}{\cos^2\theta}+\frac{1}{\sin^2\theta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\phi}-\frac{1}{4}\right)-\frac{1}{4}\right)+\frac{\hbar^2}{2MR^2}\right]dt\right\} \tag{91}$$

Cylindrical:

$$= \frac{1}{R^3} \int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathcal{D}\theta(t) \sin\theta\cos\theta \int\limits_{\phi_1(t')=\phi_1'}^{\phi_1(t'')=\phi_1''} \mathcal{D}\phi_1(t) \int\limits_{\phi_2(t')=\phi_2'}^{\phi_2(t'')=\phi_2''} \mathcal{D}\phi_2(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\left(\dot{\theta}^2+\cos^2\theta\dot{\phi}_1^2+\sin^2\theta\dot{\phi}_2^2-\omega^2\frac{1-\cos^2\theta\sin^2\phi_2}{\cos^2\theta\sin^2\phi_2}\right)\right.\right.$$

$$\left.\left. -\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sin^2\theta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi_1}+\frac{k_2^2-\frac{1}{4}}{\sin^2\phi_1}-\frac{1}{4}\right)+\frac{1}{\cos^2\theta}\left(\frac{k_3^2-\frac{1}{4}}{\cos^2\phi_2}-\frac{1}{4}\right)-1\right)\right]dt\right\} \tag{92}$$

Sphero-Elliptic:

$$= \frac{1}{R^3} \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t) \sin^2\chi \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot{\chi}^2+\sin^2\chi(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)(\dot{\mu}^2+\dot{\nu}^2)-\omega^2\tan\chi^2)\right.\right.$$

$$\left.\left. -\frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}+\frac{k_2^2-\frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}+\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\right)+\frac{\hbar^2}{2MR^2}\right]dt\right\} \tag{93}$$

Elliptic-Cylindrical I

$$= \frac{1}{R^3} \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\left((k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)(\dot{\mu}^2+\dot{\nu}^2)+\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu\,\dot{\phi}^2-\omega^2\frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right)\right.\right.$$

$$-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\mathrm{sn}^2\mu\,\mathrm{sn}^2\nu}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\phi}\right)+\frac{k_3^2-\frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right)$$

$$\left.\left. +\frac{\hbar^2}{8MR^2}\left(4+\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}\left(\frac{\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu}{\mathrm{sn}^2\mu}+k'^4\frac{\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu}{\mathrm{dn}^2\nu}\right)\right)\right]dt\right\} \tag{94}$$

Elliptic-Cylindrical II

$$
=\frac{1}{R^3}\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathscr{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)\,\mathrm{cn}\,\nu\,\mathrm{cn}\,\nu\int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathscr{D}\phi(t)
$$

$$
\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\left((k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2)+\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu\,\dot\phi^2-\omega^2\,\frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right)\right.\right.
$$

$$
-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\left(\frac{k_1^2-\frac14}{\cos^2\phi}+\frac{k_2^2-\frac14}{\sin^2\phi}\right)+\frac{k_3^2-\frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\right)
$$

$$
\left.\left.+\frac{\hbar^2}{8MR^2}\left(4+\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}\left(\frac{\mathrm{sn}^2\mu\,\mathrm{dn}^2\mu}{\mathrm{cn}^2\mu}+\frac{\mathrm{sn}^2\nu\,\mathrm{dn}^2\nu}{\mathrm{cn}^2\nu}\right)\right)\right]dt\right\} \tag{95}
$$

Ellipsoidal:

$$
=\frac{1}{R^3}\int\limits_{\varrho_1(t')=\varrho_1'}^{\varrho_1(t'')=\varrho_1''}\mathscr{D}_{MP}\varrho_1(t)\int\limits_{\varrho_2(t')=\varrho_2'}^{\varrho_2(t'')=\varrho_2''}\mathscr{D}_{MP}\varrho_2(t)\int\limits_{\varrho_3(t')=\varrho_3'}^{\varrho_3(t'')=\varrho_3''}\mathscr{D}_{MP}\varrho_3(t)\,\frac{(\varrho_2-\varrho_1)(\varrho_3-\varrho_2)(\varrho_3-\varrho_1)}{8\sqrt{P(\varrho_1)P(\varrho_2)P(\varrho_3)}}
$$

$$
\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\sum_{i=1}^{3}g_{\varrho_i\varrho_i}\dot\varrho_i^2-V_1(\vec s)-\Delta V_{PF}(\vec s)\right]dt\right\}. \tag{96}
$$

In ellipsoidal coordinates the notation is for the metric and the quantum potential as in (49, 53), respectively, and V_1 as in (90). Note that this path integral has been written for convenience in the mid-point formulation. We consider the first three coordinate systems. In the cylindrical and spherical system we apply successively the path integral solution of the Pöschl-Teller potential. In the sphero-elliptic system the (μ,ν) subsystem is separated with some quantum numbers N, m, where we know from the spherical system and appendix B that we have for the principal quantum number $N\in\mathbb{N}_0$. Thus we obtain $(\lambda_3^2=(1+4M^2\omega^2R^4/\hbar^2)/4)$

$K^{(V_1)}(\vec s'',\vec s';T)$

Spherical, $\lambda_1=2n\pm k_1\pm k_2+1,\ \lambda_2=2m+\lambda_1\pm k_3+1;\ (\chi,\theta,\phi)\in[0,\frac{\pi}{2}]$:

$$
=\sum_{n,m,N=0}^{\infty}\mathrm{e}^{-\mathrm{i}E_lT/\hbar}\,\Psi^*_{N,m,n}(\chi',\theta',\phi';R)\,\Psi_{N,m,n}(\chi'',\theta'',\phi'';R), \tag{97}
$$

$$
\Psi_{N,m,n}(\chi,\theta,\phi;R)=(\sin^2\chi\sin\theta)^{-1/2}\,\Phi_m^{(\pm k_2,\pm k_1)}(\phi)\,\Phi_n^{(\lambda_1,\pm k_3)}(\theta)\,S_{N,m,n}^{(\lambda_2,\lambda_3)}(\chi;R), \tag{98}
$$

$$
\Phi_m^{(\pm k_2,\pm k_1)}(\phi)=\left[2(2m\pm k_1\pm k_2+1)\,\frac{m!\,\Gamma(m\pm k_1\pm k_2+1)}{\Gamma(m\pm k_1+1)\,\Gamma(m\pm k_2+1)}\right]^{1/2}
$$

$$
\times(\sin\phi)^{1/2\pm k_2}(\cos\phi)^{1/2\pm k_1}\,P_m^{(\pm k_2,\pm k_1)}(\cos2\phi), \tag{99}
$$

$$\Phi_n^{(\lambda_1, \pm k_3)}(\theta) = \left[2(2n \pm \lambda_1 \pm k_3 + 1) \frac{n!\, \Gamma(n + \lambda_1 \pm k_3 + 1)}{\Gamma(n + \lambda_1 + 1)\, \Gamma(n \pm k_3 + 1)} \right]^{1/2} \times (\sin\theta)^{1/2 + \lambda_1} (\cos\theta)^{1/2 \pm k_3} P_n^{(\lambda_1, \pm k_3)}(\cos 2\theta), \tag{100}$$

$$S_{N,m,n}^{(\lambda_2,\lambda_3)}(\chi; R) = \left[2(2(N-n-m) + \lambda_2 + \lambda_3 + 1) \times \frac{(N-n-m)!\, \Gamma((N-n-m) + \lambda_2 + \lambda_3 + 1)}{R^3\, \Gamma((N-n-m) + \lambda_2 + 1)\, \Gamma((N-n-m) + \lambda_3 + 1)} \right]^{1/2} \times (\sin\chi)^{1/2 + \lambda_2} (\cos\chi)^{1/2 + \lambda_3} P_{N-n-m}^{(\lambda_2, \lambda_3)}(\cos 2\chi), \tag{101}$$

Cylindrical, $\lambda_2 = 2n_2 \pm k_3 \pm k_4 + 1$, $\lambda_1 = 2n_1 + \lambda_3 \pm k_2 + 1$; $(\theta, \phi_{1,2}) \in [0, \frac{\pi}{2}]$:

$$= \sum_{N, n_{1,2} = 0}^{\infty} e^{-iE_N T/\hbar}\, \Psi_{N,n_1,n_2}(\theta'', \phi_1'', \phi_2''; R)\, \Psi_{N,n_1,n_2}^*(\theta', \phi_1', \phi_2'; R), \tag{102}$$

$$\Psi_{N,n_1,n_2}(\chi, \theta, \phi; R) = (\sin\theta\cos\theta)^{-1/2}\, \Phi_{n_1}^{(\pm k_2, \pm k_1)}(\phi_1)\, \Phi_{n_2}^{(\lambda_3, \pm k_3)}(\phi_2)\, S_{N,n_1,n_2}^{(\lambda_1,\lambda_2)}(\theta; R), \tag{103}$$

$$\Phi_{n_1}^{(\pm k_2, \pm k_1)}(\phi_1) = \left[2(2n_1 \pm k_1 \pm k_2 + 1) \frac{n_1!\, \Gamma(n_1 \pm k_1 \pm k_2 + 1)}{\Gamma(n_1 \pm k_1 + 1)\, \Gamma(n_1 \pm k_2 + 1)} \right]^{1/2} \times (\sin\phi_1)^{1/2 \pm k_2} (\cos\phi_1)^{1/2 \pm k_1} P_{n_1}^{(\pm k_2, \pm k_1)}(\cos 2\phi_1), \tag{104}$$

$$\Phi_{n_2}^{(\lambda_3, \pm k_3)}(\phi_2) = \left[2(2n_2 + \lambda_3 \pm k_3 + 1) \frac{n_2!\, \Gamma(n_2 + \lambda_3 \pm k_3 + 1)}{\Gamma(n_2 + \lambda_3 + 1)\, \Gamma(n_2 \pm k_3 + 1)} \right]^{1/2} \times (\sin\phi_2)^{1/2 + \lambda_3} (\cos\phi_2)^{1/2 \pm k_3} P_{n_2}^{(\lambda_3, \pm k_3)}(\cos 2\phi_2), \tag{105}$$

$$S_{N,n_1,n_2}^{(\lambda_1,\lambda_2)}(\theta; R) = \left[2(2(N-n_1-n_2) + \lambda_1 + \lambda_2 + 1) \times \frac{(N-n_1-n_2)!\, \Gamma((N-n_1-n_2) + \lambda_1 + \lambda_2 + 1)}{R^3\, \Gamma((N-n_1-n_2) + \lambda_1 + 1)\, \Gamma((N-n_1-n_1) + \lambda_2 + 1)} \right]^{1/2} \times (\sin\theta)^{1/2 + \lambda_1} (\cos\theta)^{1/2 + \lambda_2} P_{N-n_1-n_2}^{(\lambda_1, \lambda_2)}(\cos 2\theta), \tag{106}$$

Sphero-Elliptic, $\lambda_2 = 2m \pm k_1 \pm k_2 \pm k_3 + 2$, $\lambda_3^2 = (1 + 4M^2\omega^2 R^4/\hbar^2)/4$:

$$= \sum_n \sum_{m=0}^{\infty} \sum_{N=0}^{\infty} e^{-iE_l T/\hbar}\, \Psi_{N,n,m}(\chi'', \mu'', \nu''; R)\, \Psi_{N,n,m}^*(\chi', \mu', \nu'; R), \tag{107}$$

$$\Psi_{N,n,m}(\chi, \mu, \nu; R) = [\sin^2\chi\, M(\mu, \nu)]^{-1/2}\, \Psi_{nm}^{(k_1,k_2,k_3)}(\mu, \nu)\, S_{N,m,n}^{(\lambda_2,\lambda_3)}(\chi; R), \tag{108}$$

$$E_N = \frac{\hbar^2}{2MR^2} \left[(2N + 3 \pm k_1 \pm k_2 \pm k_3 + \lambda_3)^2 - 1 - \frac{M^2\omega^2 R^4}{\hbar^2} \right]. \tag{109}$$

The wave functions $\Psi_{nm}^{(k_1,k_2,k_3)}(\mu,\nu)$ are as introduced in appendix B.

It is easily checked that that in the limit $R\to\infty$ the flat space result is reproduced [19, 38]. For the remaining three coordinate systems no path integration is possible.

4.2. We consider the potential ($k_{1,2}>0$)

$$V_2(\vec{s})=-\frac{\alpha}{R}\frac{s_4}{\sqrt{s_1^2+s_2^2+s_3^2}}+\frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac14}{s_1^2}+\frac{k_2^2-\frac14}{s_2^2}\right), \tag{110}$$

which has in the three separating coordinate systems the following form

Spherical:

$$V_2(\vec{s})=-\frac{\alpha}{R}\cot\chi+\frac{\hbar^2}{2MR^2\sin^2\chi\sin^2\theta}\left(\frac{k_1^2-\frac14}{\cos^2\phi}+\frac{k_2^2-\frac14}{\sin^2\phi}\right) \tag{111}$$

Sphero-Elliptic:

$$=-\frac{\alpha}{R}\cot\chi+\frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right) \tag{112}$$

Elliptic-Cylindrical II (rotated):

$$=-\frac{\alpha}{R}\frac{k'\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu-k\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}+\frac{\hbar^2}{2MR^2}\frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\left(\frac{k_1^2-\frac14}{\cos^2\phi}+\frac{k_2^2-\frac14}{\cos^2\phi}\right). \tag{113}$$

We have the path integral representations

$K^{(V_2)}(\vec{s}\,'',\vec{s}\,';T)$

Spherical:

$$\begin{aligned}&=\frac{1}{R^3}\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathscr{D}\chi(t)\sin^2\chi\int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\mathscr{D}\theta(t)\sin\theta\int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathscr{D}\phi(t)\\&\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot\chi^2+\sin^2\chi(\dot\theta^2+\sin^2\theta\dot\phi^2))+\frac{\alpha}{R}\cot\chi\right.\right.\\&\left.\left.-\frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{1}{\sin^2\theta}\left(\frac{k_1^2-\frac14}{\cos^2\phi}+\frac{k_2^2-\frac14}{\sin^2\phi}-\frac14\right)-\frac14\right)+\frac{\hbar^2}{2MR^2}\right]dt\right\}\end{aligned} \tag{114}$$

Sphero-Elliptic:

$$\begin{aligned}&=\frac{1}{R^3}\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathscr{D}\chi(t)\sin^2\chi\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathscr{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)\\&\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot\chi^2+(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2))+\frac{\alpha}{R}\cot\chi\right.\right.\\&\left.\left.-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\sin^2\chi}\left(\frac{k_1^2-\frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}+\frac{k_2^2-\frac14}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right)-1\right)\right]dt\right\}\end{aligned} \tag{115}$$

Elliptic-Cylindrical II (rotated):

$$=\frac{1}{R^3}\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathscr{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\,\mu+k'^2\,\mathrm{cn}^2\,\nu)\,\mathrm{cn}\,\mu\,\mathrm{cn}\,\nu\int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathscr{D}\phi(t)$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}((k^2\,\mathrm{cn}^2\,\mu+k'^2\,\mathrm{cn}^2\,\nu)(\dot\mu^2+\dot\nu^2)+\mathrm{cn}^2\,\mu\,\mathrm{cn}^2\,\nu\,\dot\phi^2)\right.\right. \tag{116}$$

$$+\frac{\alpha}{R}\,\frac{k'\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu-k\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{k^2\,\mathrm{cn}^2\,\mu+k'^2\,\mathrm{cn}^2\,\nu}-\frac{\hbar^2}{2MR^2}\,\frac{1}{\mathrm{cn}^2\,\mu\,\mathrm{cn}^2\,\nu}\left(\frac{k_1^2-\frac14}{\cos^2\phi}+\frac{k_2^2-\frac14}{\cos^2\phi}\right)$$

$$\left.\left.+\frac{\hbar^2}{8MR^2}\left(4+\frac{1}{k^2\,\mathrm{cn}^2\,\mu+k'^2\,\mathrm{cn}^2\,\nu}\left(\frac{\mathrm{sn}^2\,\mu\,\mathrm{dn}^2\,\mu}{\mathrm{cn}^2\,\mu}+\frac{\mathrm{sn}^2\,\nu\,\mathrm{dn}^2\,\nu}{\mathrm{cn}^2\,\nu}\right)\right)\right]dt\right\}.$$

In the sphero-elliptic system the (μ,ν) path integration is similar as for the oscillator-case, with a remaining χ-path integration as in (74). In the spherical system, we apply in the ϕ-path integration the Pöschl-Teller, and in the θ-path integration the symmetric Pöschl-Teller path integral solution, again with a remaining χ-path integral as in (74). Thus we obtain ($\lambda_1=2n\pm k_1\pm k_2+1$, $\lambda_2=m+\lambda_1+1/2$, the expansion in sphero-elliptic coordinates remains but a formal expansion, cf. appendix B; $\phi\in[0,\frac{\pi}{2}]$, $(\chi,\theta)\in[0,\pi]$)

$$K^{(V_2)}(\vec s'',\vec s';T)$$

Spherical:

$$=\sum_{n,m=0}^{\infty}\sum_{N=1}^{\infty}e^{-iE_NT/\hbar}\,\Psi_{N,n,m}(\chi'',\theta'',\phi'';R)\,\Psi^*_{N,n,m}(\chi',\theta',\phi';R), \tag{117}$$

$$\Psi_{N,n,m}(\chi,\theta,\phi;R)=(\sin\chi)^{-1}\,\Phi_n^{(\pm k_2,\pm k_1)}(\phi)\,\Phi_m(\theta)\,S_N(\chi;R), \tag{118}$$

$$\Phi_n^{(\pm k_2,\pm k_1)}(\phi)=\left[2(2n\pm k_1\pm k_2+1)\,\frac{n!\,\Gamma(n\pm k_1\pm k_2+1)}{\Gamma(n+1\pm k_1)\,\Gamma(n+1\pm k_2)}\right]^{1/2}$$
$$\times(\sin\phi)^{1/2\pm k_2}(\cos\phi)^{1/2\pm k_1}\,P_n^{(\pm k_2,\pm k_1)}(\cos 2\phi),$$

$$\Phi_m(\theta)=\sqrt{(m+\lambda_1+1)\,\frac{\Gamma(m+2\lambda_1+1)}{m!}}\,P_{m+\lambda_1}^{-\lambda_1}(\cos\theta), \tag{119}$$

Sphero-Elliptic:

$$=\sum_n\sum_{m=0}^{\infty}\sum_{N=0}^{\infty}e^{-iE_NT/\hbar}\,\Psi_{N,n,m}(\chi'',\mu'',\nu'';R)\,\Psi^*_{N,n,m}(\chi',\mu',\nu';R), \tag{120}$$

$$\Psi_{N,n,m}(\chi,\mu,\nu;R)=(\sin^2\chi\,\mathrm{sn}\,\mu\,\mathrm{cn}\,\mu\,\mathrm{cn}\,\nu\,\mathrm{dn}\,\nu)^{-1/2}\,\Phi_{n,m}^{(k_1,k_2,\pm\frac12)}(\mu,\nu)\,S_N(\chi;R). \tag{121}$$

The P_ν^μ is a Legendre function. The wave functions S_N and the energy-spectrum are given by ($\tilde N=N+\lambda_2+1/2$, $a=\hbar^2/M\alpha$, $\sigma_N=R/a\tilde N$)

$$S_N(\chi;R)=\frac{1}{\Gamma(2\lambda+1)}\left[\frac{\sigma_N^2+\tilde N^2}{R^3\tilde N^2}\,\frac{\Gamma(\tilde N+\lambda+\frac12)\,\Gamma(\mathrm{i}\sigma_N+\lambda+\frac12)}{N!\,\Gamma(\mathrm{i}\sigma_N-\lambda+\frac12)}\right]^{1/2}$$
$$\times(2\sin\chi)^{\lambda+1/2}\exp[\mathrm{i}\chi(\mathrm{i}\sigma_N-N)]\,{}_2F_1(-N,\lambda+\tfrac12+\mathrm{i}\sigma_N;2\lambda+1;1-\mathrm{e}^{2\mathrm{i}\chi}),\tag{122}$$

$$E_N=\hbar^2\frac{\tilde N^2-1}{2MR^2}-\frac{M\alpha^2}{2\hbar^2\tilde N^2}\,.\tag{123}$$

In the rotated elliptic-cylindrical II system no path integral solution is possible.

4.3. We consider the potential ($k_{1,2,3}>0$)

$$V_3(\vec s)=\frac{\hbar^2}{2M}\left[\frac{1}{2\sqrt{s_1^2+s_2^2}}\left(\frac{k_1^2-\frac14}{\sqrt{s_1^2+s_2^2}+s_1}+\frac{k_2^2-\frac14}{\sqrt{s_1^2+s_2^2}-s_1}\right)+\frac{k_3^2-\frac14}{s_3^2}\right],\tag{124}$$

which has in spherical and cylindrical coordinates the following form

Spherical:

$$V_3(\vec s)=\frac{\hbar^2}{2MR^2\sin^2\chi}\left[\frac{k_3^2-\frac14}{\cos^2\theta}+\frac{1}{4\sin^2\theta}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\phi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\phi}{2}}\right)\right]\tag{125}$$

Cylindrical:

$$=\frac{\hbar^2}{2MR^2}\left[\frac{k_3^2-\frac14}{\cos^2\theta\cos^2\phi_2}+\frac{1}{4\sin^2\theta}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\phi_1}{2}}+\frac{k_2^2-\frac14}{\cos^2\frac{\phi_1}{2}}\right)\right].\tag{126}$$

We have the path integral representations

$K^{(V_3)}(\vec s'',\vec s';T)$

Spherical:

$$=\frac{1}{R^3}\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)\sin^2\chi\int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\mathcal{D}\theta(t)\sin\theta\int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathcal{D}\phi(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot\chi^2+\sin^2\chi(\dot\theta^2+\sin^2\theta\dot\phi^2))\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{k_3^2-\frac14}{\cos^2\theta}+\frac{1}{4\sin^2\theta}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\phi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\phi}{2}}-1\right)-\frac14\right)+\frac{\hbar^2}{2M}\right]dt\right\}\tag{127}$$

Cylindrical

$$=\frac{1}{R^3}\int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\mathcal{D}\theta(t)\sin\theta\cos\theta\int\limits_{\phi_1(t')=\phi_1'}^{\phi_1(t'')=\phi_1''}\mathcal{D}\phi_1(t)\int\limits_{\phi_2(t')=\phi_2'}^{\phi_2(t'')=\phi_2''}\mathcal{D}\phi_2(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot\theta^2+\cos^2\theta\dot\phi_1^2+\sin^2\theta\dot\phi_2^2)-\frac{\hbar^2}{2MR^2}\left(\frac{1}{\cos^2\theta}\left(\frac{k_3^2-\frac14}{\cos^2\phi}-\frac14\right)\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{4\sin^2\theta}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\phi_1}{2}}+\frac{k_2^2-\frac14}{\cos^2\frac{\phi_1}{2}}-1\right)-1\right)\right]dt\right\}.\tag{128}$$

We can solve the path integral for V_3 in spherical and cylindrical coordinates. The solutions have the form

$$K^{(V_3)}(\vec{s}'', \vec{s}'; T)$$

Spherical, $\lambda_1 = 2n \pm k_1 \pm k_2 + 1$, $\lambda_2 = 2m + \lambda_1 \pm k_3 + 1$; $(\chi, \phi) \in [0, \pi]$, $\theta \in [0, \frac{\pi}{2}]$:

$$= \sum_{n,m,l=0}^{\infty} e^{-iE_l T/\hbar} \Psi_{l,m,n}(\chi'', \theta'', \phi''; R) \Psi^*_{l,m,n}(\chi', \theta', \phi'; R), \tag{129}$$

$$\Psi_{l,m,n}(\chi, \theta, \phi; R) = (\sin\chi \sin\theta)^{-1/2} \Phi_n^{(\pm k_2, \pm k_1)}(\phi) \Phi_m^{(\lambda_1, \pm k_3)}(\theta) S_l(\chi; R), \tag{130}$$

$$\Phi_m^{(\pm k_2, \pm k_1)}(\phi) = \left[(2n \pm k_2 \pm k_1 + 1) \frac{n!\, \Gamma(n \pm k_2 \pm k_1 + 1)}{\Gamma(n \pm k_2 + 1)\, \Gamma(n \pm k_1 + 1)}\right]^{1/2}$$
$$\times \left(\sin\frac{\phi}{2}\right)^{1/2 \pm k_2} \left(\cos\frac{\phi}{2}\right)^{1/2 \pm k_1} P_n^{(\pm k_2, \pm k_1)}(\cos\phi), \tag{131}$$

$$\Phi_m^{(\lambda_1, \pm k_3)}(\theta) = \left[2(2m + \lambda_1 \pm k_3 + 1) \frac{m!\, \Gamma(m + \lambda_1 \pm k_3 + 1)}{\Gamma(m + \lambda_1 + 1)\, \Gamma(m \pm k_3 + 1)}\right]^{1/2}$$
$$\times (\sin\theta)^{1/2 + \lambda_1} (\cos\theta)^{1\ 2 \pm k_3} P_m^{(\lambda_1, \pm k_3)}(\cos 2\theta), \tag{132}$$

$$S_l(\chi; R) = \frac{1}{R^{3/2}} \sqrt{(l + \lambda_2 + \tfrac{1}{2}) \frac{\Gamma(l + 2\lambda_2 + 1)}{l!}}\, P_{l+\lambda_2}^{-\lambda_2}(\cos\chi), \tag{133}$$

$$E_l = \frac{\hbar^2}{2MR^2} [(l + \lambda_2 + \tfrac{1}{2})^2 - 1], \tag{134}$$

Cylindrical, $\lambda_1 = 2n_1 + \lambda_1 \pm k_2 + 1$, $\lambda_2 = n_2 \pm k_3 + \frac{1}{2}$; $\phi_{1,2} \in [0, \pi]$, $\theta \in [0, \frac{\pi}{2}]$:

$$= \sum_{n_1, n_2, l=0}^{\infty} e^{-iE_l T/\hbar} \Psi_{l,m,n}(\theta'', \phi_1'', \phi_2; R) \Psi^*_{l,m,n}(\theta', \phi_1', \phi_2; R), \tag{135}$$

$$\Psi_{l,m,n}(\theta, \phi_1, \phi_2; R) = (\sin\theta \cos\theta)^{-1/2} \Phi_n^{(\pm k_2, \lambda_1)}(\phi_1) \Phi_{n_2}(\phi_2) S_l^{(\lambda_2, \lambda_1)}(\theta; R), \tag{136}$$

$$\Phi_n^{(\pm k_2, \pm k_1)}(\phi_1) = \left[(2n \pm k_2 \pm k_1 + 1) \frac{n!\, \Gamma(n \pm k_2 \pm k_1 + 1)}{\Gamma(n \pm k_2 + 1)\, \Gamma(n \pm k_1 + 1)}\right]^{1/2}$$
$$\times \left(\sin\frac{\phi_1}{2}\right)^{1/2 \pm k_2} \left(\cos\frac{\phi_1}{2}\right)^{1/2 \pm k_1} P_n^{(\pm k_2, \pm k_1)}(\cos\phi_1), \tag{137}$$

$$\Phi_{n_2}(\phi_2) = \sqrt{\cos\phi_2} \sqrt{(n_2 + \lambda_1 + \tfrac{1}{2}) \frac{\Gamma(n_2 + 2\lambda_1 + 1)}{n_2!}}\, P_{n_2+\lambda_1}^{-\lambda_1}(\sin\phi_2), \tag{138}$$

$$S_l^{(\lambda_2, \lambda_1)}(\theta; R) = \frac{1}{R^{3/2}} \left[2(2l + \lambda_1 + \lambda_2 + 1) \frac{l!\, \Gamma(l + \lambda_1 + \lambda_2 + 1)}{\Gamma(l + \lambda_1 + 1)\, \Gamma(l + \lambda_2 + 1)}\right]^{1/2}$$
$$\times (\sin\theta)^{1/2 + \lambda_1} (\cos\theta)^{1/2 + \lambda_2} P_l^{(\lambda_2, \lambda_1)}(\cos 2\theta), \tag{139}$$

$$E_l = \frac{\hbar^2}{2MR^2}\left[(l+\lambda_2+\tfrac{1}{2})^2-1\right]. \tag{140}$$

4.4. The next three potentials generalize the minimally super-integrable potentials from the flat space $\mathbb{R}^3$. We have found three of such generalizations, including a "double-ring" and a "Hartmann"-potential on the sphere.

We consider the potential ($k>0$)

$$V_4(\vec{s}) = \frac{M}{2}\omega^2 R^2 \frac{s_1^2+s_2^2+s_3^2}{s_4^2} - \frac{\hbar^2}{2M}\left(\frac{k^2-\frac{1}{4}}{s_3^2}+\frac{F(s_2/s_1)}{s_1^2+s_2^2}\right), \tag{141}$$

which has in the four separating coordinate systems the following form

Spherical:

$$V_4(\vec{s}) = \frac{M}{2}\omega^2 R^2 \tan^2\chi + \frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{k^2-\frac{1}{4}}{\cos^2\theta}+\frac{F(\tan\phi)}{\sin^2\theta}\right) \tag{142}$$

Cylindrical:

$$= \frac{M}{2}\omega^2 R^2 \frac{1-\cos^2\theta\sin^2\phi_2}{\cos^2\theta\sin^2\phi_2} - \frac{\hbar^2}{2MR^2}\left(\frac{k^2-\frac{1}{4}}{\cos^2\theta\cos^2\phi_2}+\frac{F(\tan\phi_1)}{\sin^2\theta}\right) \tag{143}$$

Elliptic-Cylindrical I:

$$= \frac{M}{2}\omega^2 R^2 \frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu} + \frac{\hbar^2}{2MR^2}\left(\frac{k^2-\frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}+\frac{F(\tan\phi)}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\right) \tag{144}$$

Elliptic-Cylindrical II:

$$= \frac{M}{2}\omega^2 R^2 \frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu} + \frac{\hbar^2}{2MR^2}\left(\frac{k^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}+\frac{F(\tan\phi)}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right). \tag{145}$$

We have the path integral representations

$K^{(V_4)}(\vec{s}\,'', \vec{s}\,'; T)$

<u>Spherical</u>:

$$= \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathscr{D}\chi(t)\sin^2\chi \int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathscr{D}\theta(t)\sin\theta \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathscr{D}\phi(t)$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot\chi^2+\sin^2\chi(\dot\theta^2+\sin^2\theta\dot\phi^2)-\omega^2\tan^2\chi)\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{k^2-\frac{1}{4}}{\cos^2\theta}+\frac{F(\tan\phi)-\frac{1}{4}}{\sin^2\theta}\right)+\frac{\hbar^2}{2M}\right]dt\right\} \tag{146}$$

Cylindrical:

$$= \frac{1}{R^3} \int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathscr{D}\theta(t) \sin\theta\cos\theta \int\limits_{\phi_1(t')=\phi_1'}^{\phi_1(t'')=\phi_1''} \mathscr{D}\phi_1(t) \int\limits_{\phi_2(t')=\phi_2'}^{\phi_2(t'')=\phi_2''} \mathscr{D}\phi_2(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\left(\dot\theta^2+\cos^2\theta\,\dot\phi_1^2+\sin^2\theta\,\dot\phi_2^2-\omega^2\,\frac{1-\cos^2\theta\sin^2\phi_2}{\cos^2\theta\sin^2\phi_2}\right)\right.\right.$$

$$\left.\left. -\frac{\hbar^2}{2MR^2}\left(\frac{1}{\cos^2\theta}\left(\frac{k^2-\frac14}{\cos^2\phi_2}-\frac14\right)+\frac{F(\tan\phi_1)-\frac14}{\sin^2\theta}-1\right)\right]dt\right\} \tag{147}$$

Elliptic-Cylindrical I

$$= \frac{1}{R^3} \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathscr{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathscr{D}\phi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\left((k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2)+\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu\,\dot\phi^2-\omega^2\,\frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right)\right.\right.$$

$$-\frac{\hbar^2}{2MR^2}\left(\frac{k^2-\frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}+\frac{F(\tan\phi)}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\right) \tag{148}$$

$$\left.\left.+\frac{\hbar^2}{8MR^2}\left(4+\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}\left(\frac{\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu}{\mathrm{sn}^2\mu}+k'^4\,\frac{\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu}{\mathrm{dn}^2\nu}\right)\right)\right]dt\right\}$$

Elliptic-Cylindrical II

$$= \frac{1}{R^3} \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathscr{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)\,\mathrm{cn}\,\mu\,\mathrm{cn}\,\nu \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathscr{D}\phi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\left((k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2)+\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu\,\dot\phi^2-\omega^2\,\frac{1-\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right)\right.\right.$$

$$-\frac{\hbar^2}{2MR^2}\left(\frac{k^2-\frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}+\frac{F(\tan\phi)}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}\right)$$

$$\left.\left.+\frac{\hbar^2}{8MR^2}\left(4+\frac{1}{k^2\,\mathrm{cn}^2\mu+k'^2\,\mathrm{cn}^2\nu}\left(\frac{\mathrm{sn}^2\mu\,\mathrm{dn}^2\mu}{\mathrm{cn}^2\mu}+\frac{\mathrm{sn}^2\nu\,\mathrm{dn}^2\nu}{\mathrm{cn}^2\nu}\right)\right)\right]dt\right\}. \tag{149}$$

We can explicitly solve the first two path integrals. We denote the energy-spectrum of the ϕ-dependent problems by $E_n=\hbar^2\lambda_\phi^2/2M$. The most important case occurs when $F(\tan\phi)=\gamma=\text{constant}$. Then the quantum motion in ϕ are just circular waves and we have in this case $\Phi_n(\phi)\equiv\Phi_\nu(\phi)=e^{i\nu\phi}/\sqrt{2\pi}$ $(\nu\in\mathbb{Z})$. We obtain

$K^{(V_4)}(\vec{s}\,'', \vec{s}\,'; T)$

<u>Spherical</u>, $\lambda_2 = 2m + n \pm k + 1$, $\lambda_3^2 = (1 + 4M^2\omega^2 R^4/\hbar^2)/4$;
$(\chi, \theta) \in [0, \frac{\pi}{2}]$, $\phi \in [0, 2\pi]$:

$$= \sum_{m,l=0}^{\infty} \sum_{n\in Z} e^{-iE_l T/\hbar} \Psi_{l,m,n}(\chi'', \theta'', \phi''; R)\, \Psi^*_{l,m,n}(\chi', \theta', \phi'; R), \tag{150}$$

$$\Psi_{l,m,n}(\chi, \theta, \phi; R) = (\sin^2\chi \sin\theta)^{-1/2} \frac{e^{in\phi}}{\sqrt{2\pi}} \Phi_m^{(n,\pm k)}(\theta)\, S_l^{(\lambda_2,\lambda_3)}(\chi; R), \tag{151}$$

$$\Phi_m^{(n,\pm k)}(\theta) = \left[2(2m + n \pm k + 1) \frac{m!\,\Gamma(n_2 + n \pm k + 1)}{\Gamma(m + n + 1)\,\Gamma(n_2 \pm k + 1)}\right]^{1/2} \times (\sin\theta)^{1/2+n} (\cos\theta)^{1/2\pm k} P_m^{(n,\pm k)}(\cos 2\theta), \tag{152}$$

$$S_l^{(\lambda_2,\lambda_3)}(\chi; R) = \left[2(2l + \lambda_2 + \lambda_3 + 1) \frac{l!\,\Gamma(l + \lambda_2 + \lambda_3 + 1)}{R^3\,\Gamma(l + \lambda_2 + 1)\,\Gamma(l + \lambda_3 + 1)}\right]^{1/2} \times (\sin\chi)^{1/2+\lambda_2} (\cos\chi)^{1/2+\lambda_3} P_l^{(\lambda_2,\lambda_3)}(\cos 2\chi), \tag{153}$$

$$E_l = \frac{\hbar^2}{2MR^2}\left[(2l + \lambda_2 + \lambda_3 + 1)^2 - 1 - \frac{M^2\omega^2 R^4}{\hbar^2}\right], \tag{154}$$

<u>Cylindrical</u>, $\lambda_1 = 2n_1 + \lambda_3 \pm k + 1$, $\lambda_3^2 = (1 + 4M^2\omega^2 R^4/\hbar^2)/4$;
$\phi_1 \in [0, 2\pi]$, $(\theta, \phi_2) \in [0, \frac{\pi}{2}]$:

$$= \sum_{n_2,l=0}^{\infty} \sum_{n_1\in Z} e^{-iE_l T/\hbar} \Psi_{l,n_1,n_2}(\theta'', \phi_2'', \phi_1''; R)\, \Psi^*_{l,n_1,n_2}(\theta', \phi_2', \phi_1'; R), \tag{155}$$

$$\Psi_{l,n_1,n_2}(\theta, \phi_1, \phi_2; R) = (\sin\theta\cos\theta)^{-1/2} \frac{e^{in_1\phi_1}}{\sqrt{2\pi}} \Phi_{n_2}^{(\lambda_3,\pm k_3)}(\phi_2)\, S_l^{(n_1,\lambda_1)}(\theta; R), \tag{156}$$

$$\Phi_{n_2}^{(\lambda_3,\pm k_3)}(\phi_2) = \left[2(2n_2 + \lambda_3 \pm k_3 + 1) \frac{n_2!\,\Gamma(n_2 + \lambda_3 \pm k_3 + 1)}{\Gamma(n_2 + \lambda_3 + 1)\,\Gamma(n_2 \pm k_3 + 1)}\right]^{1/2} \times (\sin\phi_2)^{1/2+\lambda_3} (\cos\phi_2)^{1/2\pm k_3} P_{n_2}^{(\lambda_3,\pm k_3)}(\cos 2\phi_2), \tag{157}$$

$$S_l^{(n_1,\lambda_1)}(\theta; R) = \left[2(2l + n_1 + \lambda_1 + 1) \frac{l!\,\Gamma(l + n_1 + \lambda_1 + 1)}{R^3\,\Gamma(l + n_1 + 1)\,\Gamma(l + \lambda_1 + 1)}\right]^{1/2} \times (\sin\theta)^{1/2+n_1} (\cos\theta)^{1/2+\lambda_1} P_l^{(n_1,\lambda_1)}(\cos 2\theta), \tag{158}$$

$$E_l = \frac{\hbar^2}{2MR^2}\left[(2l + \lambda_1 + n_1 + 1)^2 - 1 - \frac{M^2\omega^2 R^4}{\hbar^2}\right]. \tag{159}$$

4.5. We consider the potential

$$V_5(\vec{s}) = -\frac{\alpha}{R} \frac{s_4}{\sqrt{s_1^2 + s_2^2 + s_3^2}} + \frac{\hbar^2}{2M(s_1^2 + s_2^2)}\left(\frac{\beta s_3}{\sqrt{s_1^2 + s_2^2 + s_3^2}} + F\left(\frac{s_2}{s_1}\right)\right), \tag{160}$$

which has in spherical and rotated elliptic-cylindrical II coordinates the form

Spherical:

$$V_5(\vec{s}) = -\frac{\alpha}{R}\cot\chi + \frac{\hbar^2}{2MR^2\sin^2\chi}\,\frac{\beta\cos\theta + F(\tan\phi)}{\sin^2\theta} \tag{161}$$

Elliptic-Cylindrical II (rotated):

$$= -\frac{\alpha}{R}\,\frac{k'\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu - k\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu} + \frac{\hbar^2}{2MR^2(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)}$$

$$\times\left(\frac{F(\tan\phi)\,k'^2 + \beta k'\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{cn}^2\mu} + \frac{F(\tan\phi)\,k^2 + \beta k\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu}{\mathrm{cn}^2\nu}\right). \tag{162}$$

We have the path integral representations

$$K^{(V_5)}(\vec{s}\,'', \vec{s}\,'; T)$$

Spherical

$$= \frac{1}{R^3}\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathscr{D}\chi(t)\sin^2\chi\int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\mathscr{D}\theta(t)\sin\theta\int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathscr{D}\phi(t)$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot\chi^2 + \sin^2\chi(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)) + \frac{\alpha}{R}\cot\chi\right.\right.$$

$$\left.\left. - \frac{\hbar^2}{2MR^2\sin^2\chi}\left(\frac{\beta\cos\theta + F(\tan\phi) - \frac{1}{4}}{\sin^2\theta} - \frac{1}{4}\right) + \frac{\hbar^2}{2MR^2}\right]dt\right\} \tag{163}$$

Elliptic-Cylindrical II (rotated):

$$= \frac{1}{R^3}\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathscr{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)\,\mathrm{cn}\,\mu\,\mathrm{cn}\,\nu\int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathscr{D}\phi(t)$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}((k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2 + \dot\nu^2) + \mathrm{cn}^2\mu\,\mathrm{cn}^2\nu\,\dot\phi^2)\right.\right.$$

$$+ \frac{\alpha}{R}\,\frac{k'\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu - k\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu} + \frac{\hbar^2}{2MR^2(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)}$$

$$\times\left(\frac{F(\tan\phi)\,k'^2 + \beta k'\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{cn}^2\mu} + \frac{F(\tan\phi)\,k^2 + \beta k\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu}{\mathrm{cn}^2\nu}\right)$$

$$\left.\left. + \frac{\hbar^2}{8MR^2}\left(4 + \frac{1}{k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu}\left(\frac{\mathrm{sn}^2\mu\,\mathrm{dn}^2\mu}{\mathrm{cn}^2\mu} + \frac{\mathrm{sn}^2\nu\,\mathrm{dn}^2\nu}{\mathrm{cn}^2\nu}\right)\right)\right]dt\right\}. \tag{164}$$

We can only solve the path integral in *spherical coordinates* and we get (we take $F(\tan\phi)=\gamma$ as before, where $\gamma \geqq |\beta|$, $\lambda_\pm^2 = n^2+\gamma\pm\beta$, $\lambda_2 = m+(\lambda_+ +\lambda_- +1)/2$, $\tilde{N}=N+\lambda_2+1/2$, $a=\hbar^2/M\alpha$, $\sigma_N = R/a\tilde{N}$, $(\chi,\theta)\in[0,\pi]$, $\phi\in[0,2\pi]$)

$$K^{(V_5)}(\vec{s}'',\vec{s}';T)$$

$$= \sum_{N,m=0}^{\infty}\sum_{n\in\mathbb{Z}} e^{-iE_N T/\hbar}\,\Psi_{N,m,n}(\chi'',\theta'',\phi'';R)\,\Psi^*_{N,m,n}(\chi',\theta',\phi';R), \tag{165}$$

$$\Psi_{N,m,n}(\chi,\theta,\phi;R) = (\sin^2\chi\sin\theta)^{-1/2}\,\frac{e^{in\phi}}{\sqrt{2\pi}}\,\Phi_m^{(\lambda_+,\lambda_-)}(\theta)\,S_N(\chi;R), \tag{166}$$

$$\Phi_m^{(\lambda_+,\lambda_-)}(\theta) = \left[(2m+\lambda_+ +\lambda_- +1)\,\frac{m!\,\Gamma(m+\lambda_+ +\lambda_- +1)}{\Gamma(m+\lambda_+ +1)\,\Gamma(m+\lambda_- +1)}\right]^{1/2}$$
$$\times\left(\sin\frac{\theta}{2}\right)^{1/2+\lambda_+}\left(\cos\frac{\theta}{2}\right)^{1/2+\lambda_-} P_m^{(\lambda_+,\lambda_-)}(\cos\theta), \tag{167}$$

$$S_N(\chi;R) = \frac{1}{\Gamma(2\lambda+1)}\left[\frac{\sigma_N^2+\tilde{N}^2}{R^3\tilde{N}^2}\,\frac{\Gamma(\tilde{N}+\lambda+\frac{1}{2})\,\Gamma(i\sigma_N+\lambda+\frac{1}{2})}{N!\,\Gamma(i\sigma_N-\lambda+\frac{1}{2})}\right]^{1/2}$$
$$\times(2\sin\chi)^{\lambda+1/2}\exp[i\chi(i\sigma_N-N)]\;{}_2F_1(-N,\lambda+\tfrac{1}{2}+i\sigma_N;2\lambda+1;1-e^{2i\theta}), \tag{168}$$

$$E_N = \hbar^2\,\frac{\tilde{N}^2-1}{2MR^2} - \frac{M\alpha^2}{2\hbar^2\tilde{N}^2}. \tag{169}$$

In the case when $R\to\infty$ the flat space limit is recovered [38, 56, 57].

4.6. We consider the potential ($k_{1,2,3,4}>0$)

$$V_6(\vec{s}) = F(\chi) + \frac{\hbar^2}{2M}\left(\frac{k_1^2-\frac{1}{4}}{s_1^2}+\frac{k_2^2-\frac{1}{4}}{s_2^2}+\frac{k_3^2-\frac{1}{4}}{s_3^2}+\frac{k_4^2-\frac{1}{4}}{s_4^2}\right), \tag{170}$$

which has in spherical and sphero-elliptic coordinates the form

Spherical:

$$V_6(\vec{s}) = F(\chi) + \frac{\hbar^2}{2MR^2}\left[\frac{k_4^2-\frac{1}{4}}{\cos^2\chi}+\frac{1}{\sin^2\chi}\left(\frac{k_3^2-\frac{1}{4}}{\cos^2\theta}+\frac{1}{\sin^2\theta}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2\phi}+\frac{k_2^2-\frac{1}{4}}{\sin^2\phi}\right)\right)\right] \tag{171}$$

Sphero-Elliptic:

$$= F(\chi) + \frac{\hbar^2}{2MR^2}\left[\frac{k_4^2-\frac{1}{4}}{\cos^2\chi}+\frac{1}{\sin^2\chi}\left(\frac{k_1^2-\frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}+\frac{k_2^2-\frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu}+\frac{k_3^2-\frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu}\right)\right]. \tag{172}$$

We have the path integral representations

$K^{(V_6)}(\vec{s}'', \vec{s}'; T)$

Spherical:

$$= \frac{1}{R^3} \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathscr{D}\chi(t) \sin^2\chi \int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathscr{D}\theta(t) \sin\theta \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathscr{D}\phi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot{\chi}^2 + \sin^2\chi(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)) - F(\chi) \right.\right. \tag{173}$$

$$\left.\left. - \frac{\hbar^2}{2MR^2}\left(\frac{k_4^2 - \frac{1}{4}}{\cos^2\chi} + \frac{1}{\sin^2\chi}\left(\frac{k_3^2 - \frac{1}{4}}{\cos^2\theta} + \frac{1}{\sin^2\theta}\left(\frac{k_1^2 - \frac{1}{4}}{\cos^2\phi} + \frac{k_2^2 - \frac{1}{4}}{\sin^2\phi} - \frac{1}{4}\right) - \frac{1}{4}\right) - 1\right)\right] dt\right\}$$

Sphero-Elliptic:

$$= \frac{1}{R^3} \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathscr{D}\chi(t) \sin^2\chi \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathscr{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(\dot{\chi}^2 + \sin^2\chi(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)(\dot{\mu}^2 + \dot{\nu}^2)) - F(\chi) \right.\right. \tag{174}$$

$$\left.\left. - \frac{\hbar^2}{2MR^2}\left(\frac{k_4^2 - \frac{1}{4}}{\cos^2\chi} + \frac{1}{\sin^2\chi}\left(\frac{k_1^2 - \frac{1}{4}}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu} + \frac{k_2^2 - \frac{1}{4}}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu} + \frac{k_3^2 - \frac{1}{4}}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu} - \frac{1}{4}\right) - 1\right] dt\right\}.$$

It is sufficient to state the separation steps in the path integration, because in both cases we end up with a path integral in the variable χ which is undetermined until the potential $F(\chi)$ is not specified. We obtain ($\lambda_1 = 2n \pm k_1 \pm k_2 + 1$, $\lambda_2 = 2m \pm k_3 + \lambda_1 + 1 \equiv 2l + 2 \pm k_2 \pm k_2 \pm k_3$ with $l \in \mathbb{N}_0$ the principal quantum number $l \in \mathbb{N}_0$ of the $S^{(2)}$-subsystem)

$$K^{(V_6)}(\vec{s}'', \vec{s}'; T) = (\sin\chi''\sin\chi')^{-1}\,e^{i\hbar T/2MR^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}$$

$$\times\begin{pmatrix}(\sin\theta'\sin\theta'')^{-1/2}\,\Phi_n^{(\pm k_2, \pm k_1)}(\phi'')\,\Phi_n^{(\pm k_2, \pm k_1)}(\phi')\,\Phi_m^{(\lambda_1, \pm k_3)}(\theta'')\,\Phi_m^{(\lambda_1, \pm k_3)}(\theta') \\ (M(\mu', \nu')\,M(\mu'', \nu'')]^{-1/2}\,\Psi_{ml}^{(k_1, k_2, k_3)}(\mu'', \nu'')\,\Psi_{ml}^{(k_1, k_2, k_3)*}(\mu', \nu')\end{pmatrix}$$

$$\times \frac{1}{R^3}\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathscr{D}\chi(t) \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\dot{\chi}^2 - F(\chi) - \frac{\hbar^2}{2MR^2}\left(\frac{k_4^2 - \frac{1}{4}}{\cos^2\chi} + \frac{\lambda_2^2 - \frac{1}{4}}{\sin^2\chi}\right)\right] dt\right\}, \tag{175}$$

where the $\Phi_n^{(\pm k_2, \pm k_1)}$, $\Phi_m^{(\lambda_1, \pm k_3)}$ denote Pöschl-Teller wave functions which we do not explicitly state for the spherical case, and the $\Psi_{ml}^{(k_1, k_2, k_3)}$ the wave functions of appendix B in the sphero-elliptical case, respectively.

5. Summary and Discussion

In this paper we have opened a new page in the study of super-integrable potentials, respectively potentials with accidental degeneracy, respectively Smorodinsky-Winternitz potentials, on spaces with constant positive curvature. In particular we found two Smorodinsky-Winternitz potentials on the two-dimensional sphere, and six Smorodinsky-

Winternitz potentials on the three-dimensional sphere. This has to be compared with the corresponding flat spaces were there are four Smorodinsky-Winternitz potentials (maximally super-integrable) in two-dimensional Euclidean space R^2, and five maximally and eight minimally super-integrable Smorodinsky-Winternitz potentials in three-dimensional Euclidean space $E(3)$. So there seems to be fewer potentials on the sphere as in flat space. This sounds surprising. One expects that a flat space limit of the sphere, i.e. $R \to \infty$, should reproduce the corresponding cases in flat space. For the potentials we have found this is in fact true. But the other direction, i.e. which Smorodinsky-Winternitz potential in flat space generalizes to one on a sphere, seems not to work.

A look at the number of coordinate systems reveals some of the problems one is encountering. Due to OLEVSKY [77] there are two coordinate systems on the sphere $S^{(2)}$ and six on the sphere $S^{(3)}$, in comparison to four coordinate systems on R^2 and eleven on R^3. For instance, on the sphere $S^{(2)}$ we have the polar and elliptic, and on R^2 we have the cartesian, polar, elliptic and parabolic system. Would it be possible to construct analogues of the *cartesian and the parabolic system* on the sphere $S^{(2)}$, it is reasonable that more structure would appear which would give rise to the "missing potentials". Actually, it is possible to make such a construction [51], and what is interesting that these coordinate systems are connected with a rotation with respect to the elliptic coordinate system, and one can introduce "cartesian" and "parabolic" coordinate systems on the sphere $S^{(2)}$ which are therefore contained in the elliptical one.

The above definition of new coordinate systems on $S^{(2)}$ (and similarly on $S^{(3)}$) allows the following consideration. In both potentials in Section 3 we *add* the coordinate system "cartesian" where the elliptic systems appears, and we add the coordinate system "parabolic" where the elliptic II systems appears. Hence, both potentials are separable in *three* coordinate system, exactly as their flat space counterparts are. In the case of the Coulomb potential on $S^{(3)}$ in Section 4 (the potential V_2) the obvious new specification of the rotated elliptic-cylindrical II system (which is actually the spherical prolate spheroidal system) is the "parabolic system" on the sphere. Hence, the Coulomb potential on $S^{(3)}$ is separable in *four* coordinate systems, exactly as in three-dimensional Euclidean space.

Similarly, we see that the cases of the ring-shaped oscillator V_4 and V_6 are "complete", i.e., all coordinate systems which separate the ring-shaped oscillator and V_6 in R^3 are already present on $S^{(3)}$. In the case of the HARTMANN potential [42], the "parabolic" coordinate system in $S^{(3)}$ is missing. However, we know that in the flat space limit the parabolic coordinate system emerges from the rotated elliptic-cylindrical II system; hence the Hartmann potential is separable in three coordinate systems on $S^{(3)}$, exactly as it is in R^3.

The remaining coordinate systems, in particular the cylindrical systems in $E(3)$, are not so easy to obtain, in particular the "cylindrical" and the "paraboloidal" systems. They are included in the ellipsoidal system and its degenerations.

Unfortunately we have presently not been able to construct the corresponding Smorodinsky-Winternitz potentials which are separable in these further coordinate systems, and we will come back to these question in a forthcoming publication.

Therefore several open questions remain:

[1] We do not know what is the Holt potential and the Stark effect on the sphere. However, any possible quantum mechanical solution is very likely not subject to path integration technique due the fact that the solutions will be in terms of Lamé polynomials, respectively Lamé functions.

[2] As in flat space we are faced with the problem of finding appropriate interbasis expansion for parametric coordinate systems. In the case of the free motion on R^2 and R^3 for the elliptic and spheroidal coordinate systems, respectively, they have been used in [36, 37] for a group path integration of the free motion in terms of these coordinates. In the case of elliptic coordinates involving the Jacobi elliptic functions such expansions also exist [73, 75, 83].

[3] How many coordinate systems can be defined on the three-dimensional sphere exceeding the already known six coordinate systems, and how can these coordinate systems be put into relation to the eleven coordinate systems in three-dimensional Euclidean space?

[4] How the notion super-integrability can be properly defined in a space of constant curvature, and how many super-integrable (minimally and maximally) potentials can be constructed? Do exist on the sphere (and pseudosphere) Smorodinsky-Winternitz potentials which have no counterparts in flat space?

[5] Is there a semi-linear connection between the Coulomb problem and the oscillator, and in how many dimensions such a transformation can be constructed? Such a construction would provide a generalization of the KUSTAANHEIMO-STIEFEL- [64], HURWITZ- [45], respectively Levi-Civita transformation of flat space in a space of constant curvature. This kind of transformation transforms from a cartesian coordinate system to a coordinate system of parabolic type. As far as one is only concerned with a one-dimensional problem, the corresponding non-linear transformation in hyperbolic geometry has the form

$$\frac{1}{2}(1-\coth\tau)=-\frac{1}{\sinh^2 r} \tag{176}$$

(where τ denotes the "radial" variable on the pseudosphere, and r the variable for the modified Pöschl-Teller potential), accompanied by the time-transformation $dt=ds$, with $f(r)=R^2\tanh^2 r$ [5, 6, 28, 29]. In some sense this transformation can be seen as a one-dimensional realization of the Kustaanheimo-Stiefel transformation in hyperbolic geometry (cf. also [75]).

From the solution of these questions and problems, including the case of the corresponding case of quantum motion in a hyperbolic geometry, it depends whether we will be able to develop the theory of Smorodinsky-Winternitz potentials (i.e. super-integrable potentials, i.e. systems with accidental degeneracy) with the same elegance and completeness as it is known nowadays in flat Euclidean space. This will be subject to future investigations.

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Appendix A: Elementary Path Integral Solutions

In this appendix we state our lattice definition of the path integral formulations we use. We use a definition as introduced in [27] called *P*roduct-*F*orm (PF) which reads

$$\begin{aligned}
K(\vec{q}'',\vec{q}';T)&=\int_{\vec{q}(t')=\vec{q}'}^{\vec{q}(t'')=\vec{q}''}\mathcal{D}\vec{q}(t)\sqrt{g}\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}h_{ac}(\vec{q})\,h_{cb}(\vec{q})\dot{q}^a\dot{q}^b-V(\vec{q})-\Delta V(\vec{q})\right]dt\right\}\\
&\equiv\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int d\vec{q}_j\sqrt{g(\vec{q}_j)}\\
&\quad\times\exp\left\{\frac{i}{\hbar}\prod_{j=1}^{N}\left[\frac{M}{2\varepsilon}h_{bc}(\vec{q}_j)\,h_{ac}(\vec{q}_{j-1})\,\Delta q_j^a\Delta q_j^b-\varepsilon V(\vec{q}_j)-\varepsilon\Delta V_{PF}(\vec{q}_j)\right]\right\}.
\end{aligned} \tag{177}$$

ΔV_{PF} denotes the well-defined quantum potential

$$\Delta V_{PF}=\frac{\hbar^2}{8M}\left[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_b)_{,b}+g^{ab}{}_{,ab}\right]+\frac{\hbar^2}{8M}\left(2h^{ac}h^{bc}{}_{,ab}-h^{ac}{}_{,a}h^{bc}{}_{,b}-h^{ac}{}_{,b}h^{bc}{}_{,a}\right) \quad (178)$$

arising from the specific lattice formulation for the path integral, respectively the ordering prescription for position and momentum operators in the quantum Hamiltonian. We only use the lattice formulation of (177) in this paper unless otherwise (and explicitly) stated. Of course, this lattice definition is completely equivalent with the midpoint definition [24, 40, 74, 78, 79] and others (e.g. [15, 21, 69, 72, 84]).

As we shall see, we encounter particularly in the case of the Higgs oscillator, the Pöschl-Teller and the modified PÖSCHL-TELLER potential [86] in our path integral problems. The path integral solution of the PÖSCHL-TELLER potential reads as follows (BÖHM and JUNKER [2, 3], DURU [16], INOMATA et al. [48, 50], and KLEINERT and MUSTAPIC [60], $0<x<\pi/2$)

$$\begin{aligned}
&\frac{i}{\hbar}\int_0^\infty dT e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{x}^2-\frac{\hbar^2}{2M}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2 x}+\frac{\beta^2-\frac{1}{4}}{\cos^2 x}\right)\right]dt\right\}\\
&=\frac{M}{2\hbar^2}\sqrt{\sin 2x'\sin 2x''}\,\frac{\Gamma(m_1-L_E)\,\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\,\Gamma(m_1-m_2+1)}\\
&\quad\times\left(\frac{1-\cos 2x'}{2}\,\frac{1-\cos 2x''}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\cos 2x'}{2}\,\frac{1+\cos 2x'}{2}\right)^{(m_1+m_2)/2}\\
&\quad\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{1-\cos 2x_<}{2}\right)\\
&\quad\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{1+\cos 2x_>}{2}\right)
\end{aligned} \quad (179)$$

$$=\sum_{n\in 0}\frac{\Phi_n^{(\alpha,\beta)}(x')\,\Phi_n^{(\alpha,\beta)}(x'')}{E_n-E}, \quad (180)$$

$$\begin{aligned}
\Phi_n^{(\alpha,\beta)}(x)=&\left[2(\alpha+\beta+2n+1)\,\frac{n!\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}\right]^{1/2}\\
&\times(\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x),
\end{aligned} \quad (181)$$

$$E_n=\frac{\hbar^2}{2M}(\alpha+\beta+2n+1)^2, \quad (182)$$

with $m_{1/2}=\frac{1}{2}(\beta\pm\alpha)$, $L_E=-\frac{1}{2}+\frac{1}{2}\sqrt{2ME}/\hbar$. The $P_n^{(\alpha,\beta)}$ are Jacobi polynomials. The Pöschl-Teller wave functions $\Phi^{(\alpha,\beta)}(x)$ are normalized to unity with respect to the scalar product

$$\int_0^{\pi/2}|\Phi^{(\alpha,\beta)}(x)|^2\,dx=1. \quad (183)$$

The case of the modified Pöschl-Teller potential is given by [1, 2, 32, 48, 60]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar}\int_{r(t')=r'}^{r(t'')=r''}\mathscr{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{r}^2-\frac{\hbar^2}{2M}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 r}-\frac{\nu^2-\frac{1}{4}}{\cosh^2 r}\right)\right]dt\right\}$$

$$=\frac{M}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\,\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\,\Gamma(m_1-m_2+1)}$$

$$\times(\cosh r'\cosh r'')^{-(m_1-m_2)}(\tanh r'\tanh r'')^{m_1+m_2+1/2}$$

$$\times{}_2F_1\left(-L_\nu+m_1,L_\nu+m_1+1;m_1-m_2+1;\frac{1}{\cosh^2 r_<}\right)$$

$$\times{}_2F_1(-L_\nu+m_1,L_\nu+m_1+1;m_1+m_2+1;\tanh^2 r_>) \tag{184}$$

$$=\sum_{n=0}^{N_M}\frac{\Psi_n^{(k_1,k_2)*}(r')\,\Psi_n^{(k_1,k_2)}(r'')}{E_n-E}+\int_0^\infty dp\,\frac{\Psi_p^{(k_1,k_2)*}(r')\,\Psi_p^{(k_1,k_2)}(r'')}{\hbar^2p^2/2M-E}, \tag{185}$$

$[m_{1,2}=\frac{1}{2}(\eta\pm\sqrt{-2ME}/\hbar),\ L_\nu=\frac{1}{2}(1-\nu)]$. The bound states are explicitly given by

$$\Psi_n^{(k_1,k_2)}(r)=N_n^{(k_1,k_2)}(\sinh r)^{2k_2-\frac{1}{2}}(\cosh r)^{-2k_1+3/2}$$
$$\times{}_2F_1(-k_1+k_2+\kappa,-k_1+k_2-\kappa+1;2k_2;-\sinh^2 r), \tag{186}$$

$$=\left[\frac{2n!(2\kappa-1)\,\Gamma(2k_1-n-1)}{\Gamma(2k_2+n)\,\Gamma(2k_1-2k_2-n)}\right]^{1/2}(\sinh r)^{2k_2-\frac{1}{2}}(\cosh r)^{2n-2k_1+3/2}$$
$$\times P_n^{[2k_2-1,2(k_1-k_2-n)-1]}\left(\frac{1-\sinh^2 r}{\cosh^2 r}\right), \tag{187}$$

$$N_n^{(k_1,k_2)}=\frac{1}{\Gamma(2k_2)}\left[\frac{2(2\kappa-1)\,\Gamma(k_1+k_2-\kappa)\,\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\,\Gamma(k_1-k_2-\kappa+1)}\right]^{1/2}$$

$$E_n=-\frac{\hbar^2}{2M}[2(k_1-k_2-n)-1]^2. \tag{188}$$

Here denote $k_1=\frac{1}{2}(1\pm\nu)$, $k_2=\frac{1}{2}(1\pm\eta)$, $n=0,1,\dots,N_M<k_1-k_2-\frac{1}{2}$, $\kappa=k_1-k_2-n$. The continuous states are $[\kappa=\frac{1}{2}(1+\mathrm{i}p)]$:

$$\Psi_p^{(k_1,k_2)}(r)=N_p^{(k_1,k_2)}(\cosh r)^{2k_1-\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}}$$
$$\times{}_2F_1(k_1+k_2-\kappa,k_1+k_2+\kappa-1;2k_2;-\sinh^2 r),$$

$$N_p^{(k_1,k_2)}=\frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}$$
$$\times[\Gamma(k_1+k_2-\kappa)\,\Gamma(-k_1+k_2+\kappa)\,\Gamma(k_1+k_2+\kappa-1)\,\Gamma(-k_1+k_2-\kappa+1)]^{1/2}. \tag{189}$$

We make extensively use of the solution of the Pöschl-Teller potential. For the Coulomb potential on the two- and three-dimensional sphere we need only the discrete spectrum of the modified Pöschl-Teller potential. However, we state only the wave function expansions and omit the Green function representation.

Appendix B: Path Integral Identity from the Six-Dimensional Sphere

Let us consider a particular coordinate system on the six-dimensional sphere ($0 \leqq \phi_{1,2,3} < 2\pi$, and the elliptic coordinates μ, ν as on $S^{(2)}$)

$$\left.\begin{aligned} s_1 &= \mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k')\cos\phi_1 \\ s_2 &= \mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k')\sin\phi_1 \\ s_3 &= \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')\cos\phi_2 \\ s_4 &= \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')\sin\phi_2 \\ s_5 &= \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')\cos\phi_3 \\ s_6 &= \mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k')\sin\phi_3 \end{aligned}\right\} . \tag{190}$$

According to the general theory the quantum potential has the form

$$\Delta V(\mu, \nu) = -\frac{2\hbar^2}{MR^2} - \frac{\hbar^2}{8MR^2}\left(\frac{1}{\mathrm{sn}^2\mu\,\mathrm{sn}^2\nu} + \frac{1}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu} + \frac{1}{\mathrm{dn}^2\mu\,\mathrm{dn}^2\nu} - 1\right). \tag{191}$$

We obtain the corresponding path integral formulation

$$\begin{aligned} &K(\mu'', \mu', \nu'', \nu', \phi_1'', \phi_1', \phi_2'', \phi_2', \phi_3'', \phi_3'; T) \\ &= \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathscr{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)\,\mathrm{sn}\,\mu\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu\,\mathrm{dn}\,\nu \\ &\times \int\limits_{\phi_1(t')=\phi_1'}^{\phi_1(t'')=\phi_1''} \mathscr{D}\phi_1(t) \int\limits_{\phi_2(t')=\phi_2'}^{\phi_2(t'')=\phi_2''} \mathscr{D}\phi_2(t) \int\limits_{\phi_3(t')=\phi_3'}^{\phi_3(t'')=\phi_3''} \mathscr{D}\phi_3(t) \\ &\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}\left((k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2 + \dot\nu^2)\right.\right.\right. \\ &\qquad\left.\left.\left. + \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu\,\dot\phi_1^2 + \mathrm{cn}^2\mu\,\mathrm{cn}^2\nu\,\dot\phi_2^2 + \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\dot\phi_3^2\right) - \Delta V(\mu, \nu)\right]dt\right\} \end{aligned}$$

$$\begin{aligned} &= (R^6\,M(\mu', \nu')\,M(\mu'', \nu''))^{-1/2}\,\mathrm{e}^{2\mathrm{i}\hbar T/m} \sum_{n_1, n_2, n_3 \in \mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}[n_1(\phi_1''-\phi_1') + n_2(\phi_2''-\phi_2') + n_3(\phi_3''-\phi_3')]}}{(2\pi)^3} \\ &\times \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathscr{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathscr{D}\nu(t)\,(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu) \\ &\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{MR^2}{2}(k^2\,\mathrm{cn}^2\mu + k'^2\,\mathrm{cn}^2\nu)(\dot\mu^2 + \dot\nu^2)\right.\right. \\ &\left.\left. - \frac{\hbar^2}{2MR^2}\left(\frac{n_1^2 - \frac14}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu} + \frac{n_2^2 - \frac14}{\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu} + \frac{n_3^2 - \frac14}{\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu} - \frac14\right)\right]dt\right\} \end{aligned}$$

$$\equiv \sum_{n_1, n_2, n_3 \in Z} \frac{e^{i[n_1(\phi_1'' - \phi_1') + n_2(\phi_2'' - \phi_2') + n_3(\phi_3'' - \phi_3')]}}{(2\pi R)^3}$$

$$\times \sum_m \sum_{l=0}^{\infty} \Psi_{ml}^{(n_1, n_2, n_3)}(\mu'', \nu'') \Psi_{ml}^{(n_1, n_2, n_3)*}(\mu', \nu') \exp\left[-\frac{i\hbar T}{2MR^2} l(l+4)\right], \tag{192}$$

where $M(\mu, 0) = \operatorname{sn}\mu \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{cn}\nu \operatorname{dn}\nu$. Here we have introduced the matrix elements (wave functions) $\Psi_{ml}^{(n_1, n_2, n_3)}(\mu, \nu)$ corresponding to these coordinate system in terms of the elliptic coordinates. This expansion may be on a formal level, however it provides a connection to the solution of the Higgs oscillator on the sphere by analytic continuation in the parameters $\{n_i\}_{i=1}^3 \mapsto \{k_i\}_{i=1}^3$ with the k_i some real numbers.

References

[1] M. Böhm and G. Junker, The SU(1, 1) Propagator as a Path Integral Over Noncompact Groups, Phys. Lett. **A 117** (1986) 375.

[2] M. Böhm and G. Junker, Path Integration Over Compact and Noncompact Rotation Groups, J. Math. Phys. **28** (1987) 1978.

[3] M. Böhm and G. Junker, Group Theoretical Approach to Path Integrations on Spheres, in "Path Summation: Achievements and Goals", Trieste 1987, p. 469; eds. S. Lundquist et al. (World Scientific, Singapore, 1988).

[4] D. Bonatos, C. Daskaloyannis and K. Kokkotas, Quantum-Algebraic Description of Quantum Superintegrable Systems in Two Dimensions, Phys. Rev. **A 48** (1993) R 3407.

[5] A. O. Barut, A. Inomata and G. Junker, Path Integral Treatment of the Hydrogen Atom in a Curved Space of Constant Curvature, J. Phys. A: Math. Gen. **20** (1987) 6271.

[6] A. O. Barut, A. Inomata and G. Junker, Path Integral Treatment of the Hydrogen Atom in a Curved Space of Constant Curvature II, J. Phys. A: Math. Gen. **23** (1990) 1179.

[7] M. V. Carpio-Bernido, Path Integral Quantization of Certain Noncentral Systems with Dynamical Symmetries, J. Math. Phys. **32** (1991) 1799.

[8] M. V. Carpio-Bernido, Green Function for an Axially Symmetric Potential Field: A Path Integral Evaluation in Polar Coordinates, J. Phys. A: Math. Gen. **24** (1991) 3013.

[9] M. V. Carpio-Bernido, C. C. Bernido and A. Inomata, Exact Path Integral Treatment of Two Classes of Axially Symmetric Potentials, in "Third International Conference on Path Integrals From meV to MeV", 1989, p. 442; eds.: V. Sa-yakanit et al. (World Scientific, Singapore, 1989).

[10] M. V. Carpio-Bernido and A. Inomata, Path Integral Treatment of the Harmann Potential, in "Bielefeld Encounters in Physics and Mathematics VII, Path Integrals From meV to MeV", 1985, p. 261; eds.: M. C. Gutzwiller et al. (World Scientific, Singapore, 1986).

[11] D. P. L. Castrigiano and F. Stärk, New Aspects of the Path Integrational Treatment of the Coulomb Potential, J. Math. Phys. **30** (1989) 2785.

[12] L. Chetouani, L. Guechi and T. F. Hammann, Exact Path Integral Solution of the Coulomb Plus Aharonov-Bohm Potential, J. Math. Phys. **30** (1989) 655.

[13] L. Chetouani and T. F. Hammann, Coulomb Green's Function, in a n-Dimensional Euclidean Space, J. Math. Phys. **27** (1986) 2944.

[14] L. Chetouani and T. F. Hammann, Traitement Exact des Système Coulombiens, dans le Formalisme des Intégrales de Feynman, en Coordonnées Parabolique, Nuovo Cimento **B 98** (1987) 1.

[15] B. S. DeWitt, Dynamical Theory in Curved Spaces. I. A Review of the Classical Quantum Action Principles, Rev. Mod. Phys. **29** (1957) 377.

[16] I. H. Duru, Path Integrals Over SU(2) Manifold and Related Potentials, Phys. Rev. **D 30** (1984) 2121.

[17] I. H. Duru and H. Kleinert, Solution of the Path Integral for the H-Atom, Phys. Lett. **B 84** (1979) 185.

[18] I. H. Duru and H. Kleinert, Quantum Mechanics of H-Atoms From Path Integrals, Fortschr. Phys. **30** (1982) 401.

[19] N. W. EVANS, Superintegrability in Classical Mechanics, Phys. Rev. **A41** (1990) 5666; Group Theory of the Smorodinsky-Winternitz System, J. Math. Phys. **32** (1991) 3369; Super-Integrability of the Winternitz-System, Phys. Lett. **A147** (1990) 483.
[20] R. P. FEYNMAN, Space-Time Approach to Non-Relativistic Quantum Mechanics, Rev. Mod. Phys. **20** (1948) 367.
[21] R. P. FEYNMAN and A. HIBBS, Quantum Mechanics and Path Integrals (McGraw Hill, New York, 1965).
[22] W. FISCHER, H. LESCHKE and P. MÜLLER, Changing Dimension and Time: Two Well-Founded and Practical Techniques for Path Integration in Quantum Physics, J. Phys. A: Math. Gen. **25** (1992) 3835;
Path Integration in Quantum Physics by Changing the Drift of the Underlying Diffusion Process: Application of Legendre Processes, Ann. Phys. (N.Y.) **227** (1993) 206;
Path Integration in Quantum Physics by Changing Drift and Time of the Underlying Diffusion Process, in "VI. International Conference for Path Integrals from meV to MeV", Tutzing, Germany 1992, p. 259; eds. H. Grabert et al. (World Scientific, Singapore, 1993).
[23] J. FRIŠ, V. MANDROSOV, YA. A. SMORODINSKY, M. UHLIR and P. WINTERNITZ, On Higher Symmetries in Quantum Mechanics, Phys. Lett. **16** (1965) 354;
J. FRIŠ, YA. A. SMORODINSKIĬ, M. UHLÍŘ and P. WINTERNITZ, Symmetry Groups in Classical and Quantum Mechanics, Sov. J. Nucl. Phys. **4** (1967) 444.
[24] J.-L. GERVAIS and A. JEVICKI, Point Canonical Transformations in the Path Integral, Nucl. Phys. **B110** (1976) 93.
[25] YA. A. GRANOVSKY, A. S. ZHEDANOV and I. M. LUTZENKO, Quadratic Algebras and Dynamics in Curved Spaces. I. Oscillator, Theor. Math. Phys. **91** (1992) 474.
[26] YA. A. GRANOVSKY, A. S. ZHEDANOV and I. M. LUTZENKO, Quadratic Algebras and Dynamics in Curved Spaces. II. The Kepler Problem, Theor. Math. Phys. **91** (1992) 604.
[27] C. GROSCHE, The Product Form for Path Integrals on Curved Manifolds, Phys. Lett. **A128** (1988) 113.
[28] C. GROSCHE, Path Integral Solution of a Class of Potentials Related to the Pöschl-Teller Potential, J. Phys. A: Math. Gen. **22** (1989) 5073.
[29] C. GROSCHE, The Path Integral for the Kepler Problem on the Pseudosphere, Ann. Phys. (N.Y.) **204** (1990) 208.
[30] C. GROSCHE, Separation of Variables in Path Integrals and Path Integral Solution of Two Potentials on the Poincaré Upper Half-Plane, J. Phys. A: Math. Gen. **23** (1990) 4885.
[31] C. GROSCHE, Coulomb Potentials by Path-Integration, Fortschr. Phys. **40** (1992) 695.
[32] C. GROSCHE, Path Integral Solution of Scarf-Like Potentials, Nuovo Cimento **B108** (1993) 1365.
[33] C. GROSCHE, Path Integral Solution of Two Potentials Related to the SO(2, 1) Dynamical Algebra, J. Phys. A: Math. Gen. **26** (1993) L279.
[34] C. GROSCHE, Path Integral Solution of a Class of Explicitly Time-Dependent Potentials, Phys. Lett. **A182** (1993) 28.
[35] C. GROSCHE, On the Path Integral in Imaginary Lobachevsky Space, J. Phys. A: Math. Gen. **27** (1994) 3475.
[36] C. GROSCHE, Path Integration and Separation of Variables in Spaces of Constant Curvature in Two and Three Dimensions, DESY preprint, Fortschr. Phys. **42** (1994) 509.
[37] C. GROSCHE, Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae, DESY Report, DESY 95-021, February 1995, to be published by World Scientific.
[38] C. GROSCHE, G. S. POGOSYAN and A. N. SISSAKIAN, Path Integral Discussion for Smorodinsky-Winternitz Potentials, I. Two- and Three-Dimensional Euclidean Space, Fortschr. Phys. **43** (1995) 453.
[39] C. GROSCHE, G. S. POGOSYAN and A. N. SISSAKIAN, Path Integral Discussion for Smorodinsky-Winternitz Potentials, III. Two- and Three-Dimensional Pseudosphere, DESY Report, in preparation.
[40] C. GROSCHE and F. STEINER, Path Integrals on Curved Manifolds, Zeitschr. Phys. **C36** (1987) 699.
[41] C. GROSCHE and F. STEINER, Classification of Exactly Solvable Feynman Path Integrals, in Proceedings of the "Fourth International Conference on Path Integrals from meV to MeV", Tutzing, Germany 1992, p. 276; H. Grabert, A. Inomata, L. S. Schulman and U. Weiss (eds.) (World Scientific, Singapore, 1993), Table of Feynman Path Integrals, to appear in: Springer Tracts in Modern Physics.

[42] H. HARTMANN, Bewegung eines Körpers in einem ringförmigen Potentialfeld, Theoret. Chim. Acta **24** (1972) 201.
[43] C. R. HOLT, Construction of New Integrable Hamiltonians in Two Degrees of Freedom, J. Math. Phys. **23** (1982) 1037.
[44] P. W. HIGGS, Dynamical Symmetries in a Spherical Geometry, J. Phys. A: Math. Gen. **12** (1979) 309.
[45] A. HURWITZ, Mathematische Werke, Band II, pp. 641 (Birkhäuser, Basel, 1933).
[46] L. INFELD, On a New Treatment of Some Eigenvalue Problems, Phys. Rev. **59** (1941) 737.
[47] L. INFELD and A. SCHILD, A Note on the Kepler Problem in a Space of Constant Negative Curvature, Phys. Rev. **67** (1945) 121.
[48] A. INOMATA, H. KURATSUJI and C. C. GERRY, Path Integrals and Coherent States of SU(2) and SU(1, 1) (World Scientific, Singapore, 1992).
[49] A. INOMATA, Alternative Exact-Path-Integral-Treatment of the Hydrogen Atom, Phys. Lett. **A101** (1984) 253.
[50] A. INOMATA and R. WILSON, Path Integral Realization of a Dynamical Group, Lecture Notes in Physics **261**, p. 42 (Springer, Berlin-Heidelberg, 1985);
Factorization-Algebraization-Path Integration and Dynamical Groups, in "Symmetries in Science II", eds.: B. Gruber and R. Lenczewski, p. 255 (Plenum Press, New York, 1986).
[51] A. A. IZMEST'EV, G. S. POGOSYAN, A. N. SISSAKIAN and P. WINTERNITZ, Contraction of Lie Algebras and Separation of Variables. I. Two-Dimensional Sphere. CRM preprint, Montreal 1995.
[52] G. JUNKER, Remarks on the Local Time Rescaling in Path Integration, J. Phys. A: Math. Gen. **23** (1990) L881.
[53] E. G. KALNINS, Separation of Variables for Riemannian Spaces of Constant Curvature (Longman Scientific & Technical, Essex, 1986).
[54] E. G. KALNINS and W. MILLER JR. and P. WINTERNITZ, The Group $O(4)$, Separation of Variables and the Hydrogen Atom, SIAM J. Appl. Math. **30** (1976) 630.
[55] N. KATAYAMA, A Note on a Quantum-Mechanical Harmonic Oscillator in a Space of Constant Curvature, Nuovo Cimento **B107** (1992) 763.
[56] M. KIBLER and C. CAMPIGOTTO, Classical and Quantum Study of a Generalized Kepler-Coulomb System, Int. J. Quantum Chem. **45** (1993) 209.
[57] M. KIBLER, L. G. MARDOYAN and G. S. POGOSYAN, On a Generalized Kepler-Coulomb System: Interbasis Expansions, Int. J Quantum Chem. **52** (1994) 1301.
[58] H. KLEINERT, How to do the Time Sliced Path Integral for the H Atom, Phys. Lett. **A120** (1987) 361.
[59] H. KLEINERT, Path Integrals in Quantum Mechanics, Statistics and Polymer Physics (World Scientific, Singapore, 1990).
[60] H. KLEINERT and I. MUSTAPIC, Summing the Spectral Representations of Pöschl-Teller and Rosen-Morse Fixed-Energy Amplitudes, J. Math. Phys. **33** (1992) 643.
[61] G. KLEPPE, Green's Function for Anti de Sitter Space Gravity, University of Alabama, UA-HEP9403, gr-qc/9406005, June 1994.
[62] H. A. KRAMERS and G. P. ITTMANN, Zur Quantelung des asymmetrischen Kreisels (I–III), Zeitschr. Phys. **53** (1929) 553; **58** (1929) 217; **60** (1930) 663.
[63] YU. A. KUROCHKIN and V. S. OTCHIK, Analogue of the Runge-Lenz Vector and Energy Spectrum in the Kepler Problem on the Three-Dimensional Sphere, Vesti Akad. Nauk. BSSR **3** (1993) 56 (in Russian).
[64] P. KUSTAANHEIMO and E. STIEFEL, Perturbation Theory of Kepler Motion Basedon Spinor Regularization, J. Rein. Angew. Math. **218** (1965) 204.
[65] T. D. LEE, Particle Physics and Introduction to Field Theory (Harwood Academic Publishers, Chur, 1981).
[66] H. I. LEEMON, Dynamical Symmetries in a Spherical Geometry, J. Phys. A: Math. Gen. **A12** (1979) 489.
[67] I. LUKÁČS, On Complete Set of the Quantum Mechanical Observables on a Two-Dimensional Sphere, Theor. Math. Phys. **14** (1973) 271.
[68] I. LUKÁČS and YA. A. SMORODINSKIĬ, Wave Functions for the Asymmetric Top, Sov. Phys. JETP **30** (1970) 728.
[69] D. W. MCLAUGHLIN and L. S. SCHULMAN, Path Integrals in Curved Spaces, J. Math. Phys. **12** (1971) 2520.

[70] A. A. MAKAROV, YA. A. SMORODINSKY, KH. VALIEV and P. WINTERNITZ, A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries, Nuovo Cimento **A52** (1967) 1061.
[71] M. F. MANNING and N. ROSEN, A Potential Function for the Vibrations of Diatomic Molecules, Phys. Rev. **44** (1933) 953.
[72] I. M. MAYES and J. S. DOWKER, Canonical Functional Integrals in General Coordinates, Proc. Roy. Soc. (London) **A327** (1972) 131;
Hamiltonian Orderings and Functional Integrals, J. Math. Phys. **14** (1973) 434;
The Canonical Quantization of Chiral Dynamics, Nucl. Phys. **B29** (1971) 259.
[73] W. MILLER JR., Symmetry and Separation of Variables (Adison-Wesley, Reading, 1977).
[74] M. M. MIZRAHI, The Weyl Correspondence and Path Integrals, J. Math. Phys. **16** (1975) 2201.
[75] F. MÖGLICH, Beugungserscheinungen an Körpern von ellipsoidischer Gestalt, Ann. Phys. **83** (1927) 609.
[76] P. M. MORSE and H. FESHBACH, Methods of Theoretical Physics (McGraw Hill, New York, 1953).
[77] M. N. OLEVSKIĬ, Triorthogonal Systems in Spaces of Constant Curvature in which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables, Math. Sb. **27** (1950) 379 (in Russian).
[78] J. C. D'OLIVO and M. TORRES, The Canonical Formalism and Path Integrals in Curved Spaces, J. Phys. A: Math. Gen. **21** (1988) 3355;
The Weyl Ordering and Path Integrals in Curved Spaces, in "Path Summation: Achievements and Goals", Trieste 1987, p. 481, eds.: S. Lundquist et al. (World Scientific, Singapore, 1988).
[79] M. OMOTE, Point Canonical Transformations and the Path Integrals, Nucl. Phys. **B120** (1977) 325.
[80] V. S. OTCHIK and V. M. RED'KOV, Quantum Mechanical Kepler Problem in Space with Constant Curvature, Minsk 1983, preprint No. 298 (in Russian).
[81] N. K. PAK and I. SÖKMEN, General New-Time Formalism in the Path Integral, Phys. Rev. **A30** (1984) 1629.
[82] K. PAK and I. SÖKMEN, A New Exact Path Integral Treatment of the Coulomb and the Morse Potential Problems, Phys. Lett. **A100** (1984) 327.
[83] J. PATERA and P. WINTERNITZ, A New Basis for the Representations of the Rotation Group. Lamé and Heun Polynomials, J. Math. Phys. **14** (1973) 1130.
[84] A. PELSTER and A. WUNDERLIN, On the Generalization of the Duru-Kleinert-Propagator Transformations, Zeitschr. Phys. **B89** (1992) 373.
[85] G. S. POGOSYAN, A. N. SISSAKIAN and S. I. VINITSKY, Interbasis "Sphere-Cylinder" Expansions for the Oscillator in the Three-Dimensional Space of Constant Positive Curvature, in "Frontiers of Fundamental Physics", p. 429, eds. M. Barone, F. Selleri (Plenum Publishing, New York, 1994).
[86] G. PÖSCHL and E. TELLER, Bemerkungen zur Quantenmechanik des anharmonischen Oszillators, Zeitschr. Phys. **83** (1933) 143.
[87] V. N. PERVUSHIN, G. S. POGOSYAN, A. N. SISSAKIAN and S. I. VINITSKY, The Equation for Quasiradial Functions in the Momentum Representation on a Three-Dimensional Sphere, Phys. At. Nucl. **56** (1993) 56.
[88] E. SCHRÖDINGER, A Method of Determining Quantum Mechanical Eigenvalues and Eigenfunctions, Proc. Roy. Irish Soc. **46** (1941) 9;
Further Studies on Solving Eigenvalue Problems by Factorization, Proc. Roy. Irish Soc. **46** (1941) 183;
The Factorization of the Hypergeometric Equation, Proc. Roy. Irish Soc. **47** (1941) 53.
[89] L. S. SCHULMAN, Techniques and Applications of Path Integration (John Wiley & Sons, New York, 1981).
[90] YA. A. SMORODINSKIĬ AND, I. I. TUGOV, On Complete Sets of Observables, Sov. Phys. JETP **23** (1966) 434.
[91] F. STEINER, Space-Time Transformations in Radial Path Integrals, Phys. Lett. **A106** (1984) 356.
[92] F. STEINER, Exact Path Integral Treatment of the Hydrogen Atom, Phys. Lett. **A106** (1984) 363.
[93] F. STEINER, Path Integrals in Polar Coordinates From eV to GeV, in "Path Integrals From meV to MeV", Bielefeld 1985, p. 335; eds. M. C. Gutzwiller et al. (World Scientific, Singapore, 1986).
[94] A. F. STEVESON, Note on the "Kepler Problem" in a Spherical Space, and the Factorization Method of Solving Eigenvalue Problems. Phys. Rev. **59** (1941) 842.
[95] S. N. STORCHAK, Path Reparametrization in a Path Integral on a Finite-Dimensional Manifold, Theor. Math. Phys. **75** (1988) 610;
Rheonomic Homogeneous Point Transformation and Reparametrization in the Path Integral, Phys. Lett. **A135** (1989) 77.

[96] S. N. STORCHAK, Remarks on Quantization of the Hydrogen Atom by the Path-Integral Method, Theor. Math. Phys. **82** (1990) 32.
[97] S. I. VINITSKY, L. G. MARDOYAN, G. S. POGOSYAN, A. N. SISSAKIAN and T. A. STRIZH, A Hydrogen Atom in the Curved Space. Expansion over Free Solutions on the Three-Dimensional Sphere, Phys. At. Nucl. **56** (1993) 321.
[98] P. WINTERNITZ, L. LUKAC and YA. A. SMORODINSKY, Quantum Mechanics of the Little Group of the Poincaré Group, Sov. J. Nucl. Phys. 7 (1968) 139.
[99] A. YOUNG and C. DEWITT-MORETTE, Time Substitution in Stochastic Processes as a Tool in Path Integration, Ann. Phys. **169** (1986) 140.

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