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MATRIX OF FINITE TRANSLATIONS IN OSCILLATOR BASIS

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Abstract

It is shown that the probability amplitude of quantum transitions under the action of a sudden homogeneous field coincides with the Charlier function up to a phase factor. The Charlier function is expressed in terms of the matrix of the operator of finite translations with respect to the basis of a linear operator. This formula is used to derive two summation theorems for the Charlier functions.

Instead of Introduction. In October, 1993 it will be a year since the decease of an outstanding scientist, Ya.A. Smorodinsky. Contacts of physicists and mathematicians of the present generation with Yakov Abramovich were highly fruitful and instructive for them as his intellectual scope was very wide. Besides, he understood and appreciated everything beautiful, which attracted people to him.

This short note is a tribute of the greatest respect of the authors to the charming man with wide-ranging erudition.

Formulation of the Problem. Here we will calculate the probability amplitude of transition of a linear oscillator from an n- to a k-state under the action of a suddenly switched-on homogeneous field $U = -Fx$. This problem is interesting for the following reasons: a) is generalizes one of the problems expounded in [1]; b) the amplitude is expressed only through the Charlier function; c) the solution of this problem provides a representation for the Charlier function that accents the symmetry rather than combinatorics; d) this representation gives new results.

When the field is switched on, the oscillator remains linear but with a shifted center of equilibrium, which means that in units the $m = \hbar = \omega = 1$ amplitude of transition $n \to k$ is described by the expression

$$\mathcal{P}_{kn} = \int_{-\infty}^{\infty} \bar{H}_k(x - F)\bar{H}_k(x)\, dx \tag{1}$$

where $\bar{H}_m$ are wave functions of steady states of the linear oscillator.

When $n = 0$, the amplitude (1) is calculated by integration by parts as a result of which the probability of transition $0 \to k$ is determined by the Poisson distribution [1]

$$w_{k0} = \frac{b^k e^{-b}}{k!} \tag{2},$$

where $b = F^2/2$.

Our further goal is to compute the amplitude (1) at arbitrary n.

Method of Calculation. The integral (1) is not so simple as it seems at first sight; however, it can easily be calculated by using the operator of finite translation and the technique of second quantization. Writing

$$\bar{H}_n(x - F) = e^{-F\partial_x}\bar{H}_k(x)$$

we can represent the amplitude (1) as the matrix of the operator of finite translation over the basis of linear oscillator (in ref. [2] the diagram technique was developed for computing the matrix of finite rotations in the basis of a multidimensional isotropic oscillator). In the Dirac notation we have

$$\mathcal{P}_{kn} = \langle n | e^{-F\partial_x} | k \rangle$$

Introduce the creation and annihilation operators

$$\hat{a}^+ = (x - \partial_x)/\sqrt{2}, \quad \hat{a} = (x + \partial_x)/\sqrt{2}$$

which obey the equation

$$-F\partial_x = F(\hat{a}^+ - \hat{a})/\sqrt{2}$$

As $[\hat{a}, \hat{a}^+] = 1$, the Hausdorff-Baker formula holds true, i.e.

$$e^{-F\partial_x} = e^{-b/2} e^{\sqrt{b}\,\hat{a}^+} e^{-\sqrt{b}\,\hat{a}}$$

Now we use the complete set of intermediate oscillator states and write the amplitude in the form

$$\mathcal{P}_{kn} = e^{-b/2} \sum_{m=0}^{\infty} \langle n | e^{\sqrt{b}\,\hat{a}^+} | m \rangle \langle m | e^{-\sqrt{b}\,\hat{a}} | k \rangle \tag{4}$$

Using the known formulae

$$\hat{a}^+ | \mu \rangle = \sqrt{\mu + 1}\, | \mu + 1 \rangle, \quad \hat{a} | \mu \rangle = \sqrt{\mu}\, | \mu - 1 \rangle$$

we can easily prove that

$$\begin{aligned} \langle n | e^{\sqrt{b}\,\hat{a}^+} | m \rangle &= \sqrt{\frac{m!}{n!}} \binom{n}{m} (\sqrt{b})^{n-m}, \\ \langle m | e^{-\sqrt{b}\,\hat{a}} | k \rangle &= \sqrt{\frac{m!}{k!}} \binom{k}{m} (-\sqrt{b})^{k-m} \end{aligned}$$

Binomial coefficients in these formulae vanish for $m > k$ and $m > n$, respectively, and therefore the sum in (4) is truncated at minimal n or k. The result is a product of an exponential factor and a polynomial.

Transition Probability. As follows from the previous section, the amplitude (1) can be represented in the following final form

$$\mathcal{P}_{nk} = (-1)^k \bar{C}_n(k, b) \tag{5},$$

where $\bar{C}_n(k, b)$ is given by the expression

$$\bar{C}_n(k, b) = \left(\frac{b^{n+k} e^{-b}}{n!k!} \right)^{1/2} C_n(k, b) \tag{6},$$

and $C_n(k, b)$ is the following polynomial

$$C_n(k, b) = \sum_m (-1)^m \binom{n}{m} \binom{k}{m} \frac{m!}{b^m} \tag{7}$$

The polynomial (7) is known in mathematics as the Charlier polynomial. It belongs to the class of so-called orthogonal polynomials of a discrete variable [3]. It is fixed if the parameter . and one of the indices are fixed (the second index represents a discrete variable).

WE will call $\bar{C}_n(k,b)$ the Charlier function. Apart from the obvious symmetry in indices n and k, the Charlier functions obey the orthonormalization condition

$$\sum_{k=0}^{\infty} \bar{C}_m(k,b)\bar{C}_n(k,b) = \delta_{mn} \tag{8}$$

and the recurrence formula

$$\sqrt{b(n+1)}\bar{C}_{n+1}(k,b) + (k-n-b)\bar{C}_n(k,b) + \sqrt{bn}\bar{C}_{n-1}(k,b) = 0 \tag{9}$$

(The relations (8) and (9) are usually written in terms of the Charlier polynomials.) For the transition probability we have

$$w_{kn} = \left(\bar{C}_n(k, F^2/2)\right)^2 \tag{10}$$

Note that at $n = 0$ formula (10) reduces to the Poisson distribution, formula (2), for the probability w_{k0}. (The model with the generalized Poisson distribution was considered in ref. [4], as well, where the distributions were expressed via the Laguerre polynomials connected with the Charlier polynomials as follows: $C_n(k,b) = (-b)^n n! L_n^{k-n}(b)$). The orthonormalization condition (8) allows us to verify the validity of formula (10). Besides, formulae (8) and (9) can be used to compute the expectation value and dispersion for the distribution (10). It is easy to show that $\bar{k} = F^2/2 + n$ and $D = \overline{k^2} - \bar{k}^2 = (2n+1)F/2$. When $n = 0$, we arrive at the relation $\bar{k} = D$ typical of the Poisson distribution.

One more interesting property is to be noted: in the limit $b = 0$ the Charlier function transforms into the Kronecker symbol. Indeed, when $n > k$ $(n < k)$ in the limit $b = 0$ the leading term in (7) is proportional to b^{-k} (b^{-n}), and, as follows from (6), the Charlier function vanishes; whereas at $n = k$ in the same limit the Charlier function equals unity.

Symmetry instead of Combinatorics. Comparison of the relations (3) and (5) leads to the following interesting representation for the Charlier functions:

$$\bar{C}_n(k,b) = (-1)^k \langle n | e^{-\sqrt{2b}\partial_x} | k \rangle \tag{11}$$

Whereas the definition (6) accents the combinatorics, the representation (11) transfers this accent to the symmetry thus stressing the connection of the Charlier functions with finite translations. Formula (11) implies that the Charlier function coincides (within a phase factor) with the matrix of operator of finite translations in the basis of a linear oscillator. This point of view may be a basis of the theory of Charlier functions. It is not difficult to verify that it immediately gives the above-listed properties of Charlier functions.

It is important that this approach is not only a more customary view of old things. It can also be used to derive new results. Indeed, if we start with the obvious identity

$$e^{-F_1\partial x} e^{F_2 \partial x} = e^{(-F_1+F_2)\partial x}$$

and pass in it from operators to matrices in the basis of a linear oscillator, we can easily derive the summation theorem

$$\bar{C}_n\left(k, (\sqrt{b_1} - \sqrt{b_2})^2\right) = \sum_{m=0}^{\infty} \bar{C}_n(m, b_1)\bar{C}_k(m, b_2)$$

that is a generalization of the orthonormalization condition (8) provided $b_1 = b_2$.

Consider one more identity,

$$e^{-F_1\partial x} e^{-F_2\partial x} = e^{-(F_1+F_2)\partial x}$$

Then, by analogy, we easily deduce one more summation theorem,

$$\bar{C}_n\left(k,(\sqrt{b_1}+\sqrt{b_2})^2\right)=\sum_{m=0}^{\infty}(-1)^m\bar{C}_n(m,b_1)\bar{C}_k(m,b_2),$$

that for a particular case, $b_1=b_2=b$, gives

$$\bar{C}_n(k,4b)=\sum_{m=0}^{\infty}(-1)^m\bar{C}_n(m,b)\bar{C}_k(m,b)$$

Thus, we see that the representation (11) is in fact not only beautiful but also useful.

Instead of Conclusion. It was a surprise for us to come across Charlier polynomials, one of the five "bricks" of the theory of orthogonal polynomials of the discrete variable. As is known, Ya.A.Smorodinsky was carried away by that theory, thus being a great expert in it. It may happen that our results would evoke a smile as if he were listening a good anecdote. In any case we regret that cannot know the corresponding opinion of Yakov Abramovich himself.

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