Physics Letters B 321 (1994) 381–384
North-Holland

PHYSICS LETTERS B

β-function for the φ^4-model in variational perturbation theory

A.N. Sissakian, I.L. Solovtsov [1] and O.P. Solovtsova

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Head Post Office P.O. Box 79, Moscow 101000, Russian Federation

Received 5 July 1993
Editor: R. Gatto

We consider the renormalization procedure for the φ^4-model in the variational perturbation theory. The nonperturbative β-function is derived in the framework of the proposed approach. The obtained result is in agreement with the four-loop approximation and has an asymptotic behaviour as $g^{3/2}$ for a large coupling constant.

A central problem of quantum field theory is to go beyond the scope of standard perturbation theory. A great amount of study is devoted to the development of nonperturbative methods. Many of the latter are based on the variational procedure for finding the leading contribution. The idea of applying the variational methods in quantum field theory has a long history [1,2] (see also [3,4]). In the last few years approaches with different modifications have found many applications. In this paper, we shall consider a method of the variational perturbation theory (VPT) [5–7]. In the framework of VPT it is possible to represent the investigated quantity in the form of a series and it is possible to influence the properties of convergence of this series through certain variational parameters. Thus it becomes possible to optimize of the VPT series from the viewpoint of a better approximation. Our method is formulated in terms of Gaussian functional quadratures (like in perturbation theory). Also, we shall construct the VPT so that for its Nth order only those diagrams that compose the Nth order of standard perturbation theory will be required.

The massless φ^4-model in four dimensions has the Euclidean action

$$S[\varphi] = S_0[\varphi] + S_I[\varphi], \tag{1}$$

where

$$S_0[\varphi] = \frac{1}{2}\int \mathrm{d}x\ \varphi(-\partial^2)\,\varphi, \tag{2}$$

$$S_I[\varphi] = \frac{(4\pi)^2}{4!}\, g \int \mathrm{d}x\ \varphi^4. \tag{3}$$

As is well known, the series of perturbation theory for the generating functional of the Green's functions

$$W[J] = \int D\varphi\ \exp\Big\{-S[\varphi] + \int \mathrm{d}x\ J\cdot\varphi\Big\}. \tag{4}$$

diverges. A formal argument consists in a meaningless functional integral for the negative coupling constant. The function $W[J]$ as a function of g is not analytic at $g = 0$. The concrete asymptotic behavior of higher-order terms of perturbation theory can be determined by the functional saddle-point method (the large parameter

[1] E-mail address: solovtso@theor.jinrc.dubna.su.

Elsevier Science B.V.
SSDI 0370-2693(93)E1572-F

is the number of the series term) [8–11]. The main contribution to the functional integral (4) comes from the configurations of the fields φ which correspond to the positive power of the large saddle-point parameter. However, in this case, the functional (3) can not be considered as the perturbative term in the comparison with expression (2), which appears as divergence of the perturbation series.

The idea of the VPT method consists in the construction of a new effective functional interaction S_I'. We expect that this functional can be considered as a small value when compared with a new functional S_0'. In realization of this idea we should ensure the possibility of the calculations. Practically, we must use only the Gaussian functional integrals, i.e. the form of $\widetilde{S}[\varphi]$ should be such that the functional integral in (4) can be reduced to Gaussian quadratures.

Let us consider the VPT-functional

$$\widetilde{S}[\varphi] = \theta^2 S_0^2[\varphi] \tag{5}$$

and rewrite the total action (1) as

$$S[\varphi] = S_0'[\varphi] + \eta S_I'[\varphi], \tag{6}$$

where

$$S_0'[\varphi] = S_0[\varphi] + \widetilde{S}[\varphi], \tag{7}$$

and

$$S_I'[\varphi] = S_I[\varphi] - \widetilde{S}[\varphi]. \tag{8}$$

In this case, the expansion of expression (4) is carried out in powers of η. After all calculations we should put $\eta = 1$. The parameter θ^2 in eq. (5) is a parameter of variational type. The initial functional (4) certainly does not depend on this parameter. We may take θ^2 so as to provide the best approximation with a finite number of VPT series terms. Different methods of the optimization were considered in [6].

It is convenient to define the new parameter t by the relation

$$\theta^2 = 4\, C_s \frac{(4\pi)^2}{4!}\, g \cdot t. \tag{9}$$

Here $C_s = 4!/(16\pi)^2$ is a constant entering into the Sobolev inequality (see, for example, refs. [12,13] and ref. [14])

$$\int \mathrm{d}x\, \varphi^4 \leqslant C_s \left[\int \mathrm{d}x\, \varphi\, (-\partial^2)\, \varphi \right]^2. \tag{10}$$

The parameter t is fixed if we require the contribution of higher order terms of the VPT series to be minimal. This way of determining a variational parameter called the asymptotic optimization of VPT series gives the value $t = 1$ [7].

After expansion in powers of η we obtain that the remainder contains the $\widetilde{S}[\varphi]$ in the exponental and consequently, we have a nongaussian form of the functional integral. However, the problem is easily solved by implementing the Fourier transformation. As a result, the Green's function $G_{2\nu}$ in the Nth order of VPT takes the following form:

$$G_{2\nu}^{(N)} = \int_0^\infty \mathrm{d}\alpha\, \alpha^{\nu-1} \exp(-\alpha - \theta^2\alpha^2) \sum_{n=0}^{N} \eta^n \alpha^{2n} \sum_{k=0}^{n} \frac{(\theta^2)^{n-k}}{(n-k)!} \frac{g_{2\nu}^k}{\Gamma(2k+\nu)}, \tag{11}$$

where the functions $g_{2\nu}^k$ are ordinary perturbative coefficients for the Green's function $G_{2\nu}$. To calculate them, the standard Feynman diagrams can be used.

It should be stressed that the expansion of expression (11) in powers of the coupling constant g contains all powers of g. The first N terms of this expansion coincide with N terms of a perturbative series.

Let us consider the procedure of renormalization. Instead of the field φ and the coupling constant g we introduce the bare field φ_0 and the bare coupling constant g_0. The field φ_0 is connected with the renormalized field by the relation: $\varphi_0 = Z^{1/2}\varphi$. The divergent constants Z and g_0 are obtained from the VPT expansion. The constant Z can be calculated by using the propagator G_2. We will employ the constant Z in the first order of the VPT series. From eq. (11) we find

$$Z^{(1)} = \Gamma(1)\, J_1(\theta_0^2) + \eta\, \theta_0^2 \Gamma(3)\, J_3(\theta_0^2)\,, \tag{12}$$

where we define

$$J_\nu(\theta^2) = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}\alpha\, \alpha^{\nu-1} \exp(-\alpha - \alpha^2\theta^2)\,. \tag{13}$$

The function $J_\nu(\theta^2)$ is normalized by the condition $J_\nu(0) = 1$. The connected part of the four-point Green's function in the second order of VPT has the form

$$-\,G_4^{(2)}(\mu^2) = \eta\, g_0\, J_4(\theta_0^2) + \eta^2 \left[g_0\, \frac{\theta_0^2}{1!}\, \frac{\Gamma(6)}{\Gamma(4)}\, J_6(\theta_0^2) - \tfrac{3}{2}\, g_0^2\, J_6(\theta_0^2)\, \ln \frac{\Lambda^2}{\mu^2} \right]. \tag{14}$$

In this expression we wrote out only the divergent part we need in the following. We use the renormalization scheme with a symmetric normalization point μ^2. For the bare coupling constant g_0 we write down the VPT expansion $g_0 = g\,(1 + \eta\,\alpha + \dots)$. The VPT expansions for θ_0^2 and $J_\nu(\theta_0^2)$ are introduced in a similar manner. The divergent coefficient α is defined by expressions (12), (14) and the requirement for the function $-Z^2 G_4(\mu^2)$ being finite. If we change the normalization point $\mu \to \mu'$ and use the bare coupling constant being independent of μ we find the connection between g and g'

$$g' = g + \eta\, \beta(g)\, \ln \frac{\mu'^2}{\mu^2}\,, \tag{15}$$

where the Gell-Mann–Low function is expressed as

$$\beta(g) = \tfrac{3}{2}\, g^2 \frac{J_6(\theta^2)/J_4(\theta^2)}{1 - \theta^2\{[\Gamma(6)\, J_6(\theta^2)/\Gamma(4)\, J_4(\theta^2)] - 2\,[\Gamma(3)\, J_3(\theta^2)/\Gamma(1)\, J_1(\theta^2)]\}}\,. \tag{16}$$

Here the parameter θ^2 is connected with the renormalized coupling constant g by eq. (9) with the optimal value $t = 1$.

The expansion of the β-function (16) in the perturbation series contains all powers of the coupling constant g. It is interesting to compare the first coefficients of the VPT β-function (16) with the well-known coefficients of perturbation theory. From (16) we get

$$\beta(g) = 1.5\, g^2 - 2.25\, g^3 + 14.63\, g^4 - 134.44\, g^5 + \dots\,. \tag{17}$$

In the considered massless case, we use counterterms containing only divergent parts. In the framework of the dimensional regularization this conforms only to the pole part for counterterms [15]. The corresponding β-function in the four-loop approximation looks as follows [16]

$$\beta_{\text{perturb.}}(g) = 1.5\, g^2 - 2.83\, g^3 + 16.27\, g^4 - 135.80\, g^5 + \dots\,. \tag{18}$$

Note that in constructing the β-function (16) we used only the lowest order of VPT. For this approximation the expressions (17) and (18) are in agreement.

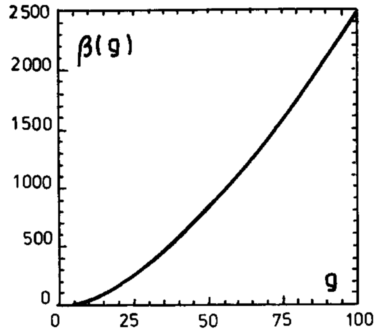

Fig. 1. The behaviour of the β-function [eq. (16)].

As follows from expression (16), the β-function is monotonously increasing and has no the ultraviolet stable point (fig. 1). For a large coupling constant, the β- function has the asymptotic behaviour

$$\beta(g) \simeq \frac{3}{10} \frac{\sqrt{\pi}}{\frac{3}{8}\pi - 1} g^{3/2}. \tag{19}$$

The degree of g in eq. (19) is larger than the linear increase of the β-function obtained in ref. [17], and is smaller than the square increase found in ref. [18].

We would like to thank V.G. Kadyshevsky, D.I. Kazakov, G.V. Efimov, O.Yu. Shevchenko A.A. Vladimirov and also A. Carlyle and N. Mellor for interest in this work and useful discussions. This work was supported in part by the Russian Found of Fundamental Researches under Grant No. 93-02-3754.

References

[1] L.I. Schiff, Phys. Rev. 130 (1963) 458.
[2] G. Rosen, Phys. Rev. 173 (1968) 1632.
[3] L. Polley and E.L. Pottinger, eds., Variational Calculation in Quantum Field Theory, Proc. of the Workshop (Wangerooge, Germany, 1987), (World Scientific, Singapore, 1988).
[4] G.V. Efimov, Problems of Quantum Theory of Nonlocal Interactions (Moscow, 1985).
[5] A.N. Sissakian and I.L. Solovtsov, Phys. Lett. A 157 (1991) 261.
[6] A.N. Sissakian and I.L. Solovtsov, Z. Phys. C 54 (1992) 263.
[7] A.N. Sissakian, I.L. Solovtsov and O.Yu. Shevchenko, Phys. Lett. B 297 (1992) 305; B 313 (1993) 367.
[8] C.S. Lam, Nuovo Cim. A 47 (1966) 451;
J.S. Langer, Ann. Phys. 44 (1967) 108;
J.C. Collins and D. Soper, Ann. Phys. 112 (1978) 209;
G. Auberson, G. Mennessier and G. Manoux, Nuovo Cim. A 48 (1978) 1.
[9] L.N. Lipatov, Sov. J. JETP 72 (1977) 411.
[10] D.I. Kazakov and D.V. Shirkov, Fortschr. Phys. 28 (1980) 465.
[11] J. Zinn-Justin, Phys. Rep. 70 (1981) 109.
[12] G. Parisi, Phys. Lett. B 68 (1977) 361.
[13] N.N. Khuri, Phys. Lett. B 150 (1985) 199.
[14] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Clarendon Press, Oxford, 1989).
[15] G. 't Hooft, Nucl. Phys. B 61 (1973) 455;
S. Weinberg, Phys. Rev. D 8 (1973) 3497.
[16] A.A. Vladimirov, D.V. Shirkov, Sov. J. UPhN 129 (1979) 407.
[17] P. Castoldi and C. Schomblond, Nucl. Phys. B 139 (1978) 269.
[18] D.I. Kazakov, O.V. Tarasov and D.V. Shirkov, Sov. J. Theor. Math. Phys. 38 (1979) 15.