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NONPERTURBATIVE β-FUNCTION IN QUANTUM CHROMODYNAMICS

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We propose a method by which it is possible to go beyond the scope of quantum chromodynamics perturbation theory. By using a new small parameter we formulate a systematic nonperturbative expansion and derive a renormalization β-function in quantum chromodynamics.

Quantum chromodynamics (QCD) is defined by the following basic gauge-invariant Lagrangian density

$$L = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \bar{q}^A_\alpha (i\hat{D} - \hat{M})_{\alpha\beta} q^A_\beta \,, \tag{1}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \,, \tag{2}$$

$$\hat{D}_{\alpha\beta} = \delta_{\alpha\beta} \partial_\mu - i g t^a_{\alpha\beta} A^a_\nu \,. \tag{3}$$

For the quantum theory of QCD the Lagrangian (1) is to be added with a gauge fixing term $L_{\text{g.f.}}(A)$ which is required to insure a proper quantization procedure. Besides, one must add a Faddeev–Popov ghost term $L_{\text{FP}} = L_0(\varphi) + g L_I(A, \varphi)$ that preserves unitarity and depends on gluon A^a_μ and ghost φ^a fields.

We rewrite the Lagrangian density in the form

$$L = L_2(A, q, \varphi) + g L_3(A, q, \varphi) + g^2 L_4(A) \,, \tag{4}$$

where

$$\begin{aligned} L_2(A, q, \varphi) &= L_0(A) + L_{\text{g.f.}}(A) + L_0(q) + L_0(\varphi) \,, \\ L_3(A, q, \varphi) &= L_3(A) + L_I(A, q) + L_I(A, \varphi) \,, \\ L_4(A) &= -\frac{1}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu \,. \end{aligned} \tag{5}$$

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The Lagrangian $L_2(A,q,\varphi)$ is a sum of free Lagrangians of the Yang–Mills fields defined in the covariant α_G-gauge by using the gauge fixing term of the form $L_{\text{g.f.}} = -\frac{1}{2\alpha_G}(\partial_\mu A^a_\mu)^2$ and free Lagrangians of the quark and ghost fields. The term $L_3(A,q,\varphi)$ generates the three-line vertices:

$$L_3(A) = -\frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) f^{abc} A^b_\mu A^c_\nu \,,$$

$L_I(A,q)$ and $L_I(A,\varphi)$ generate the three-gluon, quark-gluon-quark and ghost-gluon-ghost vertices, respectively. These interactions are the Yukawa type interactions. The term $L_4(A)$ gives the four-gluon vertices. This term requires some transformation in our approach we shall consider below. Now we give a brief description of some phenomenological results related to QCD at large distances.

Hadronic spectroscopy has been described phenomenologically by the static quark-antiquark confining potential (see, for example, Ref. 1)

$$V(r) = -\frac{4}{3}\frac{\alpha_S^{PT}(r)}{r} + a^2 r \,. \tag{6}$$

The function α_S^{PT} in Eq. (6) is the Fourier transform of the perturbative running coupling constant. The potential (6) is Coulomb-like at short distances in accordance with the asymptotic freedom. At large distances, the nonperturbative effects are dominant and responsible for confinement of colored states and the formation of hadrons. The second term in (6) describes the quark interaction at large distances and cannot be calculated in the framework of the standard QCD perturbation theory. The static quark-antiquark potential is linear in r for large distances as expected from the Wilson area law or the string picture of hadrons.

The quark potential in momentum space can be written as

$$V(q^2) = -\frac{16\pi}{3}\frac{\alpha_S(q^2)}{q^2} \,, \tag{7}$$

where $\alpha_S(q^2)$ is the invariant charge that describes the regions for large and small q^2.

If we assume that the invariant charge has the singular ir asymptotics $\alpha_S(q^2) \sim q^{-2}$, we obtain $V(r) \sim r$ at large distances. This asymptotic behavior corresponds to the increase of $\lambda = \alpha_S/(4\pi)$ and for the Gell-Mann–Low function we have $\beta(\lambda) \to -\lambda$ for a large coupling constant. This ir picture of the QCD arises in many different approaches. In particular, in Refs. 2–4 the ir properties of gauge theory have been derived from the Schwinger–Dyson equations and Ward identities. The renormalization β-function in the strong coupling regime has been obtained in the lattice formulation of the QCD (see Ref. 5 for a review).

In this letter, by using the method proposed in Ref. 6, we calculate the nonperturbative β-function in QCD. This method allows one to systematically determine the low energy structure in QCD using the expansion in a new small parameter.

This expansion arises in the framework of variational perturbation theory (VPT) considered in Refs. 7–11 for special choice of the VPT functional.

First of all we introduce the fields $\chi_{\mu\nu}$ and transform the term $g^2 L_4(A)$ in the Lagrangian (4) to the Yukawa type diagrams

$$\exp\left[ig^2\int dx L_4(A)\right] = \int D\chi \exp\left\{\frac{i}{2}\int dx dy\, \chi^a_{\mu\nu}[\Delta^{-1}]^{ab}_{\mu\nu;\mu_1\nu_1}\chi^b_{\mu_1\nu_1} + i\frac{g}{\sqrt{2}}\int dx\, \chi^a_{\mu\nu} f^{abc} A^b_\mu A^c_\nu\right\}, \tag{8}$$

where $\Delta(x,y)$ is the propagator of the χ-field

$$[\Delta(x,y)]^{ab}_{\mu\nu;\mu_1\nu_1} = \delta(x-y)\delta^{ab}\delta_{\mu\mu_1}\delta_{\nu\nu_1}\,. \tag{9}$$

After the χ-transformation the diagram technique contains only Yukawa type graphs corresponding to the standard Yukawa vertices of QCD (AAA, $\bar{q}Aq$, $\varphi A\varphi$) and a new $A\chi A$ vertex. The four-gluon graph appears after the expansion of the gluon propagator in the χ-field in a perturbation series and χ-integration, which result in a set of standard diagrams of the perturbation theory.

The Green functions can be written as

$$G(\cdots) = \langle G_{\text{Yuk.}}(\cdots|\chi)\rangle\,, \tag{10}$$

where

$$G_{\text{Yuk.}}(\cdots|\chi) = \int D_{\text{QCD}}[\cdots]\exp\left\{i[S(A,\chi)+S_0(q)+S_0(\varphi)+S_{\text{Yuk.}}(A,q,\varphi)]\right\}, \tag{11}$$

and

$$\langle\cdots\rangle = \int D\chi[\cdots]\exp[iS_0(\chi)]\,. \tag{12}$$

The integration measure D_{QCD} in (11) defines the standard integrations over gluon, quark and ghost fields. The functional $S_{\text{Yuk.}}(A,q,\varphi)$ contains only the Yukawa type interactions in QCD. The action $S(A,\chi)$ can be written in the form

$$S(A,\chi) = \frac{1}{2}\int dx dy A^a_\mu(x)[D^{-1}(x,y)|\chi)]^{ab}_{\mu\nu}A^b_\nu(y)\,, \tag{13}$$

where the gluon propagator $D(x,y|\chi)$ in the χ-field is defined as

$$[D^{-1}(x,y|\chi)]^{ab}_{\mu\nu} = [(-g_{\mu\nu}\partial^2+\partial_\mu\partial_\mu)\delta^{ab}+g\sqrt{2}\,f^{abc}\chi^c_{\mu\nu} + \text{gauge terms}]\delta(x-y)\,. \tag{14}$$

Using the χ-transformation we rewrite the Lagrangian (4) in the form

$$L = L'_0(A,q,\varphi,\chi) + \eta L'_I(A,q,\varphi)\,, \tag{15}$$

where

$$L_0'(A,q,\varphi,\chi) = \zeta^{-1}L(A,\chi) + \zeta^{-1}L_0(q) + \zeta^{-1}L_0(\varphi) + \xi^{-1}L_0(\chi)\,, \tag{16}$$

$$\begin{aligned} L_I'(A,q,\varphi) = {} & gL_{\text{Yuk.}}(A,q,\varphi) - (\zeta^{-1}-1)[L(A,\chi) + L_0(q) + L_0(\varphi)] \\ & - (\xi^{-1}-1)L_0(\chi)\,. \end{aligned} \tag{17}$$

Parameters ζ and ξ are the parameters of variational type. The original Lagrangian does not depend on them. Therefore, the freedom in choosing ζ and ξ can be used to improve properties of the series. It is clear that if $0 < \zeta < 1$ and $0 < \xi < 1$, we strengthen the new free Lagrangian and at the same time weaken the interaction Lagrangian. After all calculations we put $\eta = 1$. This parameter will also be written in the propagator $D(x,y|\chi)$ in a combination with the coupling constant. The VPT series for the Green function is given by

$$G(\cdots) = \sum_n G_n(\cdots)\,,$$

$$\begin{aligned} G_n(\cdots) &= \frac{1}{n!}\eta^n \int D\chi D_{\text{QCD}}[\cdots][iS_I']^n \exp[iS_0']\,, \\ &= (i\eta)^n \sum_{k=0}^{n} \frac{1}{(n-k)!k!} \int D\chi D_{\text{QCD}}[\cdots][gS_{\text{Yuk.}}(A,q,\varphi)]^k \\ &\quad \times \{(\zeta^{-1}-1)[S(A,\chi) + S_0(q) + S_0(\varphi)] + (\xi^{-1}-1)S(\chi)\}^{n-k} \exp[iS_0']\,. \end{aligned} \tag{18}$$

We redefine $L_0'(A,q,\varphi,\chi)$ for the convenience of calculations as follows:

$$\begin{aligned} L_0'(A,q,\varphi,\chi) \Rightarrow {} & [1+\kappa(\zeta^{-1}-1)][L(A,\chi) + L_0(q) + L_0(\varphi)] \\ & + [1+\kappa(\xi^{-1}-1)]L(\chi)\,. \end{aligned} \tag{19}$$

In this case, any power of $[(\zeta^{-1}-1)[S(A,\chi)+S_0(q)+S_0(\varphi)] + (\xi^{-1}-1)S(\chi)]$ in (18) can be obtained by the corresponding number of differentiations of the expression $\exp[iS_0'(A,q,\varphi,\chi,\kappa)]$ with respect to κ. After all calculations we put $\kappa = 1$.

From (18) and (19) we have

$$G_n = \eta^n \sum_{k=0}^{n} \frac{1}{(n-k)!}\left(-\frac{\partial}{\partial\kappa}\right)^{n-k} \langle g_k(\kappa)\rangle\,, \tag{20}$$

where the functions

$$\begin{aligned} g_k(\kappa) = {} & \frac{i^k}{k!}\int D_{\text{QCD}}[\cdots][gS_{\text{Yuk.}}(A,q,\varphi)]^k \\ & \times \exp\left\{i[1+\kappa(\zeta^{-1}-1)]\int dx\,[L(A,\chi) + L_0(q) + L_0(\varphi)]\right\} \end{aligned} \tag{21}$$

correspond to the Yukawa diagrams of QCD with the gluon propagator in the χ-field

$$[1+\kappa(\zeta^{-1}-1)]^{-1}D(x,y|\chi) \Rightarrow \zeta D(x,y|\chi)$$

for $\kappa = 1$. Similar factors appear in the quark and ghost propagators. The propagator of χ-field includes the factor $[1+\kappa(\xi^{-1}-1)]^{-1}$ transformed into ξ for $\kappa = 1$. The operator of differentiation $\frac{1}{l!}(-\frac{\partial}{\partial\kappa})^l$ gives the factor $(1-\zeta)^l$ for the gluon, quark and ghost propagator and $(1-\xi)^l$ for the propagator of the χ-field.

It is easy to verify that the Nth order of the VPT series contains the Nth order of a perturbation series with the correction $O(g^{N+1})$. Therefore, the VPT expansion does not contradict the perturbative results obtained for a small coupling constant.

Schematically, the structure of the VPT expansion for the Green functions can be written as

$$\begin{aligned} &1+\eta(1-\zeta)+\eta^2[(1-\zeta)^2+g^2\zeta^3+g^2\xi] \\ &\quad +\eta^3[(1-\zeta)^3+g^2\zeta^3(1-\zeta)+g^2\xi(1-\zeta)+g^2\xi(1-\xi)]+\cdots . \end{aligned} \tag{22}$$

If we choose $\xi = \zeta^3$ and $(1-\zeta)^2 \sim g^2\zeta^3$, we obtain the nth order term of the VPT series which contains the factor $(1-\zeta)^n$.

Using the dimensional regularization with $d = 4-2\varepsilon$ and one-loop diagrams in our approach for the renormalization constant associated with the renormalization coupling constant α_S in the first order of the VPT we get (the corresponding definitions can be found in Ref. 12)

$$\begin{aligned} Z_{1\mathrm{YM}} &= 1+\frac{a^2}{C}\left[N\left(\frac{17}{6}-\frac{3}{2}\alpha_G\right)-\frac{4}{3}N_f\right]\frac{1}{2\varepsilon}, \\ Z_{3\mathrm{YM}} &= 1+\frac{a^2}{C}\left[N\left(\frac{13}{3}-\alpha_G\right)-\frac{4}{3}N_f\right]\frac{1}{2\varepsilon}, \end{aligned} \tag{23}$$

where $\lambda = \alpha_S/(4\pi)$, N_f is the number of flavors and the parameter $a = 1-\zeta$ obeys the equation

$$a^2 = C\lambda(1-a)^3 \tag{24}$$

with a positive constant C. This constant is a variational parameter. Equation (24) gives $(1-\zeta) < 1$ for all values of the coupling constant.

From (23) we obtain the connection between the bare λ_0 and renormalized coupling constant λ

$$\lambda_0 = \mu^{2\varepsilon}Z_{1\mathrm{YM}}^2 Z_{3\mathrm{YM}}^{-3}\lambda = \lambda\mu^{2\varepsilon}\left[1-\frac{a^2}{C}\frac{b_0}{\varepsilon}\right], \tag{25}$$

where $b_0 = 11-2/3N_f$.

By using Eqs. (23)–(25) and the fact that λ_0 is independent of μ for the β-function we get

$$\beta(\lambda) = \lim_{\varepsilon\to 0} \mu^2 \frac{\partial \lambda}{\partial \mu^2} = -b_0 \left(\lambda \frac{\partial}{\partial \lambda} - 1\right)(\lambda^2 \zeta^3)$$

$$= -b_0 \frac{a^4}{C^2(1-a)^2(1+a/2)} \,. \tag{26}$$

The result of calculation for $-\beta(\lambda)/\lambda$ as a function of λ for $N_f = 3$ and $C = 0.977$ is shown in Fig. 1.

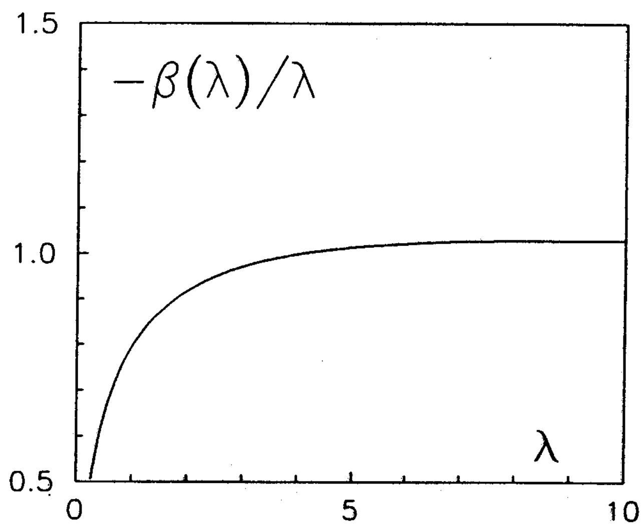

Fig. 1. The function $-\beta(\lambda)/\lambda$ vs λ.

Here we have used the VPT-functional of the harmonic type which contains the terms quadratic in the fields. The main contribution to higher orders of our expansion is defined by the large configurations of fields (like in perturbation theory). Therefore, if the variational parameter is fixed and independent of the expansion order, we obtain an asymptotic series. The Nth order of our series coincides with the Nth order of perturbative series with the accuracy $O(g^{N+1})$. As compared with the standard perturbation theory, the VPT series gives a better approximation for a quantity under consideration when the coupling constant increases. However, a different situation arises if the auxiliary parameter is chosen in each order of the expansion according to some variational principle, for example, the phenomenon of "induced convergence." The mechanism of induced convergence has been discussed in detail in Ref. 13. In Ref. 14 the convergence of an optimized δ-expansion has been proved in the cases of zero and one dimensions. Therefore, in our case the VPT series can converge if we understand its convergence as "induced convergence" and change the variational parameter in each order. We plan to discuss the convergence properties in future.

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