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Variational perturbation theory. Anharmonic oscillator

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Abstract. A nonperturbative method is suggested for calculating functional integrals. Its efficiency is demonstrated for the quantum-mechanical anharmonic oscillator. A quantity we are interested in is represented by a series, a finite number of terms of which describes not only the region of small coupling constants but well reproduces the strong coupling limit. The method is formulated only in terms of the Gaussian functional quadratures and diagrams are used of the conventional perturbation theory.

1 Introduction

Nowadays, perturbation theory is a basic method of computations in quantum field theory. Applied together with the renormalization procedure to quantum electrodynamics, the theory of electroweak interactions and quantum chromodynamics, it allows one to analyse a wide class of problems; at the same time it is known that this theory alone cannot provide an exhaustive study of the structure of a quantum-field model, which concerns not only theories with a large coupling constant but also theories with a small enough coupling constant, i.e. quantum electrodynamics.

At present, a lot of approach exist which give a way of going out of the scope of perturbation theory. One of them is summation of the series of perturbation theory [1], in which higher-order terms are described by asymptotic formulae that can be derived by the functional saddle-point method [2–4]. It is to be noted here that owing to the asymptotic nature of perturbative series, their summation is not unique and admits, in general, a functional arbitrariness. For a correct summation one should apply to an extra information, for instance, concerning the analytic properties of the sum of a series as a function of the coupling constant and its asymptotic behaviour at large coupling constants. Though for scalar models of field theory in one-and two-dimensional spaces, summation of perturbative series can be made by the Borel method, for more complex models in spaces of higher dimensionality summation is of a hypothetical character. Other approaches that do not directly employ the series of perturbation theory were considered by many authors (see, e.g., [5–8]). More convenient for application of nonperturbative methods turns out to be the effective potential. It is usually calculated by a quasiclassical method consisting in an expansion in the number of loops; however it cannot be considered as a serious going beyond perturbation theory as it is based on the coupling constant being small.

Recently, variational methods have found their extensive applications, the most popular of which is the Gaussian effective potential method (see [8, 10, 11]). Note is to be made that the variational methods as a rule do not provide a reliable estimation of the accuracy of the results obtained since it is difficult there to evaluate corrections to the "leading contribution"; in other words, it is difficult to answer the question: to what extent that contribution provided by a variational method is actually the leading contribution.*

In this paper we suggest a method for calculating path integral in quantum field theory beyond the scope of perturbation theory. Our method contains an element of the variational procedure, the so-called optimization principle, however, unlike the conventional variational method, our method expresses the quantity of interest as a series that, in principle, allows us to compute a required correction and thus to estimate the accuracy of results obtained and to analyse their stability. Series appearing in this approach will be called the series of variational perturbation theory.

Our method is formulated in terms of the Gaussian quadratures (like in perturbation theory). In fact, we shall realize a functional integral with help of the Gaussian integral definition

$$\int \mathscr{D}\varphi \exp\{i[\tfrac{1}{2}\int d\,x((\partial\varphi)^2 - M^2\varphi^2) + \int \mathscr{D}\, x J\varphi]\}$$
$$= \left[\det \frac{\partial^2 + M^2}{\partial^2 + m^2}\right]^{-1/2} \exp\left[-\frac{i}{2}\langle J\Delta J\rangle\right], \tag{1.1}$$

* The problems of stability of the Gaussian parametrization was considered in [12]

where

$$\langle J\Delta J\rangle=\int \mathrm{d}x\,\mathrm{d}y\,J(x)\,\Delta(x-y)\,J(y) \tag{1.2}$$

and

$$\Delta(k)=(k^2-M^2+\mathrm{i}0)^{-1} \tag{1.3}$$

is the poropagator in the momentum space. The polynomial in fields φ in the path integral (1.1) will be determined by variational differentiation with respect to the current $J(x)$.

It is important to stress that we shall construct the variational perturbation theory so that for its Nth order only those diagrams will be required that compose the Nth order of standard perturbation theory. Therefore, for practical computations this approach is not more difficult than the standard perturbative methods.

We shall only consider a quantum-mechanical anharmonic oscillator, a one-dimensional model of field theory with interaction, that can easily demonstrate the efficiency of the method. The approach is based on a rather universal apparatus of the path integral, and at least formally, may be extended to spaces of high dimensionality. In view of the extension of the method to quantum field theory, we shall analyse its validity for such objects as the effective potential and propagator and consider a simplifying role of the intermediate dimensional regularization. To start with, let us examine a simple numerical example.

2 A zero-dimensional analog

Consider the integral*

$$Z[g]=\int_{-\infty}^{\infty}\mathrm{d}x_1\,\mathrm{d}x_2\exp[-(S_0+gS_I)], \tag{2.1}$$

where

$$S_0=x_1^2+x_2^2=\mathbf{x}^2, \tag{2.2}$$

$$S_I=x_1^4+x_2^4. \tag{2.3}$$

S_0 is a zero-dimensional analog of the free action, and S_I is the action with interaction.

We shall work only with the Gaussian quadratures, i.e. integrals of the form

$$\int \mathrm{d}\mathbf{x}\,\mathscr{P}(x_1,x_2)\exp(-S_0), \tag{2.4}$$

where $\mathscr{P}(x_1,x_2)$ is a polynomial of two variables. Conventionally, the exponential in (2.1) is expanded in powers of the coupling constant

$$Z[g]=\sum_{n=0}^{\infty}g^nC_n, \tag{2.5}$$

$$C_n=\frac{(-1)^n}{n!}\int_{-\infty}^{\infty}\mathrm{d}x_1\,\mathrm{d}x_2(x_1^4+x_2^4)^n\exp[-(x_1^2+x_2^2)]. \tag{2.6}$$

Expansion (2.5) with coefficients (2.6) represents a conventional asymptotic series of perturbation theory. Transition from (2.1) to (2.5) is unique, however, the inverse problem of reproducing a function from its asymptotic series is arbitrary enough [13]. Therefore, without further information one cannot uniquely reproduce the function from the asymptotic expansion (2.5) and consequently the conventional perturbative series itself cannot provide a unique information on the nonperturbative region.

As indicated above, this uniqueness can be provided by assuming auxiliary conditions on the sum (a condition of that sort is given, e.g., by the Karleman' theorem [13]). However, in the case of quantum field theory it is impossible to ascertain a priori that a given function represented by a path integral obeys definite conditions guaranteeing the uniqueness of summation of perturbative series. This question requires an independent study and at present it has a solution only for some simple cases.

Here we will solve another task: how one can construct an expansion for $Z[g]$ different from a perturbative series by using only Gaussian quadratures? To what extent is this expansion valid for analysing the nonperturbative region, in particular, the strong coupling limit

$$Z[g]\underset{g\to\infty}{\sim}g^{-1/2}\,\Gamma^2(1/4)/4=3.2863\,g^{-1/2} \tag{2.7}$$

is reproduced?

If we rewrite the total action to the form

$$S=S_0+gS_I=S_0'+S_I', \tag{2.8}$$

where

$$S_0'=S_0+\theta S_0^2, \tag{2.9}$$

$$S_I'=gS_I-\theta S_0^2, \tag{2.10}$$

and θ is an arbitrary parameter for the time being, then the new expansion for the function $Z[g]$ looks as follows:

$$Z[g]=\sum_{n=0}^{\infty}Z_n[g,\theta], \tag{2.11}$$

$$Z_n[g,\theta]=\frac{1}{n!}\int \mathrm{d}\mathbf{x}(S_I')^n\exp(-S_0'). \tag{2.12}$$

It is intuitively clear that the expansion in powers of the new interaction S_I' should improve the convergence of the series (2.11) as compared to the standard perturbative one (in what follows we will show this on a rigorous mathematical basis). Besides, in view of $Z[g]$ being independent of the parameter θ, we may take the latter so as to provide the best approximation of the function $Z[g]$ with a finite number of the terms of series (2.11). Thus, θ plays the role of a variational parameter that

* We consider a two-dimensional integral since from the following consideration it is clear that a one-dimensional integral is a trivial example for this method

is determined from one or another principle of optimisation. Series like (2.11) will further be called the series of variational perturbation theory (VPT).

Consider various versions of the optimisation procedure. In the first version of optimisation (version 1) one chooses such a value of the variational parameter, $\theta=\theta_1$, at which the absolute value of the last of the considered VPT expansion terms is minimal. For instance, in the first nontrivial VPT order examined in this paper,

$$Z^{(1)}[g,\theta]=Z_0[g,\theta]+Z_1[g,\theta], \tag{2.13}$$

an optimal value of the parameter, θ_1, is found from the condition

$$\theta_1: \min_{\{\theta\}}|Z_1[g,\theta]|. \tag{2.14}$$

The best value can be obtained when there exists a root of the equation

$$Z_1[g,\theta]=0. \tag{2.15}$$

Version 2 of the optimisation procedure uses the fact that the exact value of $Z[g,\theta]$ does not depend on the parameter θ, and consequently, $\partial Z[g,\theta]/\partial\theta=0$; therefore the optimal parameter, $\theta=\theta_2$, is determined from the equation*

$$\frac{\partial Z^{(N)}[g,\theta]}{\partial\theta}=0, \tag{2.16}$$

where $Z^{(N)}[g,\theta]=\sum_{n=0}^{\infty}Z_n[g,\theta]$ is an Nth order of VPT.

And finally, in version 3 of the optimisation procedure the parameter, $\theta=\theta_3$, is fixed if we require the contribution of higher order terms of the VPT series to be minimal. This way of determining variational parameters will be called the asymptotic optimisation.

As is known, the perturbative series (2.5) converges only at one point, $g=0$, whereas for the VPT series (2.11) we have a finite domain of convergence,

$$g\leqq 2\theta, \tag{2.17}$$

The asymptotic behaviour of the terms of series (2.12) at large n is as follows:

$$Z_n\underset{n\to\infty}{\sim}\frac{\sqrt{t-1}}{2}\sqrt{\frac{\pi}{\theta}}\frac{1}{n}\left(\frac{t-1}{t}\right)^n\exp(-\sqrt{n/\theta}), \tag{2.18}$$

for $t>1$, where

$$t=2\theta/g, \tag{2.19}$$

and

$$Z_n\underset{n\to\infty}{\sim}\frac{1}{2}\sqrt{\frac{\pi}{\theta}}\frac{(-1)^n}{n}\exp(-\sqrt{n/\theta}), \tag{2.20}$$

at $t=1$.

* When (2.16) has no root, it is natural to require the absolute value of the lhs to be minimal

Thus, our approach allows us to pass from the asymptotic series of perturbation theory convergent only at one point to series having a finite region of convergence. We stress also that at $t=1$, which corresponds to the asymptotic optimisation, as seen from (2.18), the VPT series (2.11) becomes the Leibniz series, an alternating series with terms decreasing in absolute value. Series of that sort allow us to make lower and upper estimations of the series sum, which is important for physical applications. In the case we are considering the Leibniz series arise only at one value of the parameter t owing to the most simple construction of the VPT series. If we construct, for instance, a two-parameter VPT, we have a whole region of change of parameters where the VPT series is the Leibniz series. Varying these parameters we may get the most exact upper and lower estimations on the series sum.*

Now we turn back to series (2.11) with coefficients (2.12) to deal with some technical details. First, note that integral (2.12) is not Gaussian owing to the term $\theta(\mathbf{x}^2)^2$ being in the exponential. However, this integral may easily be reduced to the Gaussian quadrature by the Fourier transformation

$$\exp(-A^2)=\langle\exp(\pm iuA)\rangle=\int_{-\infty}^{\infty}\frac{du}{2\sqrt{\pi}}\exp(-u^2/4\pm iuA). \tag{2.21}$$

As a result, the terms of VPT series (2.12) may be represented in the form

$$Z_n[g,\theta]=\left\langle\sum_{k=0}^{n}\frac{(-\theta)^{n-k}}{(n-k)!}\left(\frac{d}{d\theta}\right)^{n-k}\frac{Z_k(g)}{(1+iu\sqrt{\theta})^{2k+1}}\right\rangle, \tag{2.22}$$

where $Z_k[g]$ are coefficients of the ordinary perturbative series,

$$Z_k(g)=\frac{1}{k!}\int dx[-g(x_1^4+x_2^4)]^k\exp(-\mathbf{x}^2). \tag{2.23}$$

Taking advantage of the relation

$$a^{-\nu}=\frac{1}{\Gamma(\nu)}\int_0^{\infty}d\alpha\,\alpha^{\nu-1}\exp(-a\alpha), \tag{2.24}$$

and integrating in (2.23) over u, we get

$$Z_n[g,\theta]=\int_0^{\infty}d\alpha\,\alpha^{2n}\exp(-\alpha-\alpha^2\theta)\cdot\sum_{k=0}^{n}\frac{\theta^{n-k}}{(n-k)!}\frac{Z_k(g)}{\Gamma(2k+1)}. \tag{2.25}$$

* Similar problems will be discussed in subsequent publications

Computing $Z_k(g)$ we obtain

$$Z_n[g,\theta]=\int_0^\infty d\alpha(\alpha^2\theta)^n\exp(-\alpha-\alpha^2\theta)\cdot\sum_{k=0}^{n}\frac{1}{(2k)!}\frac{(-g/\theta)^k}{(n-k)!}a_k, \tag{2.26}$$

where

$$a_k=\sum_{l=0}^{k}\frac{\Gamma(2l+1/2)}{l!}\frac{\Gamma(2(k-l)+1/2)}{(k-l)!}. \tag{2.27}$$

Introducing $t=2\theta/g$ we rewrite (2.26) in the form

$$Z_n[g,\theta]=\frac{1}{\sqrt{g}}\sqrt{\frac{2}{t}}\int_0^\infty d\alpha\,\alpha^{2n}\exp\left(-\alpha^2-\alpha\sqrt{\frac{2}{gt}}\right)\cdot\sum_{k=0}^{n}\frac{(-2/t)^k}{(2k)!(n-k)!}a_k. \tag{2.28}$$

From (2.28) and the above consideration it is seen that the optimal value of the parameter t does not depend on the coupling constant g. Therefore, the VPT series allows us to reconstruct the functional dependence $Z[g]$ when $g\to\infty$ immediately (see (2.7)).

Let us now examine which results come from the first order of VPT. From (2.26) we get

$$Z_0[g,\theta]=\pi\int_0^\infty d\alpha\exp(-\alpha-\alpha^2\theta), \tag{2.29}$$

$$Z_1[g,\theta]=\pi\int_0^\infty d\alpha\,\alpha^2\theta(1-3g/4\theta)\exp(-\alpha-\alpha^2\theta). \tag{2.30}$$

If we make use of the optimisation of version 1, we should put $Z_1[g,\theta]=0$ and as a result

$$t_1=2\theta_1/g=3/2. \tag{2.31}$$

It is interesting that optimisation of the first order by version 2, i.e. with the condition

$$\frac{\partial Z^{(1)}[g,\theta]}{\partial\theta}=0,\qquad Z^{(1)}=Z_0+Z_1, \tag{2.32}$$

gives the same value of the parameter t as version 1 ($t_2=t_1$). The results of calculations are reported in Table 1 from which it is seen that even the first order of VPT reproduces the exact value of $Z[g]$ quite well in the whole range of variation of the coupling constant g.* In Table 1 we also present the results derived with the asymptotic version of optimisation (version 3) that gives the value of the parameter $t_3=1$. In this version, the interval $(Z^{(1)}[g,\theta], Z^{(0)}[g,\theta_3])$ determines the upper and lover estimates for $Z[g]$. When $g\to\infty$, we find from (2.28)

* The limit $g\to\infty$ will be discussed somewhat later

Table 1. The result of variational perturbation theory for (2.1)

g	$Z_{\text{exact}}[g]$	$Z^{(1)}[g,\theta]$ $\theta=\theta_1=\theta_2$	$Z^{(0)}[g,\theta]$ $\theta=\theta_3$	$Z^{(1)}[g,\theta]$ $\theta=\theta_3$
0.1	2.8026	2.8001	2.8927	2.7917
1.0	1.8726	1.8584	2.0860	1.8153
10.0	0.8500	0.8374	0.9841	0.7920
100.0	0.3076	0.3017	0.3642	0.2801
1000.0	0.1017	0.0997	0.1214	0.0918
10000.0	0.0326	0.0320	0.0391	0.0294

$$Z_n[g,\theta]=\frac{1}{\sqrt{g}}\frac{\Gamma(n+1/2)}{2}\sqrt{\frac{2}{t}}\sum_{k=0}^{n}\frac{(-2/t)^k a_k}{(2k)!(n-k)!}. \tag{2.33}$$

Upon performing one of summations, in the Nth order of VPT we obtain

$$Z^{(N)}=A^{(N)}g^{-1/2},$$

where

$$A^{(N)}=\sqrt{\frac{2}{t}}\Gamma(N+3/2)\sum_{n=0}^{N}\frac{(-2/t)^k a_k}{(2k+1)!(N-k)!}. \tag{2.34}$$

In the first order we get

$$A^{(1)}=\frac{3}{4}\pi^{3/2}\sqrt{\frac{2}{t}}(1-1/2t), \tag{2.35}$$

with the optimal value

$$t_1=t_2=3/2,$$

and the coefficient equals

$$A^{(1)}=3.2115,$$

which is in good agreement with the exact value (see (2.3))

$$A_{\text{exact}}=3.2863.$$

3 Ground state energy and propagator of the anharmonic oscillator in the strong coupling limit

We will proceed from the partition function represented by the path integral

$$\exp(-TE)=\int_{\varphi(-T/2)=\varphi(T/2)}\mathcal{D}\varphi\exp[-(S_0+m^2\tilde{S}+gS_I)], \tag{3.1}$$

where

$$S_0=\tfrac{1}{2}\int_{-T/2}^{T/2}dt\,\dot{\varphi}^2,\qquad \tilde{S}=\tfrac{1}{2}\int_{-T/2}^{T/2}dt\,\varphi^2,\qquad S_I=\int_{-T/2}^{T/2}dt\,\varphi^4. \tag{3.2}$$

The ground state energy E_0 follows from (3.1) in the limit $T\to\infty$. It is convenient to pass from the functional integrals typical of the statistical mechanics to the functional integrals of the Euclidean field theory. To this end consider the quantity dE_0/dg that is expressed in terms of the four-point Euclidean Green function. So, passing to the dimensionless variables

$$\varphi\to g^{-1/6}\varphi, \quad t\to g^{-1/3}t, \quad \omega^2=g^{-2/3}m^2$$

from (3.1) we obtain

$$dE_0/dg=g^{-2/3}G_4(0), \tag{3.3}$$

where

$$G_4(0)=N^{-1}\int\mathscr{D}\varphi\,\varphi^4(0)\exp[-(S_0+\omega^2\tilde{S}+S_I)], \tag{3.4}$$

$$N=\int\mathscr{D}\varphi\exp[-(S_0+\omega^2\tilde{S}+S_I)]. \tag{3.5}$$

Here S_0, $\tilde{S}$, and S_I are given by (3.2) but integration runs over t from minus to plus infinity.

In what follows we will be interested in the strong coupling limit,

$$g/m^3\to\infty \quad (\omega^2\to 0).$$

Defining the functional

$$A[\varphi]=\theta S_0[\varphi]+\kappa\tilde{S}[\varphi], \tag{3.6}$$

with arbitrary parameters θ and κ for the time being, we rewrite (3.4) in the form of a VPT series,

$$\begin{aligned}G_4(0)&=N^{-1}\int\mathscr{D}\varphi\,\varphi^4(0)\\&\quad\cdot\exp[-(S_0+\omega^2\tilde{S}+A^2)-(S_I-A^2)]\\&=N^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int\mathscr{D}\varphi\,\varphi^4(0)[A^2-S_I]^n\\&\quad\cdot\exp[-(S_0+\omega^2\tilde{S}+A^2)].\end{aligned} \tag{3.7}$$

Next we will find the asymptotics of the path integral

$$\int\mathscr{D}\varphi[A^2-S_I]^n\exp[-(S_0+A^2)] \tag{3.8}$$

at large n. Changing the variables, $\varphi\to n^{1/4}\varphi$, we represent (3.8) as follows

$$n^n\int\mathscr{D}\varphi\exp[-nS_{\text{eff}}[\varphi]-n^{1/2}S_0[\varphi]], \tag{3.9}$$

where

$$S_{\text{eff}}=A^2-\ln[A^2-S_I]. \tag{3.10}$$

The functional integral (3.9) contains a large parameter n and can be calculated by the functional saddle point method [4]. The saddle-point function φ_0 is determined from the condition $\delta S_{\text{eff}}/\delta\varphi=0$ that leads to the equation

$$-\ddot{\varphi}_0+a\varphi_0-b\varphi_0^3=0, \tag{3.11}$$

where

$$\begin{aligned}&a=\kappa/\theta, \quad b=\{\theta A[\varphi_0](1-\mathscr{D}[\varphi_0])\}^{-1},\\&\mathscr{D}[\varphi_0]=A^2[\varphi_0]-S_I[\varphi_0].\end{aligned} \tag{3.12}$$

The solution to (3.11) decreasing at infinity, corresponding to a finite action, and given a major contribution to functional integral (3.9) at large n is of the form

$$\varphi_0=\pm\sqrt{\frac{2a}{b}}\,[\cosh\sqrt{a}(t-t_0)]^{-1} \tag{3.13}$$

where t_0 is an arbitrary parameter reflecting the theory to be translationality invariant. It is not difficult to compute functional (3.10) for the functional (3.13)

$$S_{\text{eff}}[\varphi_0]=1-\ln\mathscr{D}[\varphi_0], \tag{3.14}$$

where

$$\mathscr{D}[\varphi_0]=1-\tfrac{3}{4}(\theta\kappa^3)^{-1/2}. \tag{3.15}$$

Here we may take advantage of version 3 of the optimisation procedure requiring the contribution of higher-order terms being minimal, which means the condition $\mathscr{D}[\varphi_0]=0$. Therefore variational parameters θ and κ are related as follows

$$\kappa=(9/16\,\theta)^{1/3}. \tag{3.16}$$

The remaining variational parameter θ is fixed on the basis of a finite number of the VPT expansion terms; we now will restrict our consideration to the first order.

Further transformations with (3.7) will proceed as follows. Since any power of A^2 in front of the exponential of (3.7) can be obtained by differentiation, we do not introduce new diagrams but those of conventional perturbation theory. Performing intermediate dimensional regularization and reducing the functional integral with the use of (2.21) to the Gaussian form, we get

$$\begin{aligned}G_4(0)=&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n-m)!}\left(\frac{d}{d\alpha}\right)^{n-m}\\&\cdot\langle g_m(z^2)[1+iu\theta\sqrt{1-\alpha}]^{-2-2m}\rangle,\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}&g_m(z^2)=\frac{(-1)^m}{m!}\int\mathscr{D}\varphi\,\varphi^4(0)\,S_I^m\exp[-(S_0+z^2\tilde{S})],\\&z^2=[\omega^2+iu\kappa\sqrt{1-\alpha}][1+iu\theta\sqrt{1-\alpha}]^{-1}.\end{aligned} \tag{3.18}$$

Upon differentiation of (3.17) with respect to α we should put $\alpha=0$.

The functions $g_m(z^2)$ are standard expansion coefficients of $G_4(0)$ into a perturbative series and they can be determined by the standard diagram technique. From (3.17) it is seen that the Nth VPT order requires only those diagrams that are present in the Nth order of conventional perturbation theory.

In this case it is not difficult to connect expressions (3.18) with the known expansion coefficients A_n of the ground state energy E_0 in the perturbative series

$$E_0(g)=m/2+m\sum_{n=1}^{\infty}A_n(g/m^3)^n, \tag{3.19}$$

and this connection looks as follows

$$g_m(z^2)=\frac{(1+m)\,A_{1+m}}{z^{2+3m}}. \tag{3.20}$$

Numerical value of the coefficients A_n may be taken from [15].*

Then making use of (2.24) we obtain

$$G_4(0)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(1+m)\,A_{1+m}}{(n-m)!}\cdot[\Gamma(1+m/2)\,\Gamma(1+3m/2)]^{-1}\cdot\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^{n-m}F_m(\theta,\kappa,\alpha), \tag{3.21}$$

where

$$F_m(\theta,\kappa;\alpha)=\int_0^{\infty}\mathrm{d}x\,x^{m/2}\exp(-x)\int_0^{\infty}\mathrm{d}y\,y^{3m/2}\cdot\exp[-\omega^2 y-(1-\alpha)\cdot(x\theta+y\kappa)]. \tag{3.22}$$

Note that we are interested in the strong coupling limit and therefore we set $\omega^2=0$ in (3.22). However, it is to be noticed that expanding $\exp(-\omega^2 y)$ in powers of ω^2 we can determine corrections to the main contribution.

From (3.21) and (3.22) and (3.16) we obtain for the ground state energy in the Nth VPT order in the strong coupling limit

$$E_0^{(N)}=3\,g^{1/3}\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{(1+m)\,A_{1+m}}{(n-m)!}\left(\frac{16}{9}\theta\right)^{1/3+m/2}\cdot[\Gamma(1+m/2)\,\Gamma(1+3m/2)]^{-1}\,R_{n,m}(\theta), \tag{3.23}$$

where

$$R_{n,m}(\theta)=\int_0^{\infty}\mathrm{d}x\,x^{m/2}\exp(-x)\cdot\int_0^{\infty}\mathrm{d}y\,y^{3m/2}(\theta x+y)^{2(n-m)}\exp[-(\theta x+y)^2]. \tag{3.24}$$

Optimal value of the parameter θ both in the first and second versions $\theta_{1,2}\ll 1$; therefore, in the first VPT order we get from (3.23) and (3.24)

$$E_0^{(1)}=g^{1/3}(\varepsilon_0+\varepsilon_1), \tag{3.25}$$

where

$$\varepsilon_0=\tfrac{3}{2}A_1\sqrt{\pi}\,x^2, \tag{3.26}$$

$$\varepsilon_1=\frac{3}{4}A_1\sqrt{\pi}\,x^2+\frac{4\Gamma(5/4)}{\sqrt{\pi}}A_2\,x^5, \tag{3.27}$$

$$x=(16\theta/9)^{1/6}. \tag{3.28}$$

* The first VPT order requires the values, $A_1=3/4$ and $A_2=-21/8$

Upon optimisation of version 1 we obtain $x_1=0.5705$ and the ground state energy

$$E_0^{(1)}(x_1)=0.649\,g^{1/3} \tag{3.29}$$

and upon optimisation of version 2 we find $x_2=0.6062$ and

$$E_0^{(1)}(x_2)=0.660\,g^{1/3}. \tag{3.30}$$

It is easy to verify that the second VPT order contributes only several percent.

We have to compare the obtained results with the exact value [16]:

$$E_{\text{exact}}=0.668\,g^{1/3}. \tag{3.31}$$

We will here also calculate the mass parameter μ^2 connected with the two-point Green function, $\mu^{-2}=G_2(p=0)$, where

$$G_2(p=0)=\int\mathrm{d}t\int\mathscr{D}\varphi\;\varphi(t/2)\,\varphi(-t/2)\exp(-S[\varphi]). \tag{3.32}$$

Numerically, this parameter was computed in [17] in the strong coupling limit,

$$\mu^2_{\text{exact}}=3.009\,g^{2/3}. \tag{3.33}$$

The VPT series for the function $G_2(0)$ is as follows

$$G_2(0)=g^{-2/3}\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\Gamma(n+1/2-m/4)}{(n-m)!}\cdot\frac{B_m}{\Gamma(1+3m/2)}\,x^{2+3m}, \tag{3.34}$$

where B_m are dimensionless coefficients of the standard perturbation theory.* In the first VPT order we get from (3.34)

$$G_2^{(1)}=g^{-2/3}(G_{20}+G_{21}),$$

where

$$G_{20}=\frac{\sqrt{\pi}}{2}x^2, \tag{3.35}$$

$$G_{21}=\frac{\sqrt{\pi}}{4}x^2-4\cdot\Gamma(5/4)\cdot x^5. \tag{3.36}$$

Upon version 1 of optimisation ($G_{21}=0$) we find

$$\mu^2=3.128\,g^{2/3}, \tag{3.37}$$

and upon version 2 ($\partial G_2^{(1)}/\partial x=0$)

$$\mu^2=3.078\,g^{2/3}. \tag{3.38}$$

We can compare these results with exact value (3.33) and get satisfaction.

* For the considered first nontrivial VPT order we need the two value, $B_0=1$ and $B_1=-6$

With the use of the propagator $G_2(p)$ we may compute the vacuum energy by the relation [18]

$$E_0=\frac{3}{4}\int\frac{\mathrm{d}p}{2\pi}\,[1-G_{20}^{-1}(p)\cdot G_2(p)], \tag{3.39}$$

where $G_{20}(p)$ is the free propagator.

It is of interest to employ a more simple version of VPT with one variational parameter, say, κ, and θ is put zero. Just this one-parameter VPT will be used in the next section for constructing the effective potential. In the first order for the two-point Green function we obtain

$$G_2^{(1)}(p)=\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{2\sqrt{\pi}}\exp(-u^2/4)$$
$$\cdot\left\{\Delta_0(p,z^2)+\left[\frac{\mathrm{d}}{\mathrm{d}\alpha}\Delta_0(p,z^2)+\Delta_1(p,z^2)\right]\right\}, \tag{3.40}$$

where

$$\Delta_0(p,z^2)=(p^2+z^2)^{-1},$$

$$\Delta_1(p,z^2)=-\frac{6g}{z}(p^2+z^2)^{-2},$$

$$z^2=\omega^2+\mathrm{i}u\kappa\sqrt{1-\alpha}.$$

Inserting (3.40) into (3.39) we get

$$E_0^{(1)}(\kappa)=\frac{3}{8}\left\{\frac{\Gamma(3/4)}{\sqrt{\pi}}\sqrt{\kappa}\right.$$
$$\left.+\left[-\frac{1}{4}\frac{\Gamma(3/4)}{\sqrt{\pi}}\sqrt{\kappa}+\frac{3g}{2\kappa}\sqrt{\pi}\right]\right\}, \tag{3.41}$$

and upon optimisation of version 1 (the expression in brackets in (3.41) is put to zero) we find

$$E_0^{(1)}(\kappa_1)=0.645\cdot g^{1/3}, \tag{3.42}$$

whereas version 2 gives

$$E_0^{(1)}(\kappa_2)=0.634\cdot g^{1/3}. \tag{3.43}$$

And finally, we shall estimate the energy of the first excited level, E_1; to do this, we define the energy shift

$$\mu_1=E_1-E_0. \tag{3.44}$$

Then, using the spectral representation for the propagator

$$G_2(p)=2\sum_{n=0}^{\infty}\frac{\mu_n}{p^2+\mu_n^2}\,|\langle 0|\,\hat{x}\,|n\rangle|^2, \tag{3.45}$$

where matrix elements of the coordinate operator are calculated for eigenstates of the total Hamiltonian, we arrive at the following estimate for the energy shift (3.44):

$$\mu_1\leqq\mu_1^{(+)},$$
$$\mu_1^{(+)}=2G_2(t=0)/G_2(p=0). \tag{3.46}$$

By analogy with the sum rules [19], we may expect a sufficiently rapid saturation of the spectral representation (3.45), which brings μ_1 and $\mu_1^{(+)}$ closer to each other. In the first order of the one-parameter VPT we get

$$\mu_1^{(+)}=1.763\,g^{1/3}, \tag{3.47}$$

whereas the exact value is [16]:

$$\mu_1=1.726\,g^{1/3}. \tag{3.48}$$

4 Effective potential

Consider the generating functional of the Green functions*

$$W[J]=\int\mathscr{D}\varphi\exp\{\mathrm{i}[S[\varphi]+\langle J\varphi\rangle]\}, \tag{4.1}$$

where

$$\langle J\varphi\rangle=\int\mathrm{d}t\,J(t)\cdot\varphi(t), \tag{4.2}$$

$$S[\varphi]=S_0-m^2\tilde{S}-gS_I. \tag{4.3}$$

The effective potential is usually constructed in the quasiclassical approximation based on the expansion in powers of the number of loops [9]. In our case this method gives the one-loop potential

$$V_{\mathrm{eff}}^{1\text{-loop}}=\tfrac{1}{2}\sqrt{m^2+12g\varphi_0^2}, \tag{4.4}$$

that is completely unfit for the description of the nonperturbative region.

In this section we will compute the effective potential by the VPT method. To this end, we introduce variational parameter a^2 rewritting action (4.3) to the form

$$S[\varphi]=\left[S_0-m^2\tilde{S}-\frac{a^2}{T}\tilde{S}^2\right]-\left[gS_I-\frac{a^2}{T}\tilde{S}^2\right]. \tag{4.5}$$

The effective potential is obtained from the effective action when the field configurations are constant, $\varphi_0=\mathrm{const}$, and in this case the variational parameter introduced in the form a^2/T will be independent of the "volume" T of x space.

Expanding the exponential of (4.1) in powers of $gS_I-a^2\tilde{S}^2/T$ and using the above expounded procedure, we get

$$W[J]=\exp(-\mathrm{i}\pi/4)\,T^{1/2}\cdot\int_{-\infty}^{\infty}\frac{\mathrm{d}v}{2\sqrt{\pi}}\exp(\mathrm{i}Tv^2/4)$$
$$\cdot\sum_{n=0}^{\infty}\mathrm{i}^n\sum_{k=0}^{n}\frac{1}{(n-k)!}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right)^{n-k}\frac{1}{k!}$$
$$\cdot\int\mathscr{D}\varphi(-gS_I)^k\exp\{\mathrm{i}[S_0-M^2\tilde{S}+\langle J\varphi\rangle]\}, \tag{4.6}$$

where

$$M^2=m^2+\sqrt{\varepsilon}\cdot a\cdot v, \tag{4.7}$$

* We employ the pseudo-Euclidean signature in the n-dimensional space, keeping in mind applications in field theory

and upon differentiation with respect to ε we should set $\varepsilon=1$. Denoting the perturbative expansion coefficients for the functional $W[J]$ by $\omega_k[J, M^2]$:

$$\omega_k[J, M^2]=\frac{(-\mathrm{i}g)^k}{k!}\left[\int \mathrm{d}t\,\frac{\delta^4}{\delta J^4(t)}\right]^k \cdot\exp\left[-\frac{\mathrm{i}}{2}\langle J\Delta J\rangle\right], \tag{4.8}$$

where

$$\Delta(p)=(p^2-M^2+\mathrm{i}0)^{-1},$$

we obtain from (4.6) in the Nth VPT order

$$W^{(N)}[J]=\exp(-\mathrm{i}\pi/4)\,T^{1/2} \cdot\int\frac{\mathrm{d}v}{2\sqrt{\pi}}\exp(\mathrm{i}Tv^2/4) \cdot\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(n-k)!}\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right)^{n-k} \cdot\left[\det\frac{\partial^2+M^2}{\partial^2+m^2}\right]^{-1/2}\omega_k[J, M^2]. \tag{4.10}$$

The functional determinant in (4.10) is calculated by the relation $\det(\ldots)=\exp[\mathrm{Sp}\ln(\ldots)]$ and the result is

$$\left[\det\frac{\partial^2+M^2}{\partial^2+m^2}\right]^{-1/2}=\exp\left\{-\mathrm{i}\frac{T}{2}[(M^2)^{1/2}-(m^2)^{1/2}]\right\}.$$

In the first VPT order we get

$$W^{(1)}[J]=\exp(-\mathrm{i}\pi/4)\,T^{1/2} \cdot\int_{-\infty}^{\infty}\frac{\mathrm{d}v}{2\sqrt{\pi}}\exp[\mathrm{i}TS(v)] \cdot\left[1+\left(\frac{\omega_1}{\omega_0}-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\ln\omega_0\right)\right], \tag{4.12}$$

where

$$S(v)=\frac{1}{2}\frac{J^2}{M^2}+\frac{v^2}{4}-\frac{1}{2}[(M^2)^{1/2}-(m^2)^{1/2}], \tag{4.13}$$

$$\tilde{\omega}_0=\exp\left\{\mathrm{i}T\left[\frac{J^2}{2M^2}-\frac{1}{2}((M^2)^{1/2}-(m^2)^{1/2})\right]\right\}, \tag{4.14}$$

$$\frac{\omega_1}{\omega_0}=-\mathrm{i}gT\left[\frac{3}{4}\frac{1}{M^2}+3\frac{J^2}{(M^2)^{5/2}}+\frac{J^4}{(M^2)^4}\right]. \tag{4.15}$$

In expressions (4.12–15) we take constant sources, $J=\mathrm{const}$ which is required for constructing the effective potential. Diagrams corresponding to expression (4.15) and to the first order of the standard perturbation theory are drawn in Fig. 1.

Introducing the generating functional of the connected Green functions

$$Z[J]=(\mathrm{i}T)^{-1}\ln W[J], \tag{4.16}$$

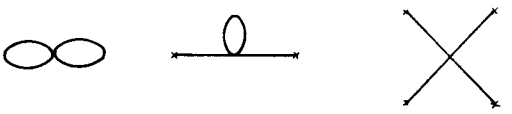

Fig. 1. Diagrams corresponding to expression (4.15)

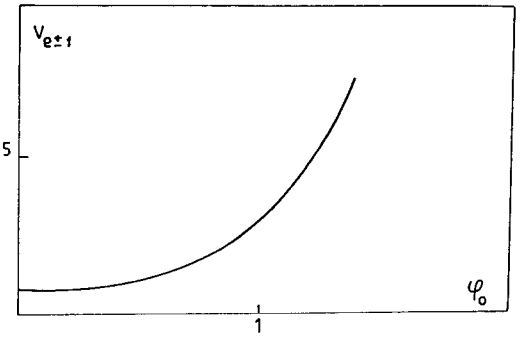

Fig. 2. The graph for the effective potential corresponding to the first VPT order (expressions (4.18–21))

we obtain for the effective potential the standard expression

$$V_{\mathrm{eff}}[\varphi_0]=J\varphi_0-Z[J], \tag{4.17}$$

where J is derived from the equation

$$\varphi_0=\mathrm{d}Z[J]/\mathrm{d}J. \tag{4.18}$$

The integrand of (4.12) contains a large parameter, T in the exponential and thus that integral may be computed by the asymptotic method of a stationary phase. Then in the first VPT order in the strong coupling limit ($m^2=0$) we get

$$Z^{(1)}[J]=Z_0[J]+Z_1[J], \tag{4.19}$$

$$Z_0[J]=\frac{3}{4}\frac{J^2}{M^2}-\frac{3}{8}(M^2)^{1/2}, \tag{4.20}$$

$$Z_1[J]=\frac{1}{14}\frac{J^2}{M^2}+\frac{1}{8}(M^2)^{1/2} -g\left[\frac{3}{4}\frac{1}{M^2}+3J^2/(M^2)^{5/2}+J^4/(M^2)^4\right], \tag{4.21}$$

where M^2 is a new variational parameter computed by the optimisation procedure. The effective potentials obtained from (4.18–21) and corresponding to the first and second versions of optimisation almost coincide with each other. The corresponding graphs are shown in Fig. 2.

To compare with numerical results for E_0 and μ^2, we should know the expansion of $V_{\mathrm{eff}}(\varphi_0)$ about the extremum. Solving the equation of optimisation, $Z_1=0$ (version 1) we get from (4.21):

$$M^2=M_0^2\left[1+\frac{4}{3}\frac{J^2}{(M_0^2)^{3/2}}+\mathcal{O}(J^4)\right], \tag{4.22}$$

where

$$M_0^2=(6g)^{2/3}, \tag{4.23}$$

and then the effective potential reads

$$V_{\text{eff}}^{(1)}(\varphi_0)=E_0^{(1)}+\frac{\mu_{(1)}^2}{2}\varphi_0^2+\mathcal{O}(\varphi_0^4), \tag{4.24}$$

where

$$E_0^{(1)}=\tfrac{3}{8}(6g)^{1/3}=0.681\cdot g^{1/3}, \tag{4.25}$$

$$\mu_{(1)}^2=M_0^2=3.302\cdot g^{2/3} \tag{4.26}$$

to be compared with the exact values given by (3.31) and (3.33). The second version of optimisation leads to the same values for E_0 and μ^2.

As for the behaviour of $V_{\text{eff}}(\varphi_0)$ for large fields φ_0, it may be found from (4.18–21) that

$$V_{\text{eff}}^{(1)}(\varphi_0)\underset{\varphi_0\to\infty}{\sim} g\varphi_0^4. \tag{4.27}$$

Note that the equality $\mu^2=M_0^2$ from a field-theoretical point of view means that the variational parameter M^2 is nothing else than the renormalised mass of the field φ. This connection will also hold true for spaces of larger dimensions.

5 Conclusion

Thus, we have proposed the method of nonperturbative calculation of functional integrals, we have called the variational perturbation theory. The method is based on the mere computation of the Gaussian functional quadratures; it does not require new diagrams and uses only those that appear in the standard perturbation theory in the same order of approximation.

Within this method, a quantity we are interested in is represented by a series whose convergence may be governed by variational parameters. This approach allows one to obtain convergent series, for instance, the Leibniz series that provides upper and lower series estimates for a given quantity.* The method implies the optimal choice of parameters. However, unlike many other variational approaches, our method allows us to compute corrections since we are dealing with a series and can always calculate a subsequent expansion term. Therefore we avoid the problem typical of variational approaches of the determination of stability and reliability of the results obtained. The proposed variational perturbation theory gives a regular method of computation of corrections, and without going beyond its scope allows us to answer the question concerning the realistic degree of dominance of the "leading contribution".

We have here considered only the anharmonic oscillator. However, owing to the functional integral formalism that allows, at least formally, the consideration of an arbitrary number of dimensions, the method may happen to work not only in quantum mechanics but also in models of quantum field theory.

* These problems will be discussed in detail in subsequent publications

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