Nonperturbative method of calculation of functional integrals

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A method is proposed to calculate functional integrals beyond the scope of perturbation theory. The method uses only the Gauss quadratures and only the diagrams of the standard perturbation theory. By using the anharmonic oscillator as an example the method is demonstrated to be very effective.

In the present paper we propose a nonperturbative method for calculating functional integrals. Here we restrict ourselves to the one-dimensional case and consider a quantum mechanical anharmonic oscillator in the limit of strong coupling. The proposed approach, as well as the standard perturbation theory, uses only the Gauss functional quadratures. However, it does not reduce to a certain way of summing an asymptotic series of perturbation theory [#1]. The proposed method contains an element of the variational procedure. However, in contrast with the ordinary variational approaches [#2] in which one encounters difficulties in evaluating whether the "main contribution" is really main, in the method under consideration the sought quantity is represented as a series allowing one to calculate a correction and evaluate the stability and reliability of the obtained results [#3].

The series arising in this approach will be called the series of variational perturbation theory.

First, let us consider the problem of calculating the ground state energy (vacuum energy) of an harmonic oscillator with the Euclidean action

$$S=S_0+m^2\tilde{S}+gS_I\,, \tag{1}$$

where

$$S_0=\tfrac{1}{2}\int \mathrm{d}t\,\dot{\varphi}^2\,,\quad \tilde{S}=\tfrac{1}{2}\int \mathrm{d}t\,\varphi^2\,,\quad S_I=\int \mathrm{d}t\,\varphi^4 \tag{2}$$

in the limit of strong coupling $g/m^3\to\infty$. For further application of the method to quantum field theory we shall calculate the quantity $\mathrm{d}E_0/\mathrm{d}g$ connected with the four-point Green function $G_4(x_i)$ by the relation

$$\frac{\mathrm{d}E_0}{\mathrm{d}g}=g^{-2/3}G_4(0)\,, \tag{3}$$

where

$$G_4(0)=N^{-1}\int \mathscr{D}\varphi\,\varphi^4(0)\exp[-(S_0+\omega^2\tilde{S}+S_I)]\,, \tag{4}$$

$$N=\int \mathscr{D}\varphi\exp[-(S_0+\omega^2\tilde{S}+S_I)]\,. \tag{5}$$

In the latter expressions we have passed to dimensionless variables making the change

$$\varphi\to g^{-1/6}\varphi\,,\quad t\to g^{-1/3}t\,,\quad \omega^2=g^{-2/3}m^2\,.$$

The limit of strong coupling corresponds to $\omega^2=0$. Now we introduce the functional

$$A=\theta S_0+\kappa\tilde{S}\,, \tag{6}$$

with as yet arbitrary parameters θ and κ. We represent the dimensionless action S in the form [#4]

$$S=S_0'+S_I'\,, \tag{7}$$

[#1] Problems related to the summation of perturbation series were discussed in refs. [1,2].

[#2] Here we mention the wide-spread method of the Gauss effective potentials [3,4].

[#3] The problems of the stability of the Gauss effective potential was considered in ref. [5].

[#4] A similar approach has been considered in ref. [6].

where

$$S_0' = S_0 + \omega^2 \tilde{S} + A^2 , \tag{8}$$

$$S_1' = S_1 - A^2 \tag{9}$$

and write down, expanding in powers of S_1', for the Green function (4) the variational series of perturbation theory,

$$G_4(0) = N^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathcal{D}\varphi \, \varphi^4(0) S_1'^n \exp(-S_0') . \tag{10}$$

The exact value of $G_4(0)$ is independent of the parameters θ and κ. Therefore, considering the finite number of terms of the series, we can choose the values of the parameters θ and κ in the optimal way. First, we require the contribution of distant terms of the series (10) to be minimal. For this purpose we should find the asymptotics of the functional integral

$$\int \mathcal{D}\varphi \, (A^2 - S_1)^n \exp[-(S_0 + A^2)] \tag{11}$$

for large numbers of n. By the change $\varphi \to n^{1/4}\varphi$ we rewrite (11) in a form allowing the use of the steepest descent method [7],

$$n^n \int \mathcal{D}\varphi \exp(-nS_{\text{eff}}[\varphi] - n^{1/2} S_0[\varphi]) , \tag{12}$$

where

$$S_{\text{eff}}[\varphi] = A^2 - \ln(A^2 - S_1) . \tag{13}$$

The saddle-point function φ corresponding to the finite action is determined from the equation $\delta S_{\text{eff}}/\delta\varphi = 0$ and has the form

$$\varphi_0 = \pm \sqrt{\frac{2a}{b}} \frac{1}{\operatorname{ch}[\sqrt{a}\,(t - t_0)]} , \tag{14}$$

where

$$a = \kappa/\theta , \quad b = \{\theta(1 - \mathcal{D}[\varphi_0])\}^{-1} ,$$

$$\mathcal{D}[\varphi_0] = 1 - \tfrac{3}{4}(\theta\kappa^3)^{-1/2}$$

and the arbitrary parameter t_0 reflects the translational invariance of the theory. As is seen from (11), the contribution of distant terms of the series will be minimal if $A^2[\varphi_0] = S_1[\varphi_0]$. This condition leads to the following relation between the variational parameters θ and κ,

$$\kappa = (9/16\theta)^{1/3} . \tag{15}$$

Thus, there remains one free parameter θ which we fix by taking a finite number of terms of the series (10) using the relation $\partial G_4^{(N)}(0)/\partial\theta = 0$, which is naturally fulfilled for an exact value of the function $G_4(0)$.

Now let us discuss some technical details of the calculation of the functional integral in (10). First, note that due to the presence of the term A^2 in the exponent (see expression (9)) we deal with an integral of the non-Gaussian type. This problem is easily solved by using the Fourier transform

$$\exp(-A^2) = \int_{-\infty}^{\infty} \frac{du}{2\sqrt{\pi}} \exp(-\tfrac{1}{4}u^2 \pm iuA) , \tag{16}$$

which preserves in the exponent only terms quadratic in the fields φ. Second, in calculating the expression $(A^2 - S_1)^n$ we shall use the fact that any power of A^2 can be determined by the corresponding number of differentiations of the expression $\exp(-\alpha A^2)$ with respect to the parameter α. This procedure, from the point of view of the diagram technique for Green functions, allows one not to take into consideration new diagrams but only ordinary ones. Thus, for the Nth order of our approximation we need only those diagrams which form the Nth order of ordinary perturbation theory.

As a result of some calculations, on the basis of (3) and expression (10) for the ground level energy in the limit of strong coupling we find

$$E_0^{(N)} = 3g^{1/3} \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{(1+m)A_{1+m}}{(n-m)!} (\tfrac{16}{9}\theta)^{1/3+m/2} \times R_{n,m}(\theta) [\Gamma(1+\tfrac{1}{2}m)\Gamma(1+\tfrac{3}{2}m)]^{-1} , \tag{17}$$

where

$$R_{n,m}(\theta) = \int_0^{\infty} \mathrm{d}x \, x^{m/2} \, \mathrm{e}^{-x} \times \int_0^{\infty} \mathrm{d}y \, y^{3m/2} (\theta x + y)^{2(n-m)} \exp[-(\theta x + y)^2] .$$

The coefficients A_n in (17) are the ordinary coefficients of the perturbation series

$$E_0(g)=\tfrac{1}{2}m+\sum_{n=1}^{\infty}A_n(g/m^3)^n .$$

Their values can be found, for instance, in ref. [8].

Expression (17) can be simplified having in mind that the optimal value of the parameter $\theta \ll 1$. Then, in the first nontrivial order we get

$$E_0^{(1)}=g^{1/3}\{\tfrac{9}{4}\sqrt{\pi}\,A_1x^2+[4\Gamma(\tfrac{5}{4})/\sqrt{\pi}]\,A_2x^5\}, \tag{18}$$

where

$$x=(\tfrac{16}{9}\theta)^{1/6}, \quad A_1=\tfrac{3}{4}, \quad A_2=-\tfrac{21}{8}.$$

Solving the optimisation equation $\partial E_0^{(1)}/\partial x=0$, we find $\theta=0.028$ and the corresponding energy value

$$E_0^{(1)}=0.660g^{1/3}. \tag{19}$$

One can easily verify that the contribution of subsequent orders in fact is small and amounts to several per cent. The obtained value (19) is to be compared with the exact one [9],

$$E_0^{\text{exact}}=0.668g^{1/3}. \tag{20}$$

The next quantity to be considered is the value of the propagator $G_2(p=0)=\mu^{-2}$,

$$G_2(0)=\int \mathrm{d}t \int \mathcal{D}\varphi\,\varphi(t)\varphi(0)\,\mathrm{e}^{-S}. \tag{21}$$

Using the same method as for calculating the ground level energy for (21) in the Nth order we get the series

$$G_2^N(0)=\tfrac{1}{2}\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{\Gamma(n+\frac{1}{2}(1-\frac{1}{2}m))}{(n-m)!}\frac{B_m}{\Gamma(1+\frac{3}{2}m)}x^{2+3m}, \tag{22}$$

where the B_m are the usual dimensionless coefficients of perturbation theory for the quantity (21). From (22) in the first nontrivial order we find

$$\mu_{(1)}^2=3.078g^{2/3}. \tag{23}$$

The exact numerical value of this quantity [10] is

$$\mu_{\text{exact}}^2=3.009g^{2/3}. \tag{24}$$

Finally, we shall consider the construction of the nonperturbative effective potential within the proposed method. We shall start with the generating functional of the Green functions

$$W[J]=\int \mathcal{D}\varphi\,\exp\{\mathrm{i}(S[\varphi]+\langle J\varphi\rangle)\}. \tag{25}$$

In analogy with the previous case we shall introduce the variational parameter having written the action in the form [#5]

$$S[\varphi]=[S_0-m^2\tilde{S}-(a^2/T)\tilde{S}^2]-[gS_{\mathrm{I}}-(a^2/T)\tilde{S}^2]. \tag{26}$$

Introducing a variational parameter as the ratio a^2/T we aim at calculating the effective potential that arises from the effective action at constant field configuration. In this case, the parameter a^2 in (26) will be independent of T.

Expanding further the integrand exponent in (25) in powers of the new interaction action equal to the second bracket in (26), we get the expression

$$W^{(N)}[J]=\exp(-\tfrac{1}{4}\mathrm{i}\pi)T^{1/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}v}{2\sqrt{\pi}}\exp(\tfrac{1}{4}\mathrm{i}Tv^2)\times\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(n-k)!}\left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right)^{n-k}\left(\det\frac{-\partial^2-M^2}{-\partial^2-m^2}\right)^{-1/2}\times\omega_k[J,M^2], \tag{27}$$

where $\omega_k[J,M^2]$ are the usual coefficients of perturbation theory for the functional (26) which are determined by the standard diagrams with the propagator

$$\Delta(k)=(k^2-M^2+\mathrm{i}0)^{-1}, \quad M^2=m^2+\sqrt{\epsilon}\,av.$$

To calculate the effective potential it suffices to use only the constant source $J=\text{const}$. The numerical integral in (27) contains a large parameter in the exponent and can be calculated by the stationary phase method. As a result, in the first order for the generating functional of coupled Green functions $Z[J]=(\mathrm{i}T)^{-1}\ln(W[J])$ in the limit of strong coupling we get

$$Z^{(1)}[J]=J^2/M^2-\tfrac{1}{4}(M^2)^{1/2}-g\left(\frac{3}{4}\frac{1}{M^2}+3\frac{J^2}{(M^2)^{5/2}}+\frac{J^4}{(M^2)^4}\right). \tag{28}$$

[#5] In the framework of field theory we work in the pseudo-Euclidean metric in the space of n dimensions and T is the volume of the one-dimensional x-space.

Here J is the variational parameter that can be found from the optimisation condition $\partial Z^{(1)}/\partial M^2=0$. The effective potential is constructed with the use of (28) in a standard way,

$$V_{\text{eff}}(\varphi_0)=J\varphi_0-Z[J]\,,$$

where J is found from the equation $\varphi_0=\mathrm{d}Z[J]/\mathrm{d}J$. For the comparison with the numerical values (20) and (24) it is sufficient to find the expansion $V^{(1)}_{\text{eff}}(\varphi_0)$ in the vicinity of the extremum. Eq. (28) yields

$$V^{(1)}_{\text{eff}}(\varphi_0)=E_0^{(1)}+\tfrac{1}{2}\mu^2_{(1)}\varphi_0^2+\mathrm{O}(\varphi_0^4)\,, \qquad (29)$$

where

$$E_0^{(1)}=\tfrac{3}{8}(6g)^{1/3}=0.681g^{1/3}\,,$$

$$\mu^2_{(1)}=(6g)^{2/3}=3.302g^{2/3}\,. \qquad (30)$$

Expression (30) should be compared with the exact values (20) and (24).

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