SPECTROSCOPY OF A SINGULAR LINEAR OSCILLATOR

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It is shown that a delta-shaped term $V(x)=\Omega\delta(x)$ changes radically the linear oscillator spectroscopy and in some limiting cases leads to the production of anomalous double degenerate energy levels for one-dimensional quantum mechanics and to the falling of a particle onto the centre.

1. Introduction

In the programme of investigations to be performed at LEP of much importance will be the quark physics at small distances, in particular, the toponium physics [1]. In this connection it is interesting to elucidate how sensitive physical observables in quantum mechanics are to the change of a potential in the vicinity of the coordinate origin.

In the present paper this problem is investigated in the simple model of the singular linear oscillator

$$U(x)=\tfrac{1}{2}\mu\omega^2x^2+\Omega\delta(x) .$$

In other words, to the linear oscillator hamiltonian is added the delta-shaped term $V(x)=\Omega\delta(x)$ and the changes in the oscillator energy spectrum following after this addition are found.

2. Wavefunctions

The Schrödinger equation for the singular linear oscillator can be transformed into

$$\frac{d^2\Phi(z)}{dz^2}+[\lambda+\tfrac{1}{2}-\tfrac{1}{4}z^2-\gamma\delta(x)]\Phi(z)=0 . \qquad (1)$$

The following notation is used:

$$z=(2\mu\omega/\hbar)^{1/2}x , \quad \lambda=E/\hbar\omega-\tfrac{1}{2} ,$$

$$\gamma=(\Omega/\hbar)(2\mu/\hbar\omega)^{1/2} .$$

The parabolic cylinder functions $D_\lambda(z)$ and $D_\lambda(-z)$ are linear independent solutions of eq. (1) at $|z|\neq 0$ so that

$$\Phi_1=AD_\lambda(z)+BD_\lambda(-z) , \quad z>0 ,$$

$$\Phi_2=CD_\lambda(z)+ED_\lambda(-z) , \quad z<0 . \qquad (2)$$

Solutions (2) should satisfy the standard conditions

$$\lim_{|z|\to\infty}\Phi_i(z)=0 , \qquad (3)$$

$$\lim_{z\to 0+}\Phi_1(z)=\lim_{z\to 0-}\Phi_2(z)\equiv\Phi(0) , \qquad (4)$$

$$\left(\frac{d\Phi_1}{dz}\right)_{z=0+}-\left(\frac{d\Phi_2}{dz}\right)_{z=0-}=\gamma\Phi(0) . \qquad (5)$$

It follows from the asymptotic formulae [2]

$$D_\lambda(z)\xrightarrow{z\to\infty}e^{-z^2/4}z^\lambda ,$$

$$D_\lambda(z)\xrightarrow{z\to-\infty}-\frac{\sqrt{2\pi}}{\Gamma(-\lambda)}e^{i\pi\lambda}e^{z^2/4}z^{-\lambda-1} ,$$

that the condition (3) holds only if $B=C=0$.

Thus,

$$\Phi_1(z)=AD_\lambda(z) , \quad \Phi_2(z)=ED_\lambda(-z) . \qquad (6)$$

The function $D_\lambda(z)$ is expressed through the degenerate hypergeometric functions [2],

$$D_\lambda(z)=2^{\lambda/2}\mathrm{e}^{-z^2/4}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(1-\lambda))}F(-\tfrac{1}{2}\lambda;\tfrac{1}{2};\tfrac{1}{2}z^2)\right.$$

$$\left.+\frac{z}{\sqrt{2}}\frac{\Gamma(1-\frac{1}{2})}{\Gamma(1-\frac{1}{2}\lambda)}F(\tfrac{1}{2}(1-\lambda);\tfrac{3}{2};\tfrac{1}{2}z^2)\right). \qquad (7)$$

At negative integers $\lambda=N, D_\lambda(z)$ is connected with the Hermite polynomials [2],

$$D_N(z)=2^{-N/2}\mathrm{e}^{-z^2/4}H_N(z/\sqrt{2}). \qquad (8)$$

Now taking into account (7) and (14) we have

$$\frac{A-E}{\Gamma(\frac{1}{2}(1-\lambda))}=0. \qquad (9)$$

Two cases are possible:

(a) $\lambda=2n+1, \quad n=0,1,...,$

(b) $A=E$.

In case (a), as follows from (8) and (6),

$$\Phi_1=A\,2^{-n-1/2}\mathrm{e}^{-z^2/4}H_{2n+1}(z/\sqrt{2}), \quad z>0,$$

$$\Phi_2=-E\,2^{-n-1/2}\mathrm{e}^{-z^2/4}H_{2n+1}(z/\sqrt{2}), \quad z<0.$$

Substituting these functions into (5) and considering that (a) $\Phi(0)=0$, we get that $A=-E$. Thus, in case (a) the singular oscillator states coincide with odd states of an ordinary linear oscillator,

$$\Phi_\lambda^{(-)}(x)=\left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^{2n+1}(2n+1)!}}$$

$$\times\exp[-(\mu\omega/2\hbar)x^2]H_{2n+1}(\sqrt{\mu\omega/\hbar}\,x). \qquad (10)$$

In case (b) the wavefunction (6) is even in z and can be written down by a single formula

$$\Phi_\lambda^{(+)}(z)=AD_\lambda(|z|). \qquad (11)$$

3. Energy spectrum

In the previous section we have found that odd states of the linear oscillator do not react on the inclusion of the delta-shaped potential. The situation changes for even states. Indeed, from (11), (7) and (5) one can easily derive the transcendental equation

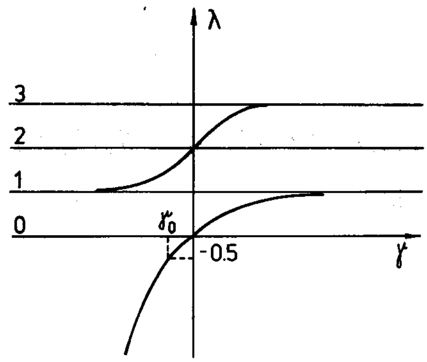

Fig. 1. Dependence of index λ on the parameter γ ($\gamma_0=-0.6$).

$$-\gamma=\Gamma(\tfrac{1}{2}(1-\lambda))/\Gamma(-\tfrac{1}{2}\lambda), \qquad (12)$$

defining the dependence of the energy spectrum, described by the wavefunctions (11), on the parameter γ and consequently on Ω. It follows from (12) that at $\gamma=0$ the index λ runs through non-negative even values. The dependence of the first even levels $\lambda(0)=0$ and $\lambda(0)=2$ on the parameter γ is depicted according to (12) by a diagram (fig. 1). With changing parameter γ the remaining excited levels with positive parity for which $\lambda(0)=4,6$ behave qualitatively in the same manner as the level $\lambda(0)=2$. Therefore, we can draw the following conclusions:

(a) Upon including the delta-potential, the initially excited even levels $\lambda(0)=2,4,6,...$ acquire additions with the sign of the parameter γ.

(b) With increasing $|\gamma|$ these excited levels approach odd levels $\lambda=3,5,7,...$ and $\lambda=1,3,5,...$, for $\gamma>0$ and $\gamma<0$, respectively.

(c) The normal level of the oscillator $\lambda(0)=0$ raises with increasing γ and approaches the first odd level $\lambda(0)=1$.

(d) With changing γ towards negative values, the normal level lowers gradually and at $\gamma=\gamma_0=-0.6$ enters into a delta-shaped well and then goes downwards ($E_0\to-\infty$).

(e) At $|\gamma|=\infty$ even excited levels fuse with odd ones, i.e. there appear double degenerate energy levels anomalous for the one-dimensional quantum mechanics. An anomalous level occurs also in the limit $\gamma=\infty$ upon fusing of the normal level of the oscillator with its first odd level $\lambda=1$.

Apart from (10) the anomalous level is described by the even wavefunction

$$\Phi_\lambda^{(+)}(x)=\left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4}\frac{\exp[-(\mu\omega/2\hbar)x^2]}{\sqrt{2^{2n+1}(2n+1)!}}$$

$$\times H_{2n+1}(\sqrt{\mu\omega/\hbar}\,|x|)\,. \qquad (13)$$

4. Falling onto the centre

As $\gamma\to-\infty$ the normal level $E_0\to-\infty$, i.e. a particle falls onto the centre. Let us show that one can derive an explicit form of the function $|\Phi|^2$ in this limiting case. Let the wavefunction (11) obey the normalisation condition

$$\int_{-\infty}^{\infty}|\Phi_\lambda^{(+)}(x)|^2\,\mathrm{d}x=1\,.$$

Here, we use the formula [3]

$$\int_0^{\infty}|D_\lambda(z)|^2\,\mathrm{d}z=\sqrt{\tfrac{1}{8}\pi}\,\frac{\psi(\frac{1}{2}(1-\lambda))-\psi(-\frac{1}{2}\lambda)}{\Gamma(-\lambda)}\,,$$

in which ψ is a logarithmic derivative of the gamma-function, and determine the normalisation factor A:

$$\Phi_\lambda^{(+)}(x)$$

$$=\left(\frac{4\mu\omega}{\pi\hbar}\right)^{1/4}\left(\frac{\Gamma(-\lambda)}{\psi(\frac{1}{2}(1-\lambda))-\psi(-\frac{1}{2}\lambda)}\right)^{1/2}D_\lambda(|x|)\,.$$

Let us pass in (14) to the limit $\gamma\to-\infty$, i.e. to $\lambda\to-\infty$. Since, according to ref. [3],

$$D_\lambda(z)\xrightarrow{\lambda\to-\infty}\frac{1}{\sqrt{2}}\,\mathrm{e}^{-\lambda/2}(-\lambda)^{\lambda/2}\exp(-\sqrt{-\lambda}\,|z|)\,,$$

then using the asymptotic expression for the gamma-function [2]

$$\Gamma(-\lambda)\xrightarrow{\lambda\to-\infty}\sqrt{2\pi}\,\mathrm{e}^{\lambda}(-\lambda)^{-\lambda-1/2}\,,$$

the formula [4]

$$\psi(y)\xrightarrow{y\to\infty}\ln y-1/y\,,$$

and the representation of the delta-function as the limit

$$\lim_{\lambda\to-\infty}\sqrt{-\lambda}\exp(-2\sqrt{-\lambda}\,|z|)=\delta(z)\,,$$

one can easily show that

$$\lim_{\lambda\to-\infty}|\Phi_\lambda^{(+)}(x)|^2=\delta(x)\,.$$

This relation implies that a particle is localised at the coordinate origin as $\gamma\to-\infty$.

5. Conclusion

We have proved that with the delta-shaped interaction $\Omega\delta(x)$ added, the spectroscopy of even states of the linear oscillator changes completely and in some limiting cases, which have been thoroughly discussed above, this leads to the appearance of double degenerate energy levels anomalous for the one-dimensional quantum mechanics and to the falling of a particle onto the centre. The physical reason for such changes lies in the phenomena of reflection and capture at the singular point $x=0$ (see the boundary condition (5)). The results obtained are in qualitative agreement with those known in an analogous problem on the effects of the delta-shaped potential $\Omega\delta(x)$ on the level of an infinite potential well [5]; this agreement testifies to our results being common. Note one more interesting phenomenon that arises upon including in the initial hamiltonian the system of long-range attraction. It has been proved in ref. [6] that in the one-dimensional Bose-gas model the addition of such an interaction always leads to the production of Bose–Einstein condensation. The most striking features of the singular oscillator: the presence of double degenerate levels and falling onto the centre are inherent in the potential $U(x)=-\alpha/|x|$ describing the so-called "one-dimensional hydrogen atom" [7]. It has been proved in ref. [7] that the assertion about nondegeneracy of the discrete spectrum in one-dimensional quantum mechanics [8] is not rigorous and is violated if the potential pole is simultaneously a zero of the wavefunctions as in the "one-dimensional hydrogen atom" [7] and in the case of the singular oscillator (see (10) and (13)). From the symmetric point of view, a double degeneracy of the spectrum in the field $U=-\alpha/|x|$ is caused by the presence of a group of dynamical symmetry O(2) inherent in the "one-dimensional hydrogen atom in the vicinity of the discrete spectrum [9].

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